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Domination and Independence in Cubic Chessboard

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ABSTRACT

In this paper, we are interested in some problems on cubic chessboard with square cells (domination and independence numbers). At the beginning, these problems are examined with one type of the chess pieces. In certain technique, studying these problems will continue with two different types. In the case of one type of chess pieces which have been dealt with in this article, the type is rook or king. The pieces of the two different types in our study are: kings with rooks together.

Keywords: Domination, Independence, Cubic chessboard, Kings and Rooks.

الهيمنة والاستقلالية في لوح الشطرنج المكعب

الخلاصة

في هذا البحث نحن نهتم بمسائل الهيمنة والاستقلالية في لوح الشطرنج المكعب ذو الخلايا المربعة . في البداية سنختبر تلك المسائل بنوع واحد من قطع الشطرنج وباسلوب معين ، سنستمر بدراسة تلك المسائل بنوعين مختلفين من قطع الشطرنج مع بعض. في حالة النوع الواحد من قطع الشطرنج ستكون القطع إما من نوع الملك أو نوع القلعة. في حالة النوعين المختلفين معا ستكون القطع من نوعي الملك والقلعة مجتمعين.

INTRODUCTION

There are two classical chessboard problems one of them is placing a maximum number of one type P of pieces such that each piece does not attack other pieces. This problem is called independence number of P and denoted by $\beta(P)$. The other problem is placing a minimum number of one kind P of pieces such that all unoccupied positions are under attack. This problem is called domination number of P and denoted by $\gamma(P)$.

In $n \times n$ square chessboard (see [1], [2] and [3]) γ and β numbers are studied for Rook "R", Bishop "B" and King "K". They proved for γ that $\gamma(R) = n$, $\gamma(B) = n$ and $\gamma(K) = \left\lfloor \frac{n+2}{3} \right\rfloor^2$, and for β that $\beta(R) = n$, $\beta(B) = 2n - 2$ and $\beta(K) = \left\lfloor \frac{n+2}{2} \right\rfloor^2$.

In [4], Joe Maio and William proved that $\gamma(R) = \min\{m, n\}$ and $\beta(R) = \min\{m, n\}$ for $m \times n$ Toroidal chessboard.

In [5], Hon-Chan Chena, Ting-Yem Hob, determined the minimum number of rooks that can dominate all squares of the STC.

Harborth, Kultan, Nyaradyova and Spendelova [6], considered the triangular hexagon board, in which the cells are hexagons and the board is a triangle. Bishops attack in straight lines through the vertices of their cells, rooks attack along straight lines through the centers of the edges of their cells, and queens have both attacks. The only general upper bound they are able to give on the independence number of the queens' graph is by the rooks bound, which is $\left\lfloor \frac{2n+1}{2} \right\rfloor$

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for all n. For $n = 3, 4, 6, 7, 13, 16, 19, 25, 31$, they found that $\beta = \left\lfloor \frac{2n+1}{2} \right\rfloor - 1$, and for the other $n \le 31$, $\beta = \left\lfloor \frac{2n+1}{2} \right\rfloor$.

In [7], [8], [9], and [10], the domination (independence) number of isosceles triangular chessboard and Rhombus chessboard with square cells are studied, one type of pieces of rooks, bishops or kings is taken. Also, some results when we have two different types of pieces together, kings with rooks, kings with bishops and rooks with bishops are gotten.

The chessboard in this paper is a new model of chessboard. It is a cubic chessboard with square cells. The same classical problems on a cubic chessboard for one type and two different types of pieces are solved. For the case of two types of pieces, we place minimum number of pieces of the first kind with a fixed number of pieces of the other kind, and look for the domination and independence number of the first type. By n_P we mean the fixed number of pieces P and by N_P the number of the cells which are attacked by one piece of P and the cell location of that piece.

The Chessboard

In this work, a cubic chessboard of size n with square cells is considered. It contains six faces $F_w, w = 1,2,\dots,6$, every face is a square chessboard of length n as in Figure 1. Two types of pieces, rooks (R) and kings (K) would be used and the pieces would move and attack as usual. We mean by the length of side, the number of cells (squares) in that side. To simplify the form of our results, we will use the matrix form where r_i denote the i^{th} row measured from top to bottom, $i = 1,2,\dots,n$. Let c_j denote the j^{th} column measured from left to right, $j = 1,2,\dots,n$ in every face. The cell (square) in F_w of i^{th} row and j^{th} column is denoted by $s_{w,i,j}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots n$ and $w = 1,2,\dots,6$.

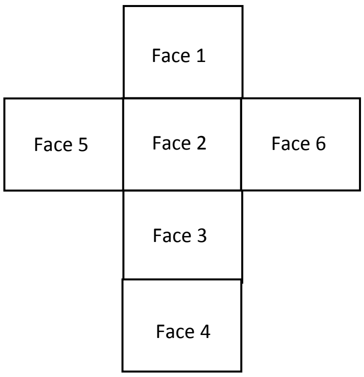


Figure (1): Cubic chessboard of size n (squares per side)

Independence and Domination of one type of pieces

In this section, the independence and domination numbers $\beta(P)$ &$\gamma(P)$ of one type of pieces are computed.

Theorem 1. , [9] *On Rectangular chessboard of size* $m \times n$
(i) $\gamma(K) = \left\lceil \frac{n}{3} \right\rceil \left\lceil \frac{m}{3} \right\rceil$ (ii) $\beta(K) = \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{m}{2} \right\rceil$

Theorem 2. *For a cubic chessboard of size* n, *we have*
$\gamma(R) = \beta(R) = n$.

Proof.

We can place the R pieces in one face such that there exist one piece only in any row or column of this face. Clearly, these R pieces dominate all cells in chessboard, so $\gamma(R) \le n$. The chessboard could not be dominated by $(n-1)$ R pieces, since there is a row which is not dominated by these pieces, thus $\gamma(R) = n$. by this distribution for R pieces, it is clear that these pieces are independent, so $\beta(R) \leq n$. We cannot add a new R piece, since two of R pieces in the same row or column are gotten. Thus, $\beta(R) = n$.

Theorem 3. *For a cubic chessboard of size* n, *we have*

$$\gamma(K) = \left\{ \begin{array}{ll} 6\left\lceil \frac{n}{3} \right\rceil^2 & , \text{ If } n \equiv 0 \ (\, mod\ 3\,) \\ \left\lceil \frac{n}{3} \right\rceil * \left\lceil \frac{4n}{3} \right\rceil + 2\left\lceil \frac{n-2}{3} \right\rceil^2 & , \text{If } n \equiv 1 \ (\, mod\ 3\,) \\ \left\lceil \frac{n}{3} \right\rceil * \left\lceil \frac{4n}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil^2 + \left\lceil \frac{n-2}{3} \right\rceil^2 & , \text{If } n \equiv 2 \ (\, mod\ 3\,) \end{array} \right\}$$

Proof.

By distributing K pieces on the first four faces of cubic chessboard (a rectangular chessboard that contains four faces F_i , $i = 1,..,4$), then there are three cases that depend on n as follows:

(i) If $n \equiv 0$ (mod 3)

The K pieces in all faces of cubic chessboard satisfy that the intersection $\mathrm{N_k}$ (the number of the cells which are attacked by one piece of K and the cell location of that piece) in each face with the other faces is empty. So, all K pieces in any face don't influence other faces (as an example, see Figure 2(b). Thus, by Theorem 1, the result is gotten.

(ii) If $n \equiv 1$ (mod 3)

Starting the distribution of the K pieces from the rightmost column in the rectangular chessboard as in Figure 2(a), results in all boundary cells in faces F_5 and F_6 would be under attack by these pieces, and the cells which are not attacked by these pieces in facesF_5 and F_6 would make two squares of dimension $n-2$. Thus by Theorem 1, the result is gotten.

(iii) If $n \equiv 2$ (mod 3)

Starting distribution of the K pieces beginning from the rightmost column in the rectangular chessboard as in Figure 2(c), would result in all boundary cells in F_6 will be under attack by these K pieces, and the cells which are not attacked by these pieces form a square chessboard of dimension $n-2$. The cells in F_5 which is not attacked by these K pieces in rectangular chessboard, forms a square chessboard, thus by Theorem 1 the result is determined.

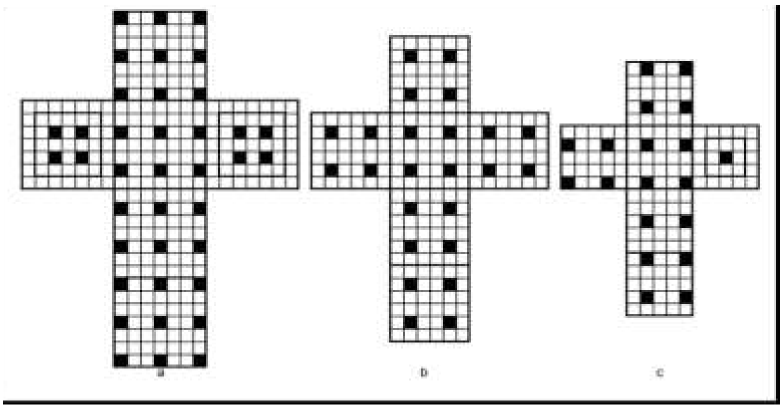


Figure (2): The domination of K pieces on cubic chessboard in case of (a) $n = 7$, (b)$n = 6$ and (c) $n = 5$.

Theorem 4. *For a cubic chessboard of size* n*, we have*

$$\beta(K)=\begin{cases}2n\left\lceil\frac{n}{2}\right\rceil+2\left\lceil\frac{n-2}{2}\right\rceil^2 & n \text{ is odd}\\ 2n\left\lceil\frac{n}{2}\right\rceil+2\left\lceil\frac{n-2}{2}\right\rceil^2 & n \text{ is even}\end{cases}$$

Proof.

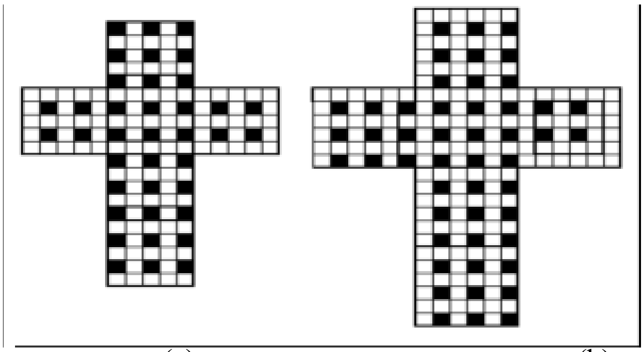
By distributing the K pieces in a rectangular chessboard that contains four faces F_i , $i = 1,..,4$, then there are two cases that depend on n as follows:

(i) If n is odd, then distribution of K pieces starting from the rightmost column in the rectangular chessboard as in Figure 3(a), would make these K pieces attack all boundary cells in faces F_5 and F_6 , and the cells which are not attacked by these pieces in the two faces F_5 and F_6 would make two squares of dimension $n-2$. Thus, by Theorem 1 we get

$$\beta(K)=2n\left\lceil\frac{n}{2}\right\rceil+2\left\lceil\frac{n-2}{2}\right\rceil^2$$

(ii) If n is even, then distribution of K pieces starting from the rightmost column in the rectangular chessboard as in Figure 3(b) ,would make these K pieces do not attack any cell in F_5 and the cells which are not attacked by these pieces in F_6 make a square of dimension $n-2$.

So, by Theorem 1, we get $\beta(K)=n^2+\left(\frac{n}{2}\right)^2+\left(\frac{n-2}{2}\right)^2=\frac{3}{2}n^2-n+1$



(a) (b)

Figure (3): The distribution of K independent pieces on cubic chessboard in case of (a) $n=5$, (b) $n=6$

Independence and Domination of Two pieces

In this section, the number of R pieces is fixed and the domination number of K pieces would be determined. The independence (domination) number of K pieces with a fixed number of rooks n_r is denoted by $\beta(K, n_r)$ ($\gamma(K, n_r)$).

Theorem 5. For K pieces with a fixed number of rooks n_r, the domination number in the cubic chessboard is given by

$$\gamma(n_r, K)=\begin{cases}\left\lceil\frac{n-n_r}{3}\right\rceil*\left\lceil\frac{3n-2n_r}{3}\right\rceil+3\left(\left\lceil\frac{n}{3}\right\rceil*\left\lceil\frac{n-n_r}{3}\right\rceil\right) & ,if\ n-n_r\equiv 0\ (mod\ 3)\\ \left\lceil\frac{n-n_r}{3}\right\rceil*\left\lceil\frac{3n-2n_r}{3}\right\rceil+2\left\lceil\frac{n}{3}\right\rceil*\left\lceil\frac{n-n_r}{3}\right\rceil+\left\lceil\frac{n-2}{3}\right\rceil*\left\lceil\frac{n-n_r-1}{3}\right\rceil & ,if\ n-n_r\equiv 1,2\ (mod\ 3)\end{cases}$$

Proof.

The R pieces in the main diagonal of one of the faces of the chessboard say F_2 are distributed, these pieces attack the rows and columns of all faces as shown in Figure 4 , and the cells which are not attacked by R pieces form three shapes one of them as rectangular chessboard of dimension $(n-n_r)(3n-n_r)$ and it is adjacent to a rectangular chessboard of dimension

$n(n-n_r)$. The others are isolated rectangular chessboard of dimension $n(n-n_r)$ as shown in Figure 4.

There are two cases depend on $n-n_r$ as follows:

(I) If $n-n_r \equiv 0\ (mod\ 3\ )$

The K pieces in a rectangular chessboard of dimension $(n-n_r)(3n-2n_r)$ are distributed. Since $n-n_r \equiv 0\ (\mathrm{mod}\ 3\ )$, then these pieces do not influence adjacent rectangular chessboard, so the other K pieces are distributed in a rectangular chessboard of dimension $n(n-n_r)$. Finally, the last K pieces in isolated rectangular chessboard of dimension $n(n-n_r)$ are distributed as shown in Figure 4(b). Thus, by Theorem 1, we get the result.

II) If $n-n_r \equiv 1,2\ (\mathrm{mod}\ 3\ )$

The K pieces in rectangular chessboard of dimension $(n-n_r)(3n-2n_r)$ are distributed, beginning from the rightmost column. These pieces attack all cells in rows and columns that are adjacent to them in F_6. Therefore, the cells which are not under attack in F_6 constitute a rectangular chessboard of dimension $(n-2)(n-n_r-1)$ as shown in Figure 4(a). Thus, by Theorem 1, we get the result.

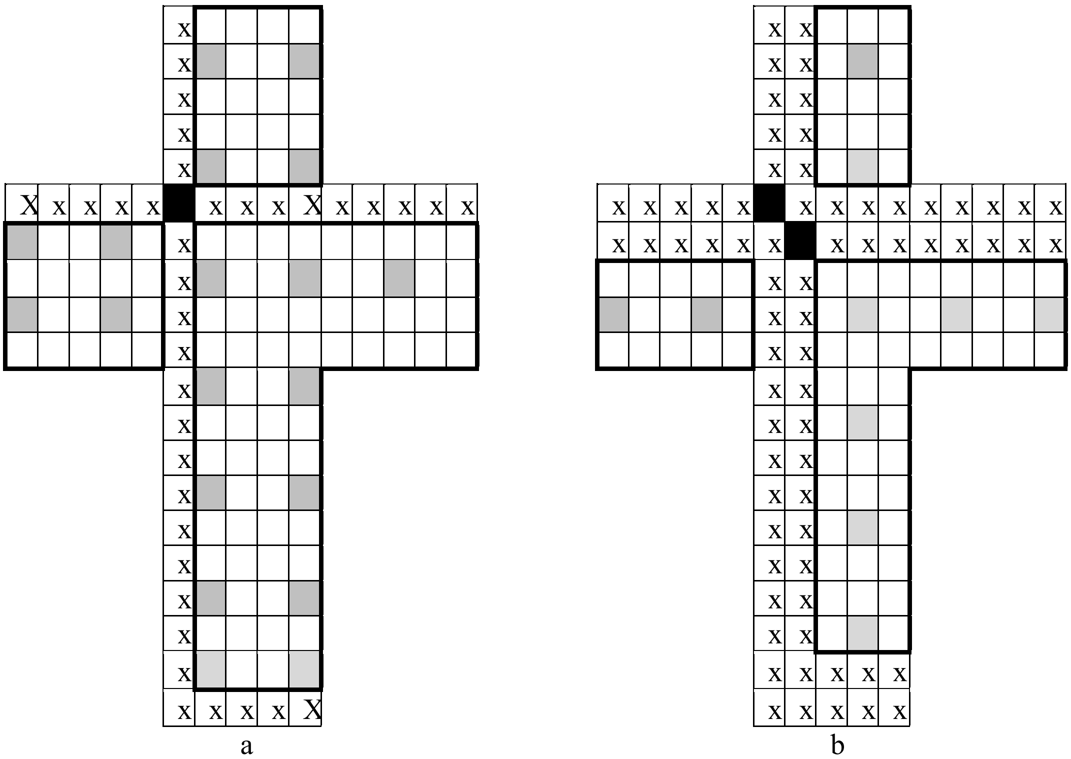


Figure (4): A domination of *K* pieces with fixed number of *R* pieces.

Theorem 6. *For K pieces with a fixed number of rooks* n_r *, the independence number where n is even and* $1 \le n_r \le \frac{n}{2}$ *in the cubic chessboard is given by*

$$\beta(n_r\,,K) \ge \frac{3}{2}n^2 - n + 1 - n_r\left(\frac{n}{2}+3\right)$$

Proof.

At first, the distribution of K pieces is as in the distribution of K pieces in Theorem 4. Our idea is to distribute the pieces of R such that they attack a minimum number of K pieces (to

keep a maximum number of K pieces on the chessboard), such that no K piece attacks any R piece.
We place the R pieces in order in cells $s_{1,2i-1,2i-1}$, $i = 1,2,..,\frac{n}{2}$.
The first piece of R attack $\frac{n}{2}$ K pieces in F_4 and three K pieces adjacent to it. So, we must remove the adjacent K pieces and we denote each cells of these pieces by “x" (as an example, see Figure 5). Continue to this procedure until $n_r = \frac{n}{2}$. Therefore, by using Theorem 4, we get $\beta(n_r , K) \geq \frac{3}{2}n^2 - n + 1 - n_r\left(\frac{n}{2} + 3\right)$

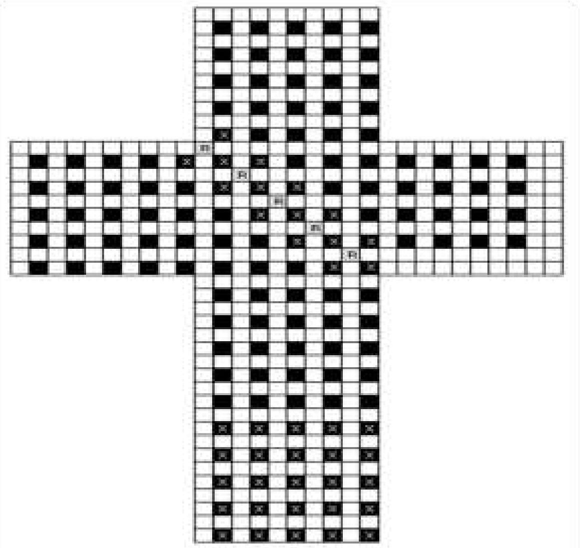


Figure (5): A distribution of K independent pieces with five R pieces.

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