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A Study on the Bounds of Relations

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Abstract

In this research we give a study on the bounds of relations and we demonstrate several important theorems.

الخلاصة

في هذا البحث قدمنا دراسة حول حد العلاقة مع بعض النظريات المهمة والامثلة.

1. Introduction

Following (Gillam ,1976). had introduced the concept of bounds of relations, the bound of a relation R is a finite relations non embeddable in R but whose proper restrictions are embeddable in R . the aim of this paper is to give a study on the bounds of relations and we prove that a birelation RS can have finitely many bounds, in this paper concludes some examples in the bounds.

2. Definitions

2.1 Let R , S be two relations of the same arity R is embeddable in S; if and only if there exists a restriction of S isomorphic with R and write $R \leq S$, (Ginsburg, 1990; Aigner, 2004)

2.2 A bound of a relation R is any finite relation A with same arity such that A is not embeddable in R, but every proper restriction of A is embeddable R. (Gillam, 1976).

2.3 A chain is a partial ordering whose elements are mutually comparable (Berge, 1958; Aigner, 2004).

2.4 Acycle is the relation with the following conditions , if there exists a minimum u and maximum V of A , then V dominates U. (Berge, 1958; Aigner, 2004).

2.5 Given two relations R , S we say that R is younger than s , if every finite restriction of R is embeddable is S (Berge, 1958)

2.6 A multirelation with base E is a finite sequence R of relations R_1 , R_2 , …. , R_h with base E , in the case where h = 2 , we will say a birelation R_1R_2 (Malitz, 1976)

2.7 Let R , S be two multirelations with the same base, S is called freely interpretable in R , if every local automorphism of R is a local automorphism of S (Malitz, 1976)

2.8 Given two finite sequences of natural numbers m , n a free operator P associates to each m-ary multirelation P(R) having the same base (Hemminger ,1986 ; Aigner, 2004) .

2.9 An n-ary relation with base E is a functional R which associates the value R $(x_1 , x_2 , \dots , x_n) = +$ or – the integer n will be called the arity of R for n = 1 say a unary , for n = 0 there exists two o – ary relations based on E denoted by (E , +) , (E , -) (Berge, 1958; Aigner, 2004).

3. We will prove several important theorems and examples

Proposition 3.1:- Let R, S be two relations of the same arity. Then the following three conditions are equivalent.

(1) Every finite restriction of R is embeddable in S ; in other words, R is younger than S .

(2) No bound of S is embeddable in R .

(3) Every bound of S admits an embedding of a bound of R .

Proof

Assume the first condition and let A be a bound of S. If $A \leq R$, then also $A \leq S$, hence A is not a bound of S. Conversely, if there exists a finite restriction A of R which is non-embeddable in S , then there exists a restriction of A , hence of R, which is a bound of S. Thus (1) and (2) are equivalents assume condition (2), and let A be a bound of S ; hence A is non-embeddable in R . Thus there exists a restriction of A which is a bound of R . Conversely if there exists a bound A of S which is embeddable in R , then A admits no embedding of any bound of R . Thus (2) and (3) are equivalent.

In particular, if R and S are finite, then the embeddability $R \leq S$ is equivalent to the condition that no bound of S is embeddable in R or again equivalent to the condition that every bound of S admits an embedding of a bound of R .

Another consequence:

Let R, S be two relations of the same arity. Then R and S have the same age iff R and S have the same bounds;

iff no bound of R is embeddable in S , nor is any bound of S embeddable In R , In particular, if R and S are finite, then R and S are isomorphic iff they have same bounds; iff no bound of R is embeddable in S nor is any bound of S embeddable in R .

Proposition 3.2:- Given two relations R, S of the same arity, if every finite restriction of R is embeddable in S , then every bound of R is a bound of S or is embeddable in S .

Proof:-

Let A be a bound of R . Assume that A is non-embeddable In S. Yet every proper **restriction of A is embeddable** In R , hence in S , Thus A is a bound of S .

The converse is false. Let R to be an infinite chain, and S to be the chain of cardinality 3 . Then every bound of R is one of the four relations with cardinalities 1, 2, 3. Hence every bound of R is a bound of S . Yet the chain of cardinality 4 is a bound of S and not a bound of R .

Another example with R and S finite. Let R to be the binary reflexive cycle of cardinality 3 . There exist four bounds of R, up to isomorphism. These are: the binary relation of cardinality 1 and value (-); the binary relation always (+) of cardinality 2; the identity relation of cardinality 2; and finally the chain of cardinality 3 . Let S to be the common extension of these four bounds, taken with disjoint bases, S taking the value (+) for every ordered pair whose terms belong to the bases of two distinct bounds. Then each bound of R is embeddable in S , yet R is non-embeddable in S .

The preceding example can be modified so as to make S infinite: add to our four bounds, a binary infinite relation always (+). Even we can make both R and S infinite: replace R by its extension to an infinite base, with the value (+) for all new ordered pairs: then there remain finitely many bounds of R and the preceding argument holds.

We have already said that for a relation with finite cardinality p , the bounds have at most cardinality p+1. For example a chain of cardinality p admits as a bound the chain of cardinality p+1,

Let us give examples where the maximum cardinality of bounds is at most p . Take the unary relation of cardinality p+q , which takes the value (+) on p elements and (-) on q elements. The bounds are the unary relation always (+) of cardinality p+1 , and the unary relation always (-) of cardinality q+1. Lett the binary cycle of cardinality p . the maximum possible cardinality of bounds is p. This number is taken on by the consecutively relation associated with a chain of cardinality p : this is a bound of the given binary cycle. Take the partial ordering formed of p component chains, each of cardinality q (p, q positive integers). Then as bounds, we have the chain of cardinality q+I , the identity of cardinality p+l , the relation of cardinality 1 and value (-), the relation of cardinality 2 always (+). Finally, assuming that $p , q \geq 2$, we have the reflexive cycle of cardinality 3 ; and the three relations of cardinality 3 , each having as proper restrictions, an identity and two chains of cardinality 2 , in a position of convergence, divergence, or succession. For p = q , the cardinality of the base is p^2 but the maximum cardinality of bounds is p+1.

Theorem 3.3

A birelation RS can have finitely many bounds, even though the first component R has Infinitely many bounds, and the second component S has finitely many bounds.

Proof :

Let S to be the chain of the natural numbers, and take R to be the consecutivity relation associated with S: hence R(x,y) = + iff y=x + 1. Then R has as bounds all finite binary cycles. However, a bound of RS has cardinality at most 3; so that RS has finitely many bounds. Indeed let A, B be two finite relations such that every restriction of AB. with cardinality 3 is embeddable in RS. Then A is a finite chain; and B takes the value (+) only for ordered pairs of consecutive elements (mod A) . Thus AB is embeddable in RS; hence AB cannot be a bound,

Even more curious is the fact that there exist two denumerable binary relation R, S each having infinitely many bounds; and yet their concatenation RS has only finitely many bounds

Proof:

For each finite set F , consider all birelations AB based on F , such that for all x, y distinct, if A(x,y) ≠ A(y,x) then B(x,x) ≠ B(y,y) ; and similarly if B(x,y) ≠ B(y,x) then A(x,x) ≠ A(y,y) . Let C be the class of those birelations. For any two finite disjoint sets F, G , two birelations belonging to C, one with base F and the other with base G , always admit a common extension with base F ∪ G , again belonging to C.

To see this, for each element x in F and each y in G , take the same value A(x,y)= A(y,x) , and similarly B(x,y) = B(y,x) , which is compatible with the values A(x,x), A(y,y), B(x,x), B(y,y) already imposed by the preceding Consequently, there exists a birelation RS with denumerable base, In with every birelation AB of the class C is embeddable. Moreover, every bound of RS has cardinality at most 2. Indeed, given a birelation AB of the finite cardinality greater than or equal to 3, If all its restrictions of cardinality2 belong to C, then AB Itself belong to C and hence is embeddable in RS .

However, because of the sign change in S(x,x) when we pass from an elemet x to an element y with R(x,y) ≠ R(y,x) , any finite cycle which is a restriction of R has even cardinality. Thus R has infinitely many bounds, which are all binary cycles of odd cardinality, with arbitrary values for R(x,x) , for each element x in such a cycle. Similarly for S .

Theorem 3.4:-

Let R, S be two relations, each of which is freely interpretable in the other. Then to each bound A of R , of cardinality strictly larger than the maximum of the arities of R and S , we can objectively associate a bound of S having the same base as A.

Proof :-

Let P be a free operator taking R into S, and z be a free operator taking S into R: Let B = P (A) , Then every proper restriction of B is embeddable in P (R) = S . Moreover z (B) = A , since both have the same proper restrictions, hence the same restrictions of cardinalities less than or equal to the arity of A . Hence B is not embeddable in S ; for otherwise z (B) = A would be embeddable in z (S)= R.

Results

(1) A finite 0-ary relation A can be a bound iff A is non-empty. However, if p = Card A , then A is a bound of R iff Card R = p-1 .

(2) For arity 1 , a finite unary relation A can be a bound iff A is non empty and is always (+) or always (-)Indeed if p = Card A and if A is always (+) for instance, then any unary relation (finite or infinite) which takes p-l times the value (+) admits A as a bound.

(3) For arity n≥ 2 , a finite n-ary relation A can be a bound iff A is non-empty. Moreover there exists an infinite relation R such that A is a bound of R .

Note first that the notion of bound is immediately extendible to multirelation

Theorem 3.5:- Let R be a finite multirelation and U a bound of R .Then there exists an extension of R to its base augmented by one element, for which U is again a bound.

Proof:-

Suppose first that R is a unary relation of cardinality p+q , taking the value (+) on p elements and (-) on q elements. Then U is, for example , the relation always (+) of cardinality p+l . We replace R by its extension which takes the value (+) on p element and (-) on q+1 elements .

Suppose now that R has at least one component with arity ≥ 2 . We shall argue

in the case that R is itself a binary relation, the proof in the general case being an immediate extension of our argument.

Let U be a bound of R . Add to the base of R a new element a , and let R_a,

be the extension of R defined by R_a (a , x) = R_a (x,a) = R_a (a , a) = + for every

x in the base |R| .Similarly add b and define R_b, by the analogous conditions with (-) instead of (+). Then U is either a bound of R_a or a bound of R_b . Indeed every proper restriction of U is embeddable in R , hence in R_a and in R_b . On the other hand, U is not embeddable in both:

Theorem 3.6 :-

Consider a finite set of finite relations A_1,,...,A_h. all of the same arity, and suppose that there exist relations of the same arity, with arbitrarily large

finite cardinality, each having the bounds A_1,,...,A_h and possibly other bounds Then there exists a denumerable relation having, among other ones, the bounds A_1,,...,A_h

Proof :-

Let R_i. (i integer) be an . ω-sequence of finite relations, whose cardinalities

are strictly increasing, such that each R_i. has at least the bounds A_1,,...,A_h, Let p_i denote the cardinality of R_i; we can assume that R_i has base {1 , 2 , ……..,p_i} . moreover, since for each i , the relation R_i. admits an embedding of all proper restrictions of A_i for instance, then letting k_1, denote the sum of the cardinalities of these proper restrictions, we can suppose that they are all embeddable in the restriction of R_i to {1,2,.. ., k_1 }. Similarly for A_2 . Let k_2 denote the sum of the cardinalities of all proper restrictions of A_2 : they are all embeddable in the restriction of R_i to {1,2,.. ., k_1+k_2 }. and so forth, There exists an infinite sequence, extracted from the sequence of the R_i , which is formed of relations having the same restriction S_1 to the singleton of I , From this first extracted sequence, we extract a second sequence, formed of relations all having the same restriction S_2 to the pair {1,2} . Iterating this we obtain, for each integer r , a relation S_r based on {1,2,...,r} , Where each S_r (r≥2) is an extension of S_{r-1} .

Let S denote the common extension of the S_r , based on all positive integers Then A_1, for instance, is a bound of S . Indeed A_1, is not embeddable in S , since otherwise it would be embeddable in some S_3 , hence in some R_i , moreover ; each proper restriction of A_1, is embeddable in S since it is embeddable in the restriction of each R_i to {1,2,.. ., k_1 }. hence in $S_{(k1)}$. , . Same argument for A_2, ...,A_h which are thus also bounds of S .

Theorem 3.7 :-

Consider again the finite relations A_i , ….. , A_h ; suppose that for each integer p , there exists a relation with cardinality greater than or equal to p , whose bounds are exactly A_i , ….. , A_h plus possibly some bounds of cardinalities ≥ p , Then there exists a denumerable relation whose bounds are exactly A_i , ….. , A_h

Proof:-

Let R_i (i positive integer) be our finite relations; which are listed by increasing values of p . We shall modify our construction in the preceding theorem , as follows. Let a sequence of all the finite relations U_j (j positive integer) with the same arity as the R_j , and let k_j br the finite cardinality of U_j.

Replace, each R_i by an isomorphic copy, we can suppose the following. For each i , if U_1, is embeddable in R_i , then U_1, is embeddable In the restriction of R_i to {1,2,.. ., k_1}. Again for each i , if U_2 is embeddable in R_i , then U_2 is embeddable in the restriction of R_i to the set: {l,2,...,k_1+ k_2} ; and so forth.

Now, construct relations S_r (r integer) as in the preceding theorem , and then we take their common extension S . Then A_1...,A_h are all bounds of S . It remains to prove that S has no other bound.

Suppose that B is a bound of S different from A_1,,...,A_h, . Then firstly, each proper restriction of B is embeddable in S , hence in the S_r for all r greater than some r(0) ; hence in all the R_i which extend $S_{r(0)}$. Secondly B cannot be a bound of R_i , for i sufficiently large, so that the integer p associated by hypothesis with R_i. is larger than the cardinality of B . Hence B is embeddable in all the R_i. which extend $S_{r(0)}$ and whoso index i is sufficiently large. Finally there exists r(1) ≥ r(0) such that, B is embeddable in all those R_i which extend $S_{r(1)}$.

From the first, there exists an integer k for which, if B is embeddable in R_i , then B is still embeddable in the restriction R_i/{ 1 , 2 , ….. .,k} . Hence B is embeddable in S_m, where m is the maximum of k and r(l) , Thus B is embeddable in S : contradiction.

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