Characterizations of α-Light mappings

Khalid .Sh. Al-Shukree
Department of Mathematics, College of Education, Al-Qadisiyh University

Abstract

In this work, we give the definition of α-Light mapping for the first time (to the best of our knowledge) and investigate some of its several properties and characterization .Also, we study the relation between this concept and the concept of light mapping and we proved that the pull back of the α-light mapping is also α-light mapping.

الخلاصة

في هذا العمل، قدمنا مفهوما جديدا حسب علمنا (التطبيق الواهن-α) مع دراسة العديد مـن الخـصائص والمميزات الخاصة بهذا المفهوم 0 ايضا درسنا العلاقة بين هذا المفهوم والتطبيق الواهن وبرهنـا بـأن لـسحب الخلفي للتطبيق الواهن- α يكون ايضا تطبيقا واهنا-α 0 .

1- Introduction

Njasted (1965) gives the definition of α-open set and studies the properties of it. Also, (Mashhour *et al.,* 1983) give the definition of α-continuous and α-open mapping and study the properties of it. Throughout this work, (X,τ) simply space X always means topological space. A subset A of (X,τ) is called α-open (Njastad, 1965). if $A \subseteq (\overline{A^{\circ}})^{\circ}$ .The complement of an α-open set is called an α-closed set. The intersection of all α-closure sets containing A is called the α-closure of A, denoted by $(\overline{A})^{\alpha}$. A subset A is called α-closed if and only if A= $(\overline{A})^{\alpha}$ a point $x \in X$ is said to be an α-interior point of A if there exists an α-open set U containing x such that $U \subseteq A$ .The set of all α-interior points of A is said to be α-interior of A (Mashhour *et al.,* 1983) and denoted by $(A^{o})^{\alpha}$ .It is clear that every open is α-open but the converse is not true. We denote the family of all α-open sets of (X,τ) by τ^{α} . It is shown in (Njastad, **1965;** Ohba & Umehara, 2000) that each of $\tau \subseteq \tau^{\alpha}$ and τ^{α} is a topology on X. A mapping *f:(X,τ)→(Y,σ)* is said to be α-continuous if $f^{-1}(V) \in \tau^{\alpha}$ for every $V \in \sigma$,and, equivalently, if for each x $\in$X and each open set V of Y containing f(x), there exists $U \in \tau^{\alpha}$ with x$\in$X such that $f(U) \subseteq V$ (Mashhour *et al.,* 1983). A mapping *f:(X,τ)→(Y,σ)* is said to be α- open if each α-open set *U* in *X,f(U)* is α-open set in *Y*.and *f* is said to be α-closed if each α-closed set *U* in *X*, *f(U)* is α-closed set in *Y* (Mashhour *et al.,* 1983)

2- α- connected spaces

Now we introduce the following definitions:-

2-1 Definitions:

1- let *X* be a space and *A,B* be non empty α-open sets , then we say that A/B is α-disconnection to X if $X = A \cup B$ and $A \cap B = \phi$.

2- let *X* be a space , *X* is said to be α-disconnected if there exists a disconnection *A/B* to *X*, other wise we say that *X* is α-connected.

2-2 Definition: Let X be a space and let *G* a non empty subset of *X* and *A,B* non empty α-open sets in *X*, then *A/B* is said to be α-disconnection of *G* if

$1-(A \bigcup B) \bigcap G = G$

$2-(A \bigcup B) \bigcap G = \phi$

other wise G is α-connected set.

2-3 Example: let $X=\{a,b,c\}$ and τ is a discrete topology defined on X note that $\{a\}/\{b,c\}$ is a α-disconnection to X.Also $\{a\}/\{b\}$ is a α-disconnection to the subset $\{a,b\}$ of X.

2-4 Theorem (Latif, 2006)**:** Every α-connected space is connected. but the converse may not be true.

2-5 Definition: A space X is said to be totally α-disconnected if for each pair of points $a,b\in X$ there is an α-disconnection A/B of X with $a\in A$ and $b\in B$.

2-6 Remark: Each totally disconnected space is totally α-disconnected space. but the converse is not true for example let $X=\{a,b,c\}$ and $\tau=\{\phi, X, \{a\}, \{b,c\}\}$.It is clear that X is totally α-disconnected space but X is not totally disconnected space because there is no exists a disconnection of $b,c\in X$.

2-7 Example (Khalid, 2004)**:** The set Q of rational numbers with the usual topology is totally α-disconnected set. (since Q is totally disconnected set with the usual topology.

Now we introduce the following definition:

2-8 Definition: A mapping $f:X\rightarrow Y$ is said to be α- homeomorphism if

1-f is bijective mapping.

2-f is α-continuous mapping.

3-f is α-open or α-closed mapping.

Now the following theorem shows that the totally α-disconnectedness is a topological property.

2-9Theorem: let X and Y be two spaces, and let $f:X\rightarrow Y$ be a α-homeomorphism .If X or Y is a totally α-disconnected so is the other.

***Proof*:** Suppose that X is totally α-disconnected and let $y_1, y_2 \in Y$ such that $y_1\neq y_2$ .since f is bijective mapping then there exist two points $x_1, x_2 \in X$ such that $x_1\neq x_2$ and $f(x_1)=y_1$, and $f(x_2)=y_2$ since X is totally α-disconnected space, then there is a α-disconnection U/V such that $x_1 \in U, x_2 \in V$ since f is α-homeomorphism then each of $f(U)$ and $f(V)$ are α-open sets in Y but $f(U)\bigcup f(V) = f(U\bigcup V) = f(X) = Y$ and since f is one-to-one then $f(U)\bigcap f(V) = f(U\bigcap V) = f(\phi) = \phi, \in X$ and $y_1 \in f(U), y_2 \in f(V)$, hence Y is totally α-disconnected space .

3- α-Light mapping

3-1 Definition (Whyburn, 1942)*:* a mapping $f: X\rightarrow Y$ is called Light mapping if $f^{-1}(y)$, is totally disconnected set (relative topology) for all $y\in Y$ with (i-e each component $f^{-1}(y)$ is singleton)

3-2 Example (Khalid, 2004)**:** let (Q,τ_U) be the space such that τ_U is the usual topology defined on the set of rational numbers Q and let $(\{k\},\tau_I)$ be indiscrete topology for $k\in R$. and let $f: (Q,\tau_U) \rightarrow (\{k\},\tau_I)$ mapping defined as follows : $f(x)=k$ for all $x \in Q$ note that $f^{-1}(k) = Q$ and since Q is totally disconnected set (2-7) , then f is Light mapping.

Now we introduce the following definition:

3-3 Definition: a mapping $f: X\rightarrow Y$ is called α-Light mapping if $f^{-1}(y)$, is totally α- disconnected set (relative topology) for all $y\in Y$.

3-4 Example: The example (3-2) we note that f is α-Light mapping it easy from (2-6).

***3-5 Theorem*:** Every Light mapping is α-Light mapping.

Proof: It is easy from (2-6), that if $f^{-1}(y)$ is totally disconnected set in X then $f^{-1}(y)$ is totally α-disconnected set in X.

3-6 Proposition: let *f:X→Y* be a α-Light mapping and let $G \subset X$, then the restriction mapping $f|_G : G \to f(G)$, is also α-Light mapping.

Proof: to show that for all $y \in f(G)$, $f^{-1}(y) \cap G$ is totally α-disconnected set in *G*. let a,b $\in f^{-1}(y) \cap G$ then a,b $\in f^{-1}(y)$ and since f is α-Light map then for all $y \in Y, f^{-1}(y)$ is totally α-disconnected set in X , that is there exists a α-disconnection *A/B* such that

$$(A \cap f^{-1}(y)) \cup (B \cap f^{-1}(y)) = f^{-1}(y) \, and$$
$$(A \cap f^{-1}(y)) \cap (B \cap f^{-1}(y)) = \phi$$

Such that *A,B* are disjoint α-open subsets of X, and $a \in A, b \in B$ now to show that A/B is α-disconnection to $f^{-1}(y) \cap G$ also .since

$$((G \cap f^{-1}(y)) \cap A) \cup ((G \cap f^{-1}(y)) \cap B) = (G \cap ((f^{-1}(y) \cap A)) \cup (G \cap (f^{-1}(y) \cap B))$$
$$= G \cap [(f^{-1}(y) \cap A) \cup (f^{-1}(y) \cap B)] = G \cap f^{-1}(y) \, and$$
$$((G \cap f^{-1}(y)) \cap A) \cap ((G \cap f^{-1}(y)) \cap B) = (G \cap (f^{-1}(y) \cap A)) \cap (G \cap (f^{-1}(y) \cap B))$$
$$= G \cap [(f^{-1}(y) \cap A) \cap (f^{-1}(y) \cap B)] = G \cap \phi = \phi$$

Such that $(G \cap f^{-1}(y)) \cap A, (G \cap f^{-1}(y)) \cap B$ are two disjoint α-open sets, hence $f^{-1}(y) \cap G$ is totally α-disconnected set, $f|_G$ is α-Light mapping.

Now we introduce the following definitions:

3-7 Definition: A mapping *f:X→Y* is said to be totally α-disconnected mapping if each totally α-disconnected set *U* in *X*, *f(U)* is totally α-disconnected set in *Y*.

3-8 Definition: A mapping *f:X→Y* is said to be inversely totally α-disconnected mapping if each totally α-disconnected set *U* in *Y*, $f^{-1}(U)$ is totally α-disconnected set in *X*.

3-9 Proposition: let $f_1 : X \to Y$ and $f_2 : Y \to Z$ be two mappings, then $f = f_2 o f_1 : X \to Z$ is α-Light mapping if f_1 is inversely totally α-disconnected and f_2 is α-Light mapping.

Proof: to prove that for $z \in Z, f^{-1}(z)$ is totally α-disconnected set in X. let $z \in Z$ then $f^{-1}(z) = (f_2 o f_1)^{-1}(z) = f_1^{-1}(f_2^{-1}(z))$ and since f_2 is α-light mapping then $f_2^{-1}(z)$ is totally α-disconnected set in Y and since f_1 is inversely totally α-disconnected mapping then $f_1^{-1}(f_2^{-1}(z))$ is totally α-disconnected set in *X*, but $f^{-1}(z) = f_1^{-1}(f_2^{-1}(z))$, then $f^{-1}(z)$ is totally α-disconnected set in X.

3-10 Theorem: let *f:X→Y* be the composition $f = f_2 o f_1$ of two mapping f_1:X→Y' and f_2:Y'→Y , then

1- If f_2 is an bijective mapping and f_1 is a α-Light mapping then *f* is a α-Light mapping.

2- If *f* is a α-Light mapping and f_2 is an injective mapping then f_1 is a α-Light map.

3- If *f* is a α-Light mapping and f_1 is a totally α-disconnected mapping then f_2 is a α-Light mapping.

Proof: 1- let $y \in Y$, since f_2 is a bijective mapping then there exists one point $y' \in Y$ such that $f_2(y')=y$ and since $f^{-1}(y) = (f_2 o f_1)^{-1}(y) = f_1^{-1}(f_2^{-1}(y)) = f_1^{-1}(f_2^{-1}(f_2(y')) = f_1^{-1}(y')$ and f_1 is α-light mapping then $f_1^{-1}(y^*)$ is totally α-disconnected set in X, so $f_1^{-1}(y)$ is totally α-light disconnected set in X and hence *f* is α-light mapping.

2- Let $y' \in Y'$, then $f_2(y') \in Y$ and since *f* is α-light mapping then $f^{-1}(f_2(y'))$ is a totally α-disconnected set in X but $f^{-1}(f_2(y')) = f_1^{-1}(f_2^{-1}(f_2(y')) = f_1^{-1}(y')$, then f_1 is a α- light mapping.

3- let $y \in Y$,since f is a α-light , then $f^{-1}(y)$ is a totally α-disconnected set in X and since f_1 is totally α-disconnected mapping then $f_1(f^{-1}(y))$ is a totally α-disconnected set in Y', but
$f_1(f^{-1}(y)) = f_1((f_2 o f_1)^{-1}(y)) = f_1(f_1^{-1}(f_2^{-1}(y))) = f_2^{-1}(y)$. Then $f_2^{-1}(y)$ is a totally α-disconnected set in Y', so f_2 is α-light mapping.

Now we will show when the product of two mappings is a α-light mapping.

***3-11 Theorem*:** let $f_1:X_1 \rightarrow Y_1$, $f_2:X_2 \rightarrow Y_2$ be two mappings , then the product mapping $f_1 x f_2 : X_1 x X_2 \rightarrow Y_1 x Y_2$ is a α-light mapping if f_1 is an bijective mapping and f_2 is a sujective α-light mapping.

Proof: let $(y_1,y_2) \in Y_1 x Y_2$, then $(f_1 x f_2)^{-1}(y_1,y_2)=(f_1^{-1} x f_2^{-1})(y_1,y_2)=f_1^{-1}(y_1) x f_2^{-1}(y_2)$ and since f_1 is bijective then there exists $x_1 \in X_1$ such that
$f_1^{-1}(y_1) = f_1^{-1}(f_1(x_1)) = x_1$, then $(f_1 \times f_2)^{-1}(y_1, y_2) = x_1 \times f_2^{-1}(y_2)$, and since f_2 is a α-light mapping , then $f_2^{-1}(y_2)$ is a totally α-disconnected set in X_2 and since $x_1 \times f_2^{-1}(y_2)$ is α- homeomorphic to $f_2^{-1}(y_2)$, then from (2-9) $x_1 \times f_2^{-1}(y_2)$ is totally α-disconnected , so, $(f_1 \times f_2)^{-1}(y_1, y_2)$ is totally α-disconnected set in $X_1 x X_2$ and hence $f_1 x f_2$ is α-light mapping.

3-12Corollary: let $f_1 : X_1 \rightarrow Y_1$, $f_2 : X_2 \rightarrow Y_2$ be two mappings such that if $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a α-light mapping, then if one of them is a surjective , then the other is a α- light mapping.

3-13 Definition (Spainer, 1966): let $f : X \rightarrow Y$ and $g:Y' \rightarrow Y$ be mappings and X' is the subspace of the product space $X x Y'$ defined as follows:-
$X' = \{(x, y') \in X \times Y' | f(x) = g(y')\}$, X' is called the fibered product of Y' and X over Y.
let $f':X' \rightarrow Y'$ be the restriction of the second projection then f' is called the pull back of f *by g.*

3-14 Theorem: The pull back of α-light mapping is also α-light mapping.

Proof: let $f:X \rightarrow Y$ be α-light mapping and $f':X' \rightarrow Y'$ be a pull back of f by $g:Y' \rightarrow Y$, now let $y' \in Y'$, $g(y') \in Y$, since f is α-light mapping then $f^{-1}(g(y^*))$ is a totally α-disconnected set in X. now for fixed $y' \in Y'$,

$f'^{-1}(y') = \{(x, y') \in X \times Y' | f'(x, y') = y'\}$ for all $x \in X$ and $f(x)=g(y')$

$= \{(x, y') \in X \times Y' | x \in f^{-1}(g(y'))\} = f^{-1}(g(y')) \times \{y'\}$ and since

$f^{-1}(g(y')) \times \{y'\}$ is α- homeomrphic to $f^{-1}(g(y'))$ and $f^{-1}(g(y'))$ is totally α-disconnected then $f^{-1}(g(y')) \times \{y'\}$ is totally α-disconnected by(2-9) $f'^{-1}(y')$, is totally α-disconnected , then f' is α-light mapping.

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