The Relationship between the Stasheff Polytope and Painted Trees Using Tubings

Dr. Shatha Asaad Salman
Applied Sciences Department, University of Technology/Baghdad.
Email:Drshatha.alnajar@gmail.com
Dr. Anwar Khaleel Faraj
Applied Sciences Department, University of Technology/Baghdad.
Email:Anwar_78_2004@yahoo.com
Alaa Sabeeh Dawood
Applied Sciences Department, University of Technology/Baghdad.
Email:yousefSadiq231@gmail.com.

ABSTRACT
    A polytope plays a central role in different areas of mathematics. It is used quite heavily in applied fields of mathematics, such as medical imaging and robotics, geometric modeling.
    A polytope has many users in modern science such as computer graphics, optimization, and search engine. It is intended for a broad audience of mathematically inclined. Therefore in this paper, we shall take a polytope with one kind of application, which is known as a Stasheff polytope. it has been applied in: moduli spaces, Erhart polynomial, physics- chemistry, and Hopf algebras.
    In this paper, a finite graph G with tubing is taken, the nodes of the graph are reduce to points in R^n and the convex hull for them are simplex, permutahedron and associahedron (Stasheff polytope) are studied.
Also, a fan graph to the painted tree is also taken and reduces its nodes to points in R^n. The converse of this result is also given with different examples to consult our results.
Keywords: Stasheff Polytope, Permutohedron, Tube, Tubing, Painted tree, fan graph.

العلاقة بين متعدد السطوح والزوايا ستاشيف والاشجار الملونة باستخدام tubings

الخـلاصـة
    متعدد السطوح والزوايا (polytope) يلعب دورا مهما في مجالات مختلفة في الرياضيات،لقد تم استخدامه بشكل كبير جدا في المجالات التطبيقية مثل التصوير الطبي والروبوتات، النماذج الهندسية، والاستخدامات في العلوم الحديثة مثلا الرسوميات الحاسوبية والامثلية. حيث ان الغرض من هذة الاستخدامات هو اكتساب اكبر عدد من الباحثين للمجال الرياضي، لذلك سناخذ نوع من انواع متعدد السطوح والزوايا والذي يعرف باسم متعدد السطوح والزوايا ستاشيف Stasheff polytope من تطبيقاتُه Erhart polynomial and Hopf algebra, moduli spaces,.
    في بحثا هذا درسنا البيان المنتهي G مع Tubing وتحويل رؤوس البيان الى نقاط في الفضاء المتجه ذو البعد n و Convex hull لتلك النقاط هي الاجسام متعددة الزوايا والسطوح Simplex, Permutahedron, associahedron,.
    وكذلك تناولنا في هذا البحث تحويل بيان fan الى شجرة ملونة وتحويل تلك الرؤوس الى نقاط في الفضاء المتجه ذو البعد n ودرسنا العكس واعطينا امثلة مختلفة لتعزيز نتائجنا.

INTRODUCTION

The Stasheff polytope (associahedron) K^n is a polytope of dimension n, represented by the convex hull of the points in R^n, where the set of vertices is in one to one correspondence with the planar binary tree having n nodes and n+1 leaves, another example of a polytope is a permutahedron P^n where the construction represented by the convex hull of the permutations of points in R^n , [7]. A lot of applications concerned with a polytope which are given in [6, 9,10]

The Stasheff polytope K^n appeared in the sixties of the last century in the work of James Dillon (Jim) Stasheff who is an American mathematician in "Hsomotopy associativity of H-spaces. I, II", and he wrote fundamental papers on higher homotopy theory and homotopy algebras. He introduced A_∞, Stasheff algebras and Stasheff polytopes. In (1980) he turned to the application of characteristic classes and other topogical and algebraic concepts in mathematical physics, first in the algebraic structure of anomalies in quantum field theory, where he worked with among others.

Berry offers in [3], a new type of painted tree made by composing binary trees on leveled trees and defines the pterahedron to be the poset of painted face trees, which we show to be the face poset of a polytope.

In this paper a new concept which is called the tube is studied and some types of graph associahedron in a graph theory which are: simplex, permutahedron and associahedron are introduced.

Also in our work we reduce the fan graph to painted tree and reduce it to a point in R^n where the converse of the above process is also given.

Different examples are taken to discuss and explain the main idea.

Preliminaries

This section presents some of the basis and fundamental concepts related to this paper.

Definition(1.1), [4]:

Let S be a subset of the vertex set for the graph G. The subgraph with vertex set S and edge set { uv │u , v ∈ S and uv ∈ E(G) }, is called the subgraph of G induced by S.
Figure.1. represent the E(S) concept of the induced subgraph.
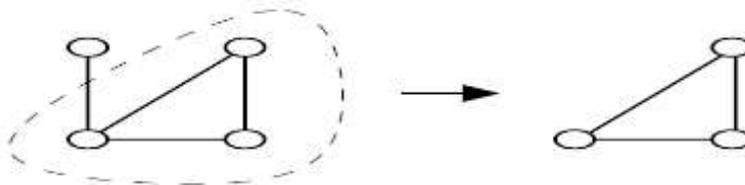
Figure(1). induced subgraph

Definition(1.2), [1]:

A graph H is a subgraph of the graph G if V(H) ⊆ V(G) and E(H) ⊆ E(G). If H≠G, then H is called a proper subgraph of G and it is denoted by H ⊂G.

Definition(1.3), [4]:

A tube u of a finite graph G is a proper, nonempty set of nodes of G whose induced graph is a proper, connected subgraph of G.
Two tubes u_1 and u_2 are:

(1) Nested if $u_1 \subset u_2$,
(2) Intersecting if $u_1 \cap u_2 \neq \emptyset$ and $u_1 \not\subset u_2$ and $u_2 \not\subset u_1$
(3) Adjacent if $u_1 \cap u_2 = \emptyset$ and $u_1 \cup u_2$ is a tube in G.

Definition (1.4), [4]:
   Tubes are compatible if they do not intersect and they are not adjacent.

Definition (1.5), [4]:
   A set of tubes of G is said to be a tubing U of G if every pair of tubes in U is compatible.

Example (1.1),[4]:
   Now, we explain the definition (1.5) by the following figures.
   Figure 2.(a) represent valid tubings and figure 2.(b) represented invalid tubings.
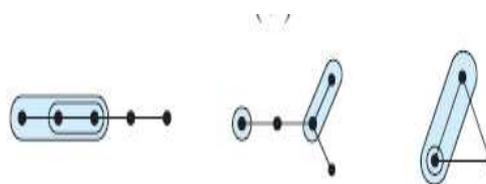
Figure 2.(a) valid tubings
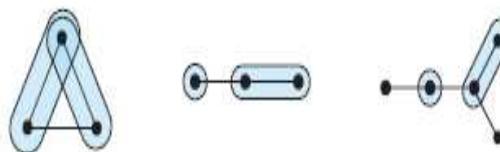
Figure 2.(b) invalid tubings

Definition (1.6), [8]:
   The face poset X(L) is the poset of cells of L ordered by inclusion, where L is the set of cells of a polytope.

Definition(1.7), [4]:
   Let G be a graph with n nodes, The graph associahedron PG is a simple, convex polytope whose face poset is based on the connected subgraphs of G. When G is a set of vertices, PG is the simplex Moreover, when G is a path, cycle or a complete graph, PG results are the associahedron, cyclohedron, and permutohedron, respectively.

Integer values of nodes associated to tubings:
   Let G be a graph with n nodes and M_G be the collection of maximal tubings of G, where each tubing U in M_G contains (n-1) compatible tubes. The smallest tube in U containing v denoted by t(v), if no tube of U contains v, then t(v) is all of G.
Define a map f_U from the nodes of G to the integers. if v = t(v) then $f_U(v) = 0$. All other nodes v of G must satisfy the recursive condition $\sum_{vi \in t(v)} f_U(vi) = 3^{|t(v)|-2}$.
For G a graph with an ordering $v_1, v_2, \ldots, v_n$ of nodes. The map c: $M_G \to R^n$ where c(U) = ($f_U(v_1), \ldots, f_U(v_n)$) is defined.

Example (2.1):

Let G be a graph with four nodes, using some steps to gives integer values of nodes associated to tubing,

1. Take the smallest tube where t(v) = 0, if the smallest tube contain one vertex.

2. All other nodes v of G must satisfy the recursive condition $\sum_{v_i \in t(v)} f_U(v_i) = 3^{\lceil t(v) \rceil - 2}$.
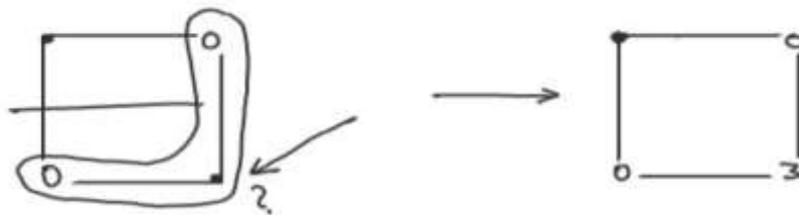
$0 + 0 + x = 3^{\lceil t(v) \rceil - 2}$
$x = 3^{3-2}$
$x = 3$.

3. Compute integer values for nodes out the tubing.

$0 + 0 + 3 + x = 3^{4-2}$
$3+x = 3^2$
$x = 9-3$
$x = 6$.

Example (2.1), [4]:

Figure.3, gives the integer values of nodes associated to tubings.

Figure.3

Theorem (2.1), [4]:

If G is a graph with n nodes. The convex hull of the points c(M_G) in R^n yields the graph associahedron PG.

Some types of graph associahedron:
The concepts of simplex, permutohedron and associahedron in graph theory are given:

Simplex, [4]:
Let G be the graph with n (disjoint) nodes, such that, each of them corresponding to choosing n-1 out of the n possible nodes.
An element of M_G is a point in R^n consisting of zeros for all coordinates except one with the value 3^{n-2} and PG the convex hull of n vertices in R^n yielding the (n-1) – simplex.

Example (3.1):
In this example we explain the simplex for n = 2, 3 and 4, for n = 2, then 1- simplex is explained in the following figure with computation of its coordinate.

For n = 3,[4], the 2- simplex is explained below.
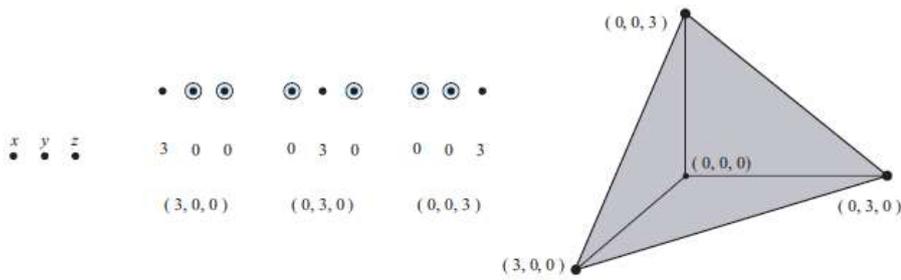
For n = 4, then 3- simplex is explained by the following.
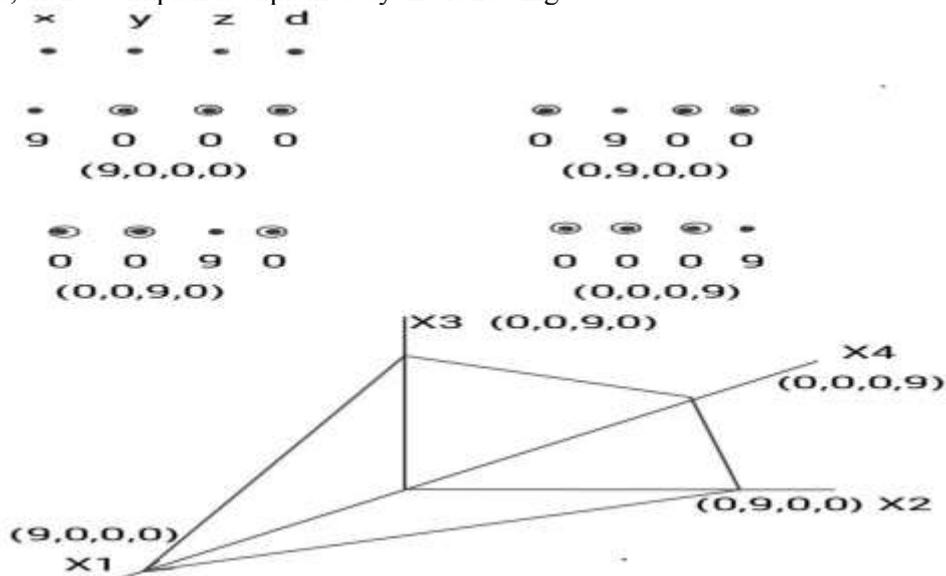
Permutahedron, [4]:
Let G be a complete graph with n nodes. Each element of M_G can be seen as a point in R^n whose values based on all permutation of the sequential nesting { 0, 1, 3^1 - 3^0, …, 3^{n-2} - 3^{n-3} }, and the convex hull of the n! vertices in R^n yielding the permutohedron.

Example (3.2):

In this example we explain the permutohedron for n = 2, 3 and 4, for n= 2, the P^1 is given below after computing the coordinate of it.
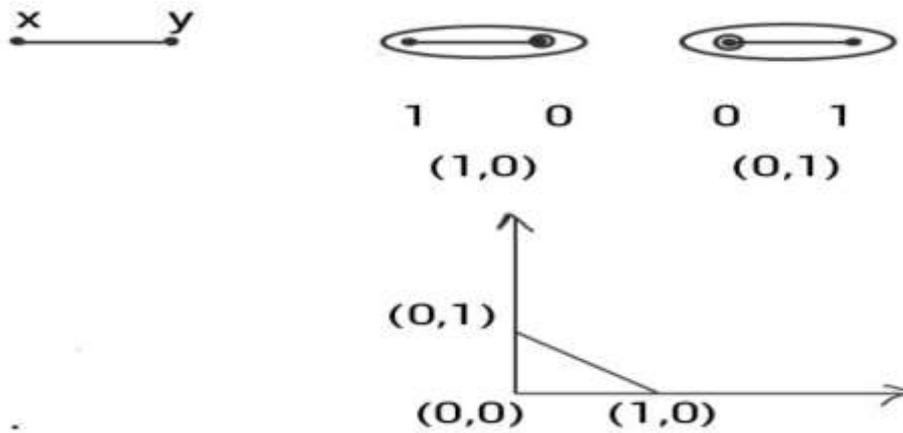
If n = 3, then P^2 is explained as follows, [4].
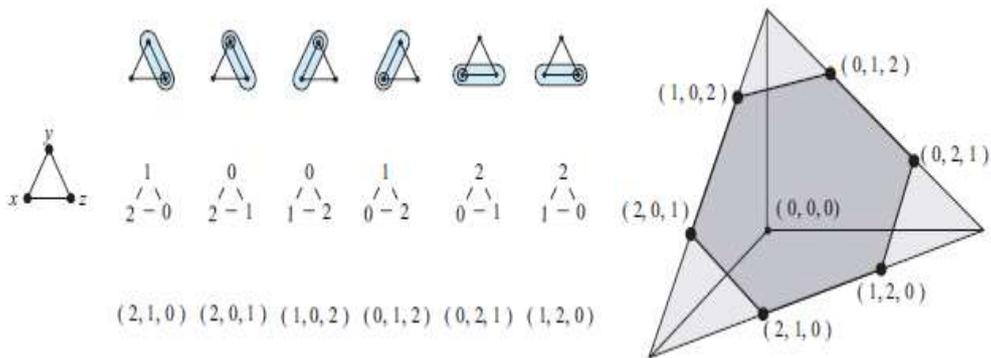
for n = 4, then coordinates of P^3 is given below.
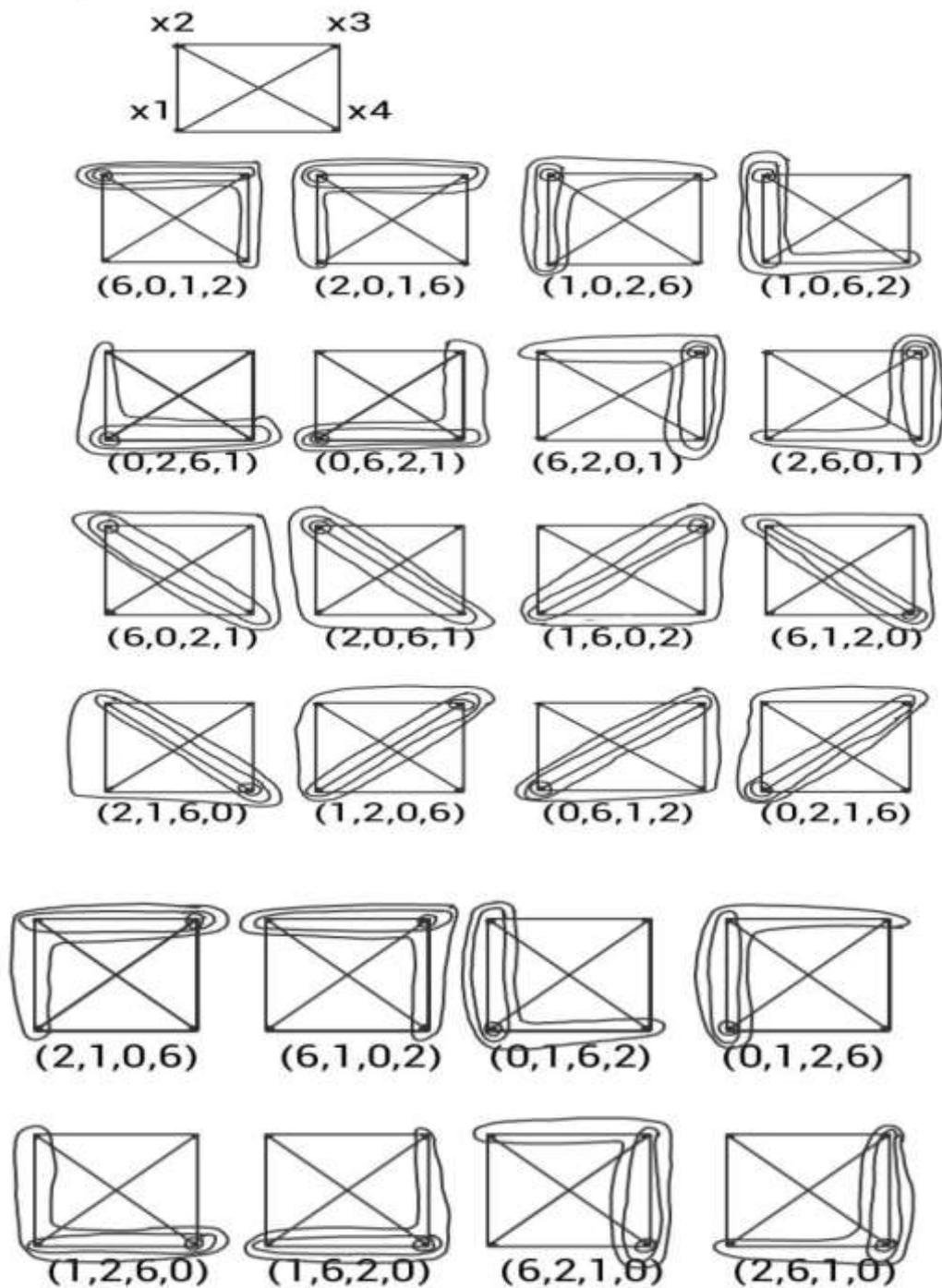
Therefore, P^3 is as follow

$$\mathcal{P}^3$$
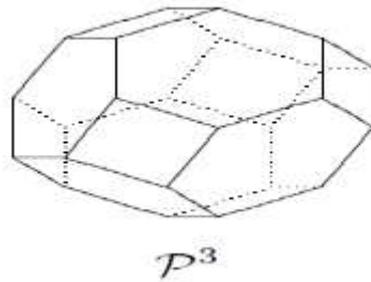
Associahedron, [4]:
    Let G be n-path. The number of such maximal tubings is in bijection with Catalan number, and the convex hull of these vertices in R^n yielding associahedron.

Example (3.3):
    In this example we explain the associahedron for n = 2, 3 and 4, for n = 2, then the computation of K^1 is given below.
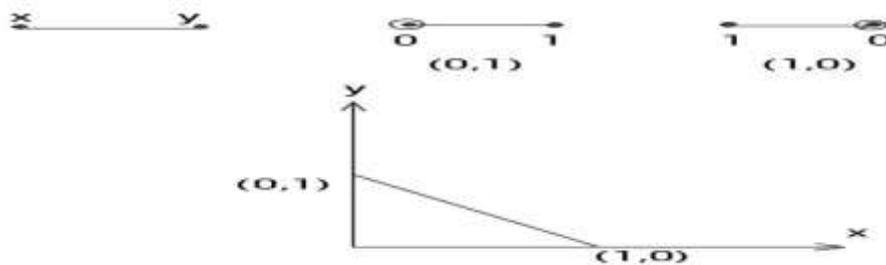
For n = 3, then the coordinates of P^3 is given below. [4]
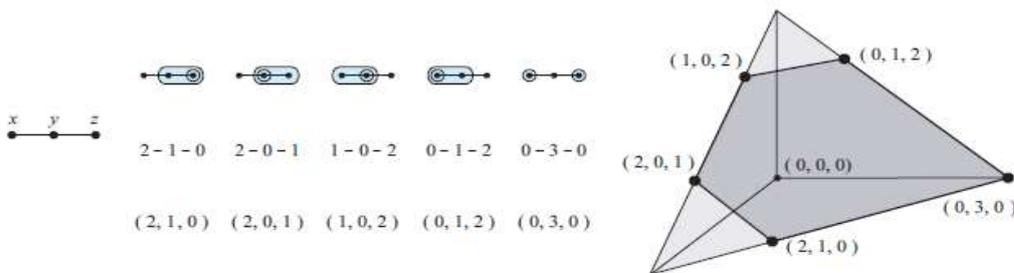
For n = 4, the evaluation of coordinate of coordinates for associahedron K^3 is done.

X Y . Z d

(0,1,2,6)   (1,0,2,6)   (2,0,1,6)

(2,1,0,6)   (6,0,1,2)   (6,1,0,2)

(6,2,0,1)   (6,2,1,0)   (0,0,3,6)

(3,0,0,6)   (0,3,0,6)   (6,0,0,3)

(6,3,0,0)   (6,0,3,0)
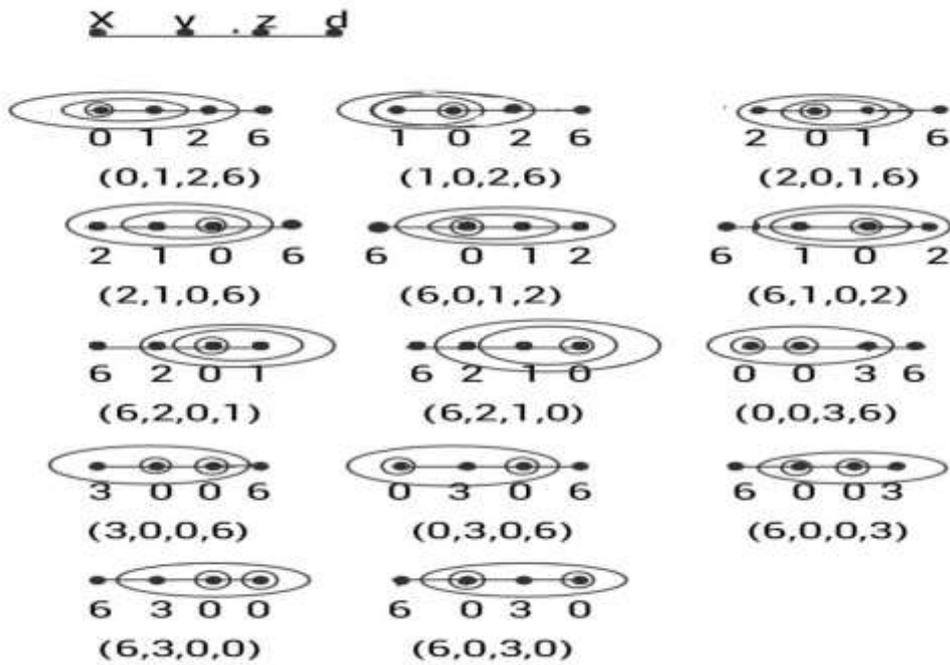
The obtained associahedron K^3 is below.
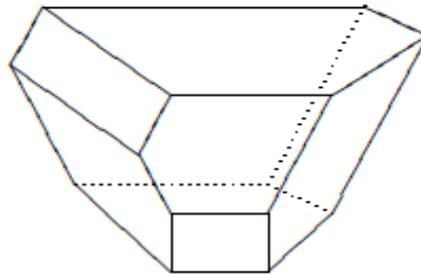
K^3

Definition (3.4), [5]:
    Cyclohedron is the graph associahedron of cycle, when n=3, the permutohedron and cyclohedron are identical in dimension two, [4].

Painted trees:
Definition (4.1), [2]:
    Composed leveled trees T_k on binary trees Y_j, is denoted by $(ToY)_n$. Where n is the interior node of painted trees.
Note (4.1), [3]:
    The number of trees in $(ToY)_n$ is $\sum_{k=0}^{n} [K! \sum_{\gamma_0 + \gamma_1 + \cdots + \gamma_n = n-k} (\prod_{i=0}^{k} C_{\gamma_i})]$
such that (K!) ways to make the leveled - painted portion of this tree with k nodes, (K+1) leaves on the top of the leveled - painted portion of this tree, and (n-k) remaining nodes to be distributed among the (k+1) binary trees that will go on the leveled - painted leaves.

Example (4.1), [2]:
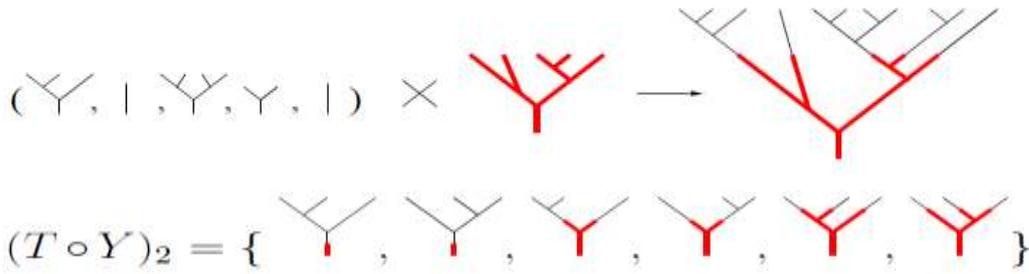
Example (4.2), [2]:

$$|(T \circ Y)_4| = 0! \ [C_4]$$
$$+ 1! \ [C_3C_0 + C_2C_1 + C_1C_2 + C_0C_3]$$
$$+ 2! \ [C_2C_0C_0 + C_1C_1C_0 + C_1C_0C_1 + C_0C_2C_0$$
$$+ C_0C_1C_1 + C_0C_0C_2]$$
$$+ 3! \ [C_1C_0C_0C_0 + C_0C_1C_0C_0 + C_0C_0C_1C_0$$
$$+ C_0C_0C_0C_1]$$
$$+ 4! \ [C_0C_0C_0C_0C_0]$$
$$= 94$$

Note (4.2), [3]:

We can compute the coordinate of the painted tree by the following steps:

a.   Labels the nodes of the painted tree by $(a_1, a_2, \ldots, a_n)$, number the nodes from left to right, 1 to n, according to the spaces above the nodes.

b.   Number the levels of the painted portion of the tree starting with n at the bottom and working up n, n-1, n-2, …

c.   Consider node i, if the node is painted, a_i= the nodes level.

Otherwise $a_i = \frac{1}{2}$ (the number of leaves to the left) (the number of leaves to the right).

A painted tree can be representing by a point in R^n where the convex hull of them represent a polytope, as seen as in figure.3.

Example (4.3), [3]:
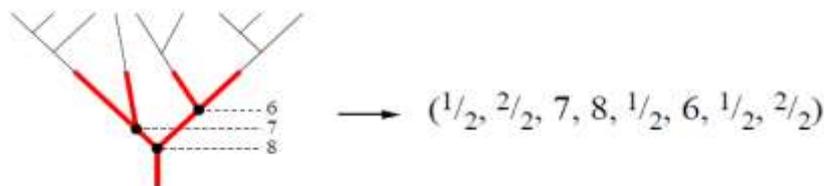
Figure.3. painted tree coordinates

How to convert the graph with the tubing to a painted tree:
We can convert the graph with the tubing by the following steps:

1-   Number the graph vertices from 0 to n, in a counter- clockwise fashion beginning with the top vertex. Identify the smallest tube containing the 0- vertex. In this work we shall call the smallest tube containing the 0- vertex t.

2-   Draw n+1 dots for the tree leaves. Number spaces between the leaves 1 to n.

3-   Note that the node under space i between tree leaves corresponds to vertex i of the graph.

4-   Using all of the non- 0 vertices inside t, connect leaves as you would to construct a (non – leveled) tree from a path graph. These edges and vertices will not be painted.

5-   Tube t indicates the location of the paint line. Edges and nodes below those constructed in the previous step will be painted.

6-   Starting with the first tube containing t and working out, add appropriate edges to the tree to create leveled nodes below the existing nodes. Work down and connect new nodes to any existing adjacent nodes. These nodes are painted.

Example (5.1):
   Let G be the a fan graph of 7 nodes with tubing then
1-Number the graph vertices from 0 to 6.
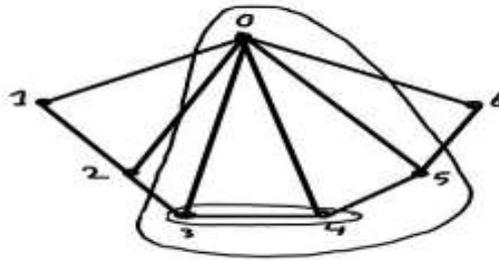
2-Draw n+1 dots for the tree leaves. Number the spaces between the leaves 1 to 6.
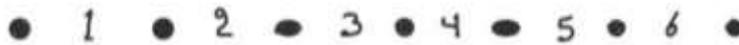
3-Edges and node between the vertices 3 and 4 .
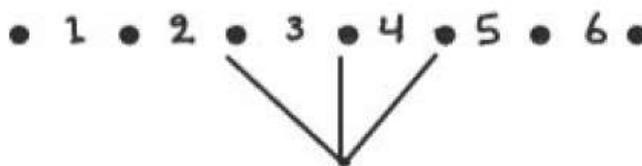
4-Add edges and node for vertex 5 (paint),
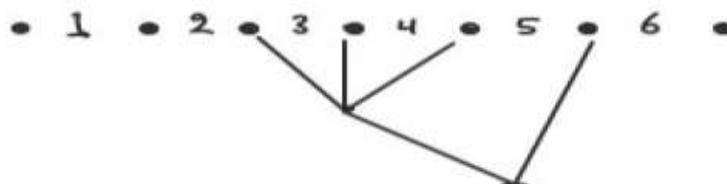
5-Add edges and node for vertex 0,
6-Add edge and node for vertex 6, add edge and node for vertex 2 and add edge and node for vertex 1.
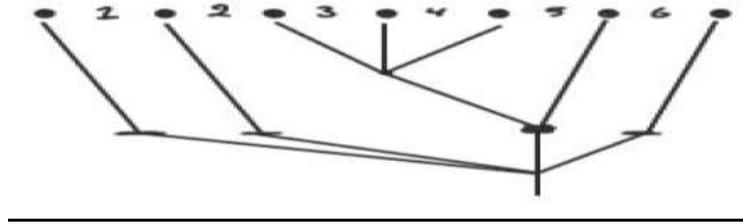
How to convert a painted tree to graph with a tubing:
The following steps give the converting the painted tree to a graph with tubing by the following:
   1- Number the spaces between the tree leaves, 1 to n from left to right.
   2- Draw a fan graph and number the vertices, 0 to n in a counter-clockwise fashion beginning with the top vertex.
   3- Consider the unpainted portions of the tree. Draw tubes as you would to construct path-graph tubing from (non- leveled) binary trees.
   4- To indicate the beginning of the paint, draw a tube containing all existing tubes and the 0-vertex.
   5- Starting with the top most painted node and working down, draw a tube for each leveled node of the tree containing all existing tubes and adding the vertex corresponding to the current node.

Example (6.1):
1-   Number the spaces between the tree leaves, from 1 to 6.
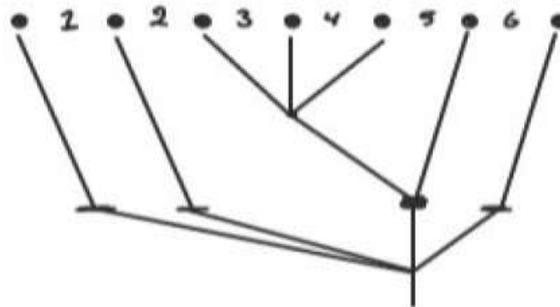
2-   Draw a fan graph and number the vertices, from 0 to 6.
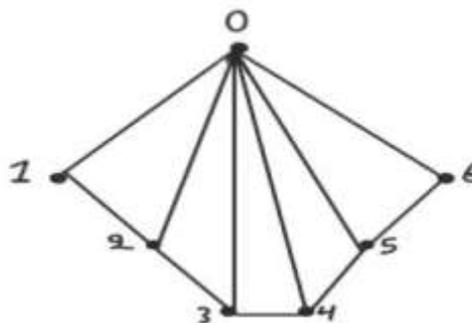
3-   Consider the unpainted portions of the tree and draw tubes as you would to construct path – graph tubings from (non- leveled) binary trees.

4-   Starting with the top most painted node and working down, draw a tube for each leveled node of the tree containing all existing tubes and adding the vertex corresponding to the current node.
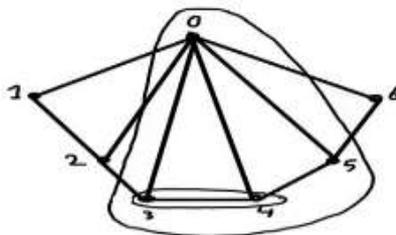
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