Unit Hydrograph, New approach

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Abstract

A new approach is presented to drive a unit hydrograph for a complex multipeaked hydrograph. The new approach is characterized with accuracy and simplicity. It is checked by using two illustrative examples.

الخلاصة

قدمت في هذا البحث طريقة جديدة لاشتقاق منحني الوحدة الزمني للتصريف لهيدروغراف معقد ذو قمم متعددة. تتميز هذه الطريقة بدقتها وبساطتها، وقد تم استخدامها في حل مثالين توضيحين فاعطت نتائج دقيقة.

1. Introduction

A unit hydrograph may be constructed from the rainfall and streameflow data of a storm with reasonably uniform rainfall intensity and without preceding or subsequent rainfall (Linsely and Franzini, 1979). The derivation of the unit hydrograph involves three steps (Linsely *et al.,* 1975): (1) separation of the base flow from the direct runoff; (2) determining the volume of runoff; (3) finding the ordinates of the unit hydrograph by dividing the ordinates of the direct runoff by the volume of direct runoff in inches. The resulting unit hydrograph should represent a unit volume (1 in., or 1 cm) of runoff.

To derive the unit hydrograph for a complex multipeaked hydrograph, deconvolution may be used. The discrete convolution equation for a linear system is written as

$$\mathbf{Q_n = \sum_{m=1}^{n\le M} P_m U_{n-m+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where
Q_n = the direct runoff
P_m = the excess rainfall
U_{n-m+1} = the unit hydrograph
The notation n ≤ M means that when n ≤ M, the terms are summed for m = 1,2,……,n and when n > M the terms are summed for m = 1,2, …., M. When P_m and U_{n-m+1}data are given, the direct runoff can be computed by using the discrete convolution equation. The reverse process is called deconvolution. It is required to derive a unit hydrograph when data on P_m and Q_n are given. When M pulses of excess rainfall and N pulses of direct runoff in the storm considered are available, then N equations are obtained for Q_n , n = 1,2, ….,N, in terms of N-M +1 unknowns values of the unit hydrograph. These equations are as follows:

$$\left.\begin{array}{l}
Q_1 = P_1U_1 \\
Q_2 = P_2U_1 + P_1U_2 \\
Q_3 = P_3U_1 + P_2U_2 + P_1U_3 \\
\ldots\ldots \\
Q_M = P_MU_1 + P_{M-1}U_2 + \ldots + P_1U_M \\
Q_{M+1} = 0 + P_MU_2 + \ldots + P_2U_M + P_1U_{M+1} \\
\ldots\ldots \\
Q_{N-1} = 0 + 0 + \ldots + 0 + 0 + P_M U_{N-M} + P_{M-1} U_{N-M+1} \\
Q_N = 0 + 0 + \ldots + 0 + 0 + 0 + P_MU_{N-M+1}
\end{array}\right\} \ldots (2)$$

Arrange in matrix form

$$\begin{bmatrix} \mathbf{P_1} & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} \\ \mathbf{P_2} & \mathbf{P_1} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} \\ \mathbf{P_3} & \mathbf{P_2} & \mathbf{P_1} & . & . & . & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} \\ . & & & & & & & & & & & & . \\ . & & & & & & & & & & & & . \\ . & & & & & & & & & & & & . \\ \mathbf{P_M} & \mathbf{P_{M-1}} & \mathbf{P_{M-2}} & . & . & . & \mathbf{P_1} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P_M} & \mathbf{P_{M-1}} & . & . & . & \mathbf{P_2} & \mathbf{P_1} & . & . & . & \mathbf{0} & \mathbf{0} \\ . & & & & & & . & & & & & & \\ . & & & & & & . & & & & & & \\ . & & & & & & . & & & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{P_M} & \mathbf{P_{M-1}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{P_M} \end{bmatrix} \begin{bmatrix} \mathbf{U_1} \\ \mathbf{U_2} \\ \mathbf{U_3} \\ . \\ . \\ . \\ \mathbf{U_{N-M+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{Q_1} \\ \mathbf{Q_2} \\ \mathbf{Q_3} \\ . \\ . \\ . \\ \mathbf{Q_M} \\ \mathbf{Q_{M+1}} \\ . \\ . \\ . \\ \mathbf{Q_{N-1}} \\ \mathbf{Q_N} \end{bmatrix} . (3)$$

or

$$[\mathbf{P}][\mathbf{U}] = [\mathbf{Q}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

When Q_n and P_m data are available and $U_{N\text{-}M+1}$ is required, the set of Eqs. (2) is overdetermined because the number of equations is greater than the number of unknowns.

In the literature, researchers have introduced several methods to solve the set of Eqs. (2). In this research a new method, which characterized with accuracy and simplicity, is proposed to solve the set of Eqs. (2).

2. Literature Review

Collins introduced a method of successive approximation to obtain a unique solution of the set of Eqs. (2) (Collins, 1939: Quoted from Chow et al., 1988). This method involves four steps:

1. Assume a unit hydrograph, and apply it to all excess- rainfall blocks of the hyetograph except the largest.
2. Subtract the resulting hydrograph from the actual direct runoff hydrograph, and reduce the residual to unit hydrograph terms.
3. Compute a weighted average of the assumed unit hydrograph and the residual unit hydrograph, and use it as the revised approximation for the next trial.
4. Repeat the previous three steps until the residual unit hydrograph does not differ by more than a permissible amount from the assumed hydrograph.

The resulting unit hydrograph may show erratic variations and even have negative values. If this occurs, a smooth curve may be fitted to the ordinates to produce an approximation of the unit hydrograph.

If a solution [U] is given for Eq.(4), then an estimate of the direct runoff hydrograph [Q_e] is obtained. Mathematically,

$$[\mathbf{P}][\mathbf{U}] = [\mathbf{Qe}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Linear programming method is used to solve Eq.(4) for [U] that minimizes the absolute value of the error between [Q] and [Q_e] and also ensures that all entries of [U] are nonnegative (Eagleson *et al.*, 1966; Deininger, 1969; Singh, 1976;Chow *et al.*, 1988, and Mays and Coles, 1980).

Bree (Bree, 1978) solved Eq.(4) for [U] by reducing the rectangular matrix [P] to a square matrix [Z] by multiplying both sides by the transpose of [P],$[P]^T$. Then both sides are multiplied by the inverse $[Z]^{-1}$ of the matrix [Z]. The resulting system is

$$[\mathbf{U}] = [\mathbf{Z}]^{-1}[\mathbf{P}]^{T}[\mathbf{Q}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where $[Z] = [P]^T[P]$.The solution determined by this method is not easy because the many repeated and blank entries in [P] create difficulties in the inversion of [Z].

Chow (Chow, et al., 1988) solved the problem by selecting a number of equations from the set of Eqs.(2) equals the number of the unit hydrograph ordinates. For example, if M = 3 and N = 11 the number of the unit hydrograph ordinates is N-M+ 1 = 11- 3+ 1 = 9. Thus (9) equations are selected and the remaining equations Eq.(10) and Eq.(11), which contain information on Q_{10} and Q_{11} are neglected. The depth of direct runoff in the derived unit hydrograph must be checked to be equal 1.00 inch (or 1 cm). In case where the derived unit hydrograph does not meet this requirement, the ordinates are adjusted by proportion so that the depth of direct runoff is 1 inch (or 1 cm).

3. The new approach

The new approach requires solution of two systems in which the number of equations is equal to the number of unknowns. The first system consists of M equations and the matrix [P] is a lower triangular matrix. The second system consists of (N- M) equations and the matrix [P] is an upper triangular matrix.

3.1 Constructing the First System

The first system consists of M equations. The matrix [Q]starts from Q_1 to Q_M. The matrix [U] starts from U_1 to U_M. The matrix [P] is lower triangular matrix. First write the main diagonal elements which equal P_1. Then write the other parallel diagonals by increasing the subscript of P by (1) until you reach the last element which is P_M. The first system is written as follows:

$$\begin{bmatrix} \mathbf{P_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} \\ \mathbf{P2} & \mathbf{P_1} & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} \\ \mathbf{P3} & \mathbf{P2} & \mathbf{P_1} & \mathbf{0} & . & . & . & \mathbf{0} & \mathbf{0} \\ . & & & & & & & & . \\ . & & & & & & & & . \\ . & & & & & & & & . \\ \mathbf{P_{M-1}} & \mathbf{P_{M-2}} & . & . & . & . & . & \mathbf{P_1} & \mathbf{0} \\ \mathbf{P_M} & \mathbf{P_{M-1}} & . & . & . & . & . & \mathbf{P_2} & \mathbf{P_1} \end{bmatrix} \begin{bmatrix} \mathbf{U_1} \\ \mathbf{U_2} \\ \mathbf{U_3} \\ \\ \\ \\ \mathbf{U_{M-1}} \\ \mathbf{U_M} \end{bmatrix} = \begin{bmatrix} \mathbf{Q_1} \\ \mathbf{Q_2} \\ \mathbf{Q_3} \\ . \\ . \\ . \\ \mathbf{Q_{M-1}} \\ \mathbf{Q_M} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

3.2 Constructing the Second System

The second system consists of (N- M) equations The matrix[Q] starts from Q_{M+1} to Q_N. The matrix [U] starts from U_2 to U_{N-M+1}. The matrix [P] is an upper triangular matrix. First write the main diagonal elements which equal P_M. Then write the other parallel diagonals by decreasing the subscript of P by (1) until you reach P_1. The other elements are given zero value. The second system is written as follows:

$$\begin{bmatrix} P_M & P_{M-1} & . & . & . & P_1 & 0 & 0 & 0 & 0 \\ 0 & P_M & P_{M-1} & . & . & . & P_1 & 0 & 0 & 0 \\ 0 & 0 & P_M & P_{M-1} & . & . & . & P_1 & 0 & 0 \\ 0 & 0 & 0 & P_M & P_{M-1} & . & . & . & P_1 & 0 \\ 0 & 0 & 0 & 0 & P_M & P_{M-1} & . & . & . & P_1 \\ 0 & 0 & 0 & 0 & 0 & P_M & P_{M-1} & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & P_M & P_{M-1} & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_M & P_{M-1} & P_{M-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_M & P_{M-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_M \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ . \\ . \\ . \\ . \\ . \\ . \\ U_{N-M} \\ U_{N-M+1} \end{bmatrix} = \begin{bmatrix} Q_{M+1} \\ Q_{M+1} \\ . \\ . \\ . \\ . \\ . \\ . \\ Q_{N-1} \\ Q_N \end{bmatrix} \quad \text{….(8)}$$

Since the matrix [U] in the second system starts with U2, there is a repeated ordinates of the derived unit hydrograph. In case the repeated ordinates of the derived unit hydrograph in the two systems are different, we select the ordinates that ensure that the depth of the direct runoff in the unit hydrograph equals 1 inch (or 1 cm).

The following are two examples to illustrate the use of the new approach.

Example 1

Find the half- hour unit hydrograph using the excess rainfall hyetograph and direct runoff hydrograph given in the following table (Chow, 1988, Example 7.4.1, pp. 216):

Time (1/2 h)	Excess rainfall (in)	Direct runoff (cfs)
1	1.06	428
2	1.93	1923
3	1.81	5297
4		9131
5		10625
6		7834
7		3921
8		1846
9		1402
10		830
11		313

Solution

M =3 and N = 11, therefore the number of pulses in the unit hydrograph is N- M+ 1= 11- 3+ 1 =9. The first system consists of M= 3 equations and it is as follows:

$$\begin{bmatrix} \mathbf{1.06} & \mathbf{0} & \mathbf{0} \\ \mathbf{1.93} & \mathbf{1.06} & \mathbf{0} \\ \mathbf{1.81} & \mathbf{1.93} & \mathbf{1.06} \end{bmatrix} \begin{bmatrix} \mathbf{U_1} \\ \mathbf{U_2} \\ \mathbf{U_3} \end{bmatrix} = \begin{bmatrix} \mathbf{428} \\ \mathbf{1923} \\ \mathbf{5297} \end{bmatrix}$$

Solving this system yields:

U_1 = 404 cfs/in, U_2 = 1079 cfs/in, U_3 = 2343 cfs/in

The second system consists of N- M= 11- 3= 8 equations and it is as follows:

$$\begin{bmatrix} \mathbf{1.81} & \mathbf{1.93} & \mathbf{1.06} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1.81} & \mathbf{1.93} & \mathbf{1.06} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1.81} & \mathbf{1.93} & \mathbf{1.06} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1.81} & \mathbf{1.93} & \mathbf{1.06} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1.81} & \mathbf{1.93} & \mathbf{1.06} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1.81} & \mathbf{1.93} & \mathbf{1.06} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1.81} & \mathbf{1.93} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1.81} \end{bmatrix} \begin{bmatrix} \mathbf{U_2} \\ \mathbf{U_3} \\ \mathbf{U_4} \\ \mathbf{U_5} \\ \mathbf{U_6} \\ \mathbf{U_7} \\ \mathbf{U_8} \\ \mathbf{U_9} \end{bmatrix} = \begin{bmatrix} \mathbf{1931} \\ \mathbf{10625} \\ \mathbf{7834} \\ \mathbf{3921} \\ \mathbf{1846} \\ \mathbf{1402} \\ \mathbf{830} \\ \mathbf{313} \end{bmatrix}$$

Solving this system yields:

$$\begin{bmatrix} \mathbf{U_2} \\ \mathbf{U_3} \\ \mathbf{U_4} \\ \mathbf{U_5} \\ \mathbf{U_6} \\ \mathbf{U_7} \\ \mathbf{U_8} \\ \mathbf{U_9} \end{bmatrix} = \begin{bmatrix} \mathbf{1079} \\ \mathbf{2343} \\ \mathbf{2506} \\ \mathbf{1460} \\ \mathbf{453} \\ \mathbf{381} \\ \mathbf{274} \\ \mathbf{173} \end{bmatrix}$$

the values are in cfs/in

Example 2

The excess rainfall and direct runoff recorded for a storm are as follows:

Time (h)	1	2	3	4	5	6	7	8	9
Excess rainfall (in)	1.0	2.0		1.0					
Direct runoff (cfs)	10	120	400	560	500	450	250	100	50

Calculate the one-hour unit hydrograph (Chow, 1988, problem 7.4.1, pp. 236)

Solution

M = 4 and N = 11, therefore, the number of pulses in the unit hydrograph is N- M+ 1= 9- 4+ 1 =6. The first system consists of M= 4 equations and it is as follows:

$$\begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 2.0 & 1.0 & 0 & 0 \\ 0 & 2.0 & 1.0 & 0 \\ 1.0 & 0 & 2.0 & 1.0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 120 \\ 400 \\ 560 \end{bmatrix}$$

Solving this system yields:

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \\ 200 \\ 150 \end{bmatrix}$$ the values are in cfs/in

The second system consists of N- M = 9- 4= 5 equations and it is as follows:

$$\begin{bmatrix} 1.0 & 0 & 2.0 & 1.0 & 0 \\ 0 & 1.0 & 0 & 2.0 & 1.0 \\ 0 & 0 & 1.0 & 0 & 2.0 \\ 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} = \begin{bmatrix} 500 \\ 450 \\ 250 \\ 100 \\ 50 \end{bmatrix}$$

Solving this system yields:

$$\begin{bmatrix} U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 150 \\ 100 \\ 50 \end{bmatrix}$$ the values are in cfs/in

4. Conclusions

1. The characteristics of the new approach are accuracy and simplicity. The two systems of equations are written in matrix form directly without passing through the set of Eqs. (2). In the resulting systems of equations the number of equations is equals the number of unknowns, hence any analytical or numerical method can be used to obtain the solution. If a software is used or a simple program is written, then there is a simplicity in entering the data of the matrix [P]since the data are either equal or zero.
2. In the new approach, all the available data (Q's and P's) are used in the solution in contrast with Chow method in which the number of equations are selected to be equal the unit hydrograph ordinates and the remaining equations are neglected. Moreover, the new approach ensures that all the unit hydrograph ordinates are nonnegative.

References

Bree, T. (1978). "The stability of parameter estimation in the general linear model", J. Hydrol., 37(12), 47-66.

Chow, V.T., Maidment, D.R., and Mays, L.W. (1988). "Applied Hydrology" McGraw- Hill, Inc., Singapore.

Linsely, R.K. and others (1975). "Hydrology for Engineers" McGraw- Hill Book Company, Inc., New York.

Linsely, R.K., and Franzini J.B. (1979). "Water Resources Engineering" 3rd edition, McGraw- Hill, Inc., New York.

Mays, L.W., and Coles,S.L. (1980). "Optimization of unit hydrograph determination" J. Hyd. Div., ASCE, 106(1), 85-97.