The Approximation Solution of Some Calculus of Variation Problems Based Euler-Lagrange Equations

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Received on: 22/10/2014 & Accepted on:8/10/2015

ABSTRACT
The proposed method transforming some of calculus of variation problems into Euler-Lagrange equations, the simplicity and effectiveness of this illustrated through some examples.
Keywords: Differential transform method; calculus of variations; Euler-Lagrange equation.

الحل التقريبي لبعض مسائل حساب التغاير بالاعتماد على معادلات اويلر-لاكرانج

الخلاصة
تم استعراض طريقة تحويل بعض مسائل حساب التغاير الى معادلات اويلر-لاكرانج وتم توضيح فعالية وبساطة هذه الطريقة من خلال عدد من الأمثلة.

INTRODUCTION

The calculus of variations is concerned with finding the maxima and minima of a certain functional. Functional minimization problems known as variational problems appear in engineering and science where minimization of functional, such as Lagrangian, strain, potential, total energy, etc., gives the laws governing the systems behavior. In optimal control theory, minimization of certain functionals gives control functions for optimum performance of the system. The brachistochrone, geodesics and isoperimetric problems have played an important role in the development of the calculus of variations [1]. The history of the calculus of variations is tightly interwoven with the history of mathematics. The field has drawn the attention of a remarkable range of mathematical luminaries, beginning with Newton, then initiated as a subject in its own right by the Bernoulli family. The first major developments appeared in the work of Euler, Lagrange and Laplace. In the nineteenth century, Hamilton, Dirichlet and Hilbert are but a few of the outstanding contributors. In modern times, the calculus of variations has continued to occupy center stage, witnessing major theoretical advances, along with wide-ranging applications in physics, engineering and all branches of mathematics [2]. Several previous works that provide survey on calculus of variations problems have been published in the open literature. In [3] they used the piecewise linear and cubic sigmoid transfer functions to solve calculus of variations problems. In [4] they have developed an accurate method for solving nonlinear variational problems. The Chebyshev wavelet FHF operational matrix of integration has been derived in details directly, and then a general formulation for the above matrix has been given. The Chebyshev wavelet operational matrix of integration is used to reduce the variational problems to solving a system of linear algebraic equations. In [5] an accurate Chebyshev finite difference method (ChFDM) for solving problems in calculus of variations is presented.

Statement of the problem

https://doi.org/10.30684/etj.34.1B.9
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The simplest form of the variational problems [6]:

$$v[y(x)] = \int_{x_0}^{x_1} F\big(x, y(x), y'(x)\big)dx\,, \qquad \dots (1)$$

Where v is the the functional that its extremum must be found. To find the extreme value ofv, the boundary points of the admissible curves are known in the following form

$$y(x_0) = \propto,\ y(x_1) = \beta, \qquad \dots (2)$$

Where $\propto$ *and* β are known. The necessary condition for the solution of the problem (1) is to satisfy the Euler–Lagrange equation

$$F_y - \frac{d}{dx}F_{y'} = 0, \qquad \dots (3)$$

The boundary statement in (3) does not have solution except in some cases, if exist it will be unique according to the boundary conditions in (2) [1]. Note that in many variational problems the existence of a solution is obvious from the physical or geometrical meaning of the problem and if the solution of Euler's equation satisfies the boundary conditions, it is unique. Also this unique extreme will be the solution of the given variational problem [1, 16].

The general form of the variational problem (1) [6] is

$$v[y_1, y_2, \dots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, \dots, y_n, {y'}_1, {y'}_2, \dots, {y'}_n)dx \qquad \dots (4)$$

With the given boundary conditions for all functions

$$y_1(x_0) = \propto_1, y_2(x_0) = \propto_2, \dots, y_n(x_0) = \propto_n, \qquad \dots (5)$$

$$y_1(x_1) = \beta_1, y_2(x_1) = \beta_2, \dots, y_n(x_1) = \beta_n \qquad \dots (6)$$

Solution of Calculus of variations problem using Differential Transform Method (DTM)

In this section, DTM will be used to find an approximate solution of calculus of variations problem. We present a definition of DTM.

Definition [7]: The differential transform of the function y(x) for the kth derivative is defined as follows:

$$\mathrm{Y(k)} = \frac{1}{\mathrm{k!}}\left[\frac{\mathrm{d^k y(x)}}{\mathrm{dx^k}}\right]_{\mathrm{x=x_0}}, \qquad \dots (7)$$

Where

y(x) is the original function and Y(k) is the transformed function. The inverse differential transform of Y(k) is defined as

$$\mathrm{y(x)} = \sum_{\mathrm{k=0}}^{\infty}(\mathrm{x - x_0})^{\mathrm{k}}\mathrm{Y(k)}\ . \qquad \dots (8)$$

Note that, the substitution of (7) into (8) gives:

$$\mathrm{y(x)} = \sum_{\mathrm{k=0}}^{\infty}(\mathrm{x - x_0})^{\mathrm{k}}\frac{1}{\mathrm{k!}}\left[\frac{\mathrm{d^k y(x)}}{\mathrm{dx^k}}\right]_{\mathrm{x=x_0}} \qquad \dots (9)$$

Which is Taylor's series for y(x) at $\mathrm{x = x_0}$ [1],[8],[9],[10],[11],[12],[13],[14].

Example [17]

The first example contains. Consider the following variational problem $\min \mathrm{j}[x(t), y(t)] = \int_0^1 (x^2 + y^2)dt,$

With the boundary conditions

$$y(0) = \propto, y'(0) = -2 \qquad \dots (10)$$

The corresponding Euler-Lagrange equation $y'' - y = 0,$

Where $\begin{cases} x'(t) = -\frac{1}{2}y(t),\ x(0) = 1 \\ y'(t) = -2x(t),\ y(1) = 0 \end{cases}$

The exact solution is: $x(t) = \frac{\cosh(1-t)}{cosh1}$

We have $y' = -2x \Rightarrow y'' = -2x'(t)$

And we have $x(t) = \frac{\cosh(1-t)}{cosh1} \Longrightarrow x'(t) = \frac{sinh(t-1)}{cosh1} \Longrightarrow y(t) = -2\frac{\sinh(t-1)}{cosh1}$

Using DTM

$y(t) = \sum_{k=0}^{n} Y(k).t^k \,, t \in [0,1]$ … (11)

$$\frac{d^n y}{dx^n} = \frac{(k+2)!}{k!} \cdot Y(k+2)$$
$$y'' = \frac{(k+2)!}{k!} \cdot Y(k+2)$$
$$= \frac{(k+2)\cdot(k+1)\cdot k!}{k!} \cdot Y(k+2)$$
$$= (k+2)\cdot(k+1)\cdot Y(k+2)$$

We get

$$\begin{cases}(k+1)(k+2)Y(k+2) - Y(k) = 0 & \ldots(12)\\ Y(0) = \propto, Y(1) = -2 & \ldots(13)\end{cases}$$

Substituting (13) into (12) and by an iterative procedure, we achieve

$$Y(k) = \{\propto/2, -1/6, \propto/24, -1/60, \propto/720, -1/2520, \propto/40320, -1/181440, \propto/3628800\}, k = 2,3,\ldots,10.$$

Substituting all of $Y(k)$ and into (11) and the 11-term of DTM Series solution of $y(t)$ can be given by

$y(t) = \propto -2t + \propto/2\, t^2 - t^3/6 + \propto t^4/24 - t^5/60 + \propto t^6/720 - t^7/2520 + \propto t^8/40320 - t^9/181440 + \propto t^{10}/3628800$ … (14)

This gives the approximation of the $y(t)$ in a series form. Now to find the constants $\propto$ the boundary conditions in (13) is imposed on the approximation solutions in (14). We have

$\propto = 1.523188281304273$ … (15)

Replacing (15) into (14), an approximation solution is obtained for $y(t)$ by the following tables:

Table (1). an approximation solution of $y(t)$ for k=10

t	$y(t) = \sum_{k=0}^{10} y(k).t^k$	$y(t) = \frac{-2\sinh(t-1)}{\cosh 1}$
0	1.523188281304273	1.523188311911530
0.1	1.330477071404862	1.330477102165283
0.2	1.151081723224859	1.151081754446303
0.3	0.983206787822903	0.983206819817760
0.4	0.825172116422569	0.825172149509178
0.5	0.675396044933011	0.675396079422822
0.6	0.532379564199213	0.532379600307790
0.7	0.394691317704033	0.394691355171603
0.8	0.260953276945440	0.260953313778318
0.9	0.129826951898996	0.129826980858764
1	-8.232652556939039e-016	0

Higher accuracy is also obtained using more components of $y(t)$, for example if *k=15* we get $\propto = 1.523188293271723$ and for *k=20* we get $\propto = 1.523188293267034$ as shown in the following table:

Table (2) an approximation solution of $y(t)$ for k=15, 20

t	$y(t)=\sum_{k=0}^{15} Y(k).t^k$	$y(t)=\sum_{k=0}^{20} Y(k).t^k$	$y(t) = \frac{-2sinh(t-1)}{cosh\,1}$
0	1.523188293271723	1.523188293267034	1.523188311911530
0.1	1.330477083432199	1.330477083427487	1.330477102165283
0.2	1.151081735432456	1.151081735427673	1.151081754446303
0.3	0.983206800332868	0.983206800327966	0.983206819817760
0.4	0.825172129358679	0.825172129353610	0.825172149509178
0.5	0.675396058411088	0.675396058405801	0.675396079422822
0.6	0.532379578273362	0.532379578267806	0.532379600307790
0.7	0.394691332172624	0.394691332166763	0.394691355171603
0.8	0.260953290821922	0.260953290815855	0.260953313778318
0.9	0.129826962262375	0.129826962256997	0.129826980858764
1	3.001209292644516e-016	2.059138837327129e-016	0

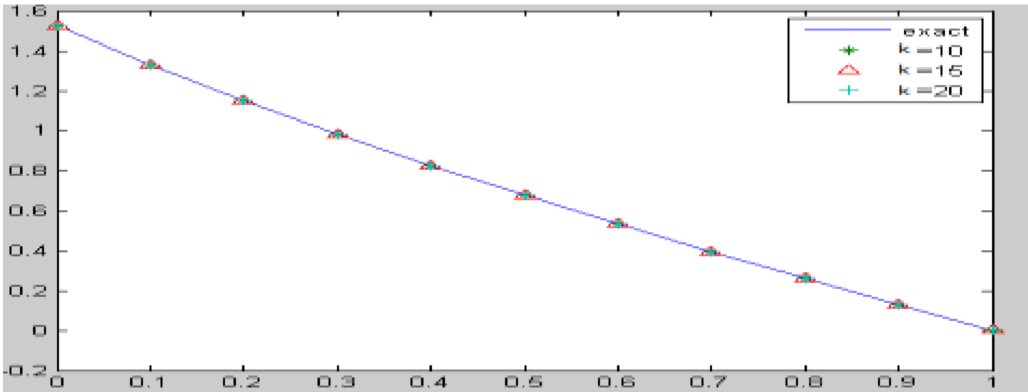


Figure (1). Conforming of approximation solution of $y(t)$ with the exact solution

It has been configured from the presents that the value of $\propto$ will be the same from *k=20* and so on, which is$\propto$= 1.523188293267034.

Example [14]

(Harmonic oscillator). Consider the following harmonic oscillator:

$\min v[y] = \int_0^1 (y'^2 - y^2)dx,$ … (16)

Subject to the boundary conditions:

$$\begin{cases} y(0) = 0 \\ y(1) = 1 \end{cases} \quad \dots (17)$$

The Euler-Lagrange equation is:

$y'' + y = 0,$ … (18)

Whose exact solution is: $y(x) = \frac{\sin x}{\sin 1}$ … (19)

Using DTM $y(x) = \sum_{k=0}^{n} Y(k).x^k \,, x \in [0,1]$ … (20)

$$y'' = \frac{(k+2)!}{k!} \cdot Y(k+2)$$
$$= \frac{(k+2)\cdot(k+1)\cdot k!}{k!} \cdot Y(k+2)$$
$$= (k+2)\cdot(k+1)\cdot Y(k+2)$$

We get

$$\begin{cases} Y(k+2) = -\frac{Y(k)}{(k+1)(k+2)} & \dots (21) \\ Y(0) = 0\,,\ Y(1) = \propto & \dots . (22) \end{cases}$$

Substituting (22) into (21) and by an iterative procedure, we achieve
$Y(k) = 0, k = 2,4,6,8,10.$
$Y(k) = \{-\propto/6\,, \propto/120\,, -\propto/5040\,, \propto/362880\,, -\propto/3628800 \quad \}, k = 3,5,7,9,11.$
Substituting all of into (20) and the 12-term of DTM Series solution of $y(x)$ can be given by:

$$y(x) = \propto x - \propto x^3/6 + \propto x^5/120 - \propto x^7/5040 + \propto x^9/362880 - \propto x^{11}/3628800 \quad \dots . (23)$$

This gives the approximation of the $y(x)$ in a series form. Now to find the constant $\propto$ the boundary conditions in (22) is imposed on the approximation solutions in (23). We have

$$\propto = 1.188395106003845, \quad \dots (24)$$

Replacing (24) into (23), an approximation solution is obtained for $y(x)$ by the following tables:

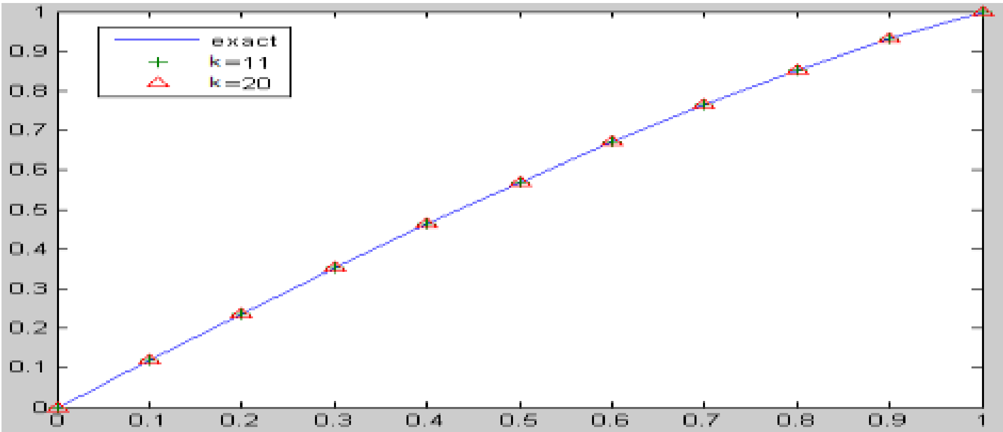
Table (2). an approximation solution of $y(x)$ for k=11

x	$y(x) = \sum_{k=0}^{11} Y(k).x^k$	$y(x) = \frac{sinx}{sin\,1}$
0	0	0
0.1	0.118641543758733	0.118641543736199
0.2	0.236097660429910	0.236097660385065
0.3	0.351194767321581	0.351194767254875
0.4	0.462782852187729	0.462782852099829
0.5	0.569746963770469	0.569746963662275
0.6	0.671018352024317	0.671018351897113
0.7	0.765585146712543	0.765585146568972
0.8	0.852502467676582	0.852502467525118
0.9	0.930901865754094	0.930901865625602
1	1	1

Higher accuracy is also obtained using more components of $y(x)$, for example if *k=20* we get $\propto = 1.188395105778121$ as shown in the following table:

Table (2). an approximation solution of $y(x)$ for k=20

x	$y(x) = \sum_{k=0}^{20} Y(k).x^k$	$y(x) = \frac{sinx}{sin\,1}$
0	0	0
0.1	0.118641543736199	0.118641543736199
0.2	0.236097660385065	0.236097660385065
0.3	0.351194767254875	0.351194767254875
0.4	0.462782852099829	0.462782852099829
0.5	0.569746963662275	0.569746963662275
0.6	0.671018351897113	0.671018351897113
0.7	0.765585146568972	0.765585146568972
0.8	0.852502467525118	0.852502467525118
0.9	0.930901865625602	0.930901865625602
1	1	1



Figure(2). Conforming of approximation solution of $y(x)$ with the exact solution when k=20

CONCLUSION

The presented study in this paper illustrates calculus of variational problems which has been solved by using DTM. Comparison of the approximate solutions and the exact solutions shows that the proposed method is efficient tool. This method can be applied to many complicated linear and non-linear case studies.

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