On Generalized Left Derivation on Semiprime Rings

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ABSTRACT

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d, then R is commutative, if any one of the following conditions hold: (1) $[d(x), F(y)] = \pm[x, y]$, (2) $[d(x), F(y)] = \pm x o y$, (3) $d(x) o F(y) = \pm x o y$, (4) $d(x) o F(y) = \pm[x, y]$, for all $x, y \in R$.

Keywords: semiprime rings**,** prime rings, generalized left derivation, generalized Jordan left derivation.

حول الاشتقاق المعمّم الايسر على الحلقات شبه الأولية

الخلاصة:

لتكن R حلقة شبه أولية طليقة الالتواء من النمط ٢، إذا احتوت R على الاشتقاق الأيسر المعمّم F المرتبط باشتقاق جوردان الأيسر d ، فأن R حلقة تبادلية، إذا تحقق احد الشروط التالية: (١) $[d(x), F(y)] = \pm[x, y]$ ، (٢) $[d(x), F(y)] = \pm x o y$ ، (٣) $d(x) o f(y) = \pm x o y$ ، (٤) $d(x) o F(y) = \pm[x, y]$ ، لكل x, y في R.

INTRODUCTION

Throughout, R will denote an associative ring with $Z(R)$. As usual, for any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol xoy denotes for anti-commutator $xy + yx$. A ring R is n-torsion free, if $nx = 0$, $x \in R$ implies $x = 0$, where n is a positive integer. Recall that a ring R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$ and semiprime if $aRa = (0)$ implies $a = 0$. A prime ring is semiprime, but the converse is not true in general. An additive mapping $d: R \longrightarrow R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Following [6] An additive mapping d : R $\longrightarrow$ R is said to be left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in R$, Clearly, every left derivation on a ring R is a Jordan left derivation, but the converse need not to be true in general. A classical result of Ashraf and Rehman in [1] asserts that any Jordan left derivation on 2-torsion free prime ring is a left derivation.

In 1991, Brešar [5] defined the following concept. An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$, such that:

$F(xy) = F(x)y + xd(y)$, for all $x, y \in R$.

And in [8] Jing and Lu, initiated the concept of generalized Jordan derivation as, an additive mapping $F : R \longrightarrow R$ is called a generalized Jordan derivation if there exists a Jordan derivation $d : R \longrightarrow R$, such that:

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$F(x^2) = F(x)x + xd(x)$, for all $x \in R$.
Every generalized derivation is a generalized Jordan derivation, but the converse is not true in general. Jing and Lu in [8] proved that the converse is true when R is 2-torsion free prime ring.

Ashraf and Ali in [3] introduced the notion of generalized left derivation as follows: an additive mapping $F : R \longrightarrow R$ is called a generalized left derivation (resp. a generalized Jordan left derivation) associated with Jordan left derivation, if there exists a Jordan left derivation $d : R \longrightarrow R$, such that: $F(xy) = xF(y) + yd(x)$ (resp. $F(x^2) = xF(x) + xd(x)$) holds for all $x, y \in R$.

Posner [9], initiated the study of the commutativity of prime rings with derivation. Following, many authors proved the commutativity of prime rings with generalized derivation, for more details (see [2, 4, 10, and 11]). And in [3] Ashraf and Ali proved that, if R is a 2-torsion free prime ring, and R admits a generalized left derivation F associated with Jordan left derivation d, then either $d = 0$ or R is commutative.

In this paper we prove the commutativity of semiprime rings with generalized left derivation associated with Jordan left derivation, under certain conditions.

The Results

Theorem 1, [12]

Let R be a 2-torsion free semiprime ring with unity. Then every generalized Jordan left derivation F associated with Jordan left derivation d, is a generalized left derivation

The following theorem plays the key role in the proofs of the main results:

Theorem 2:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d, then $d(x) \in Z(R)$, for all $x \in R$.

For the proof of the theorem, we will need the lemma below:

Lemma 3:

Suppose that R is semiprime ring and $a \in R$, such that $a[a, x] = 0$, or $[a, x]a = 0$ for all $x \in R$, then $a \in Z(R)$.

Proof:

If $a[a, x] = 0$, for all $x \in R$, *for the proof see [7]*.

Now, suppose:

$[a, x]a = 0$, for all $x \in R$ .... (1)

Replace x by xr in (1) and using (1), we get:

$[a, x]ra = 0$, for all $x, r \in R$ .... (2)

Replace r by rx in (2), to get:

$[a, x]rxa = 0$, for all $x, r \in R$.. (3)

Again, right multiplication of (2) by x, to get:

$[a, x]rax = 0$, for all $x, r \in R$.. (4)

Subtracting (3) from (4), we get:

$[a, x]r[a, x] = 0$, for all $x, r \in R$ .... (5)

By semiprimeness of R, we get:

$[a, x] = 0$, for all $x \in R$ (6)

And thus, $a \in Z(R)$. ■

Proof of Theorem 2:

We have:

$F(x^2y) = x^2F(y) + yd(x^2)$, for all $x, y \in R$ (1)

That is:

$F(x^2y) = x^2F(y) + 2yxd(x)$, for all $x, y \in R$ (2)

On the other hand, we find that:

$F(x.xy) = xF(xy) + xyd(x)$, for all $x, y \in R$ (3)

That is:

$F(x^2y) = x^2F(y) + 2xyd(x)$, for all $x, y \in R$ (4)

Comparing (2) and (4), we obtain:

$2[x, y]d(x) = 0$, for all $x, y \in R$ (5)

Since R is 2-torsion free, we have:

$[x, y]d(x) = 0$, for all $x, y \in R$ (6)

Linearizing (6) on x, we find that:

$[x, y]d(w) + [w, y]d(x) = 0$, for all $x, y, w \in R$ (7)

Replacing y by yz in (6) and using (6), we have:

$[x, y]zd(x) = 0$, for all $x, y, z \in R$ (8)

Replacing z by $d(w)z[w, y]$ in (8), we have:

$[x, y]d(w)z[w, y]d(x) = 0$, for all $x, y, z, w \in R$ (9)

Comparing (7) and (9), we obtain:

$[x, y]d(w)z[x, y]d(w) = 0$, for all $x, y, z, w \in R$ (10)

Since R is semiprime, we obtain:

$[x, y]d(w) = 0$, for all $x, y, w \in R$ (11)

By Lemma 3, we get $d(w) \in Z(R)$, for all $w \in R$. ■

As an easy consequence of Theorem 2, we have:

Corollary 4:

Let R be a 2-torsion free semiprime ring, and d: R ⟶ R, then d is left derivation if and only if it is central derivation.

Corollary 5:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d, then $d(x) \in Z(R)$, for all $x \in R$.

Proof:

Using Theorem 1, we get R admits a generalized left derivation, and by **Theorem 2**, we get $d(x) \in Z(R)$, for all $x \in R$

Another immediate consequence of Theorem 2, is:

Corollary 6:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d, such that $[d(x), F(y)] = \pm[x, y]$, for all $x, y \in R$. Then R is commutative.

Corollary 7:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d, such that $[d(x), F(y)] = \pm[x, y]$, for all $x, y \in R$. Then R is commutative

Corollary 8:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d, such that $[d(x), F(y)] = \pm x o y$, for all $x, y \in R$. Then R is commutative.

Proof:

We assume:

$[d(x), F(y)] = x o y$, for all $x, y \in R$ (1)

By Theorem (2), (1) gives:

$x o y = 0$, for all $x, y \in R$ (2)

Replace y by yr in (2), to get:

$x o (yr) = 0$, for all $x, y, r \in R$ (3)

This can be rewritten as:

$(xoy)r - y[x, r] = 0$, for all $x, y, r \in R$ (4)
Using (2), (4) gives:
$y[x, r] = 0$, for all $x, y, r \in R$ (5)
Left multiplication of (5) by $[x, r]$, and since R is semiprime, we get:
$[x, r] = 0$, for all $x, r \in R$ (6)
Thereby, we get R is commutative.
Similarly, if $[d(x),F(y)]=-xoy$, for all $x, y\in R$, then R is commutative. ■
The last ***Corollary***, led to:

Corollary 9:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d, such that $[d(x), F(y)] = \pm xoy$, for all $x, y \in R$. Then R is commutative.

Now, we will prove the main results:

Theorem 10:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d, such that $d(x)oF(y) = \pm xoy$, for all $x, y \in R$. Then R is commutative.

Proof*:*

We are given that:
$d(x)oF(y) = xoy$, for all $x, y \in R$ (1)
By Theorem (2), (1) gives:
$2d(x)F(y) = xoy$, for all $x, y \in R$ (2)
Replacing y by yz in (2), we obtain:
$2d(x)yF(z) + 2d(x)zd(y) = xo(yz)$, for all $x, y, z \in R$ (3)
Again, by Theorem (2), (3) gives:
$2yd(x)F(z) + 2d(x)zd(y) = xo(yz)$, for all $x, y, z \in R$ (4)
Since $xo(yz) = y(xoz) + [x, y]z$, then from (2) and (4), we obtain:
$2d(x)zd(y) = [x, y]z$, for all $x, y, z \in R$ (5)
In particular, for $z = x$, (5) gives:
$2d(x)xd(y) = [x, y]x$, for all $x, y \in R$ (6)
Replacing y by ry in (6), we obtain:
$2d(x)xrd(y) + 2d(x)xyd(r) = [x, ry]x$, for all $x, y, r \in R$ (7)
Again, using Theorem (2), (7) reduces to:
$2d(x)xd(y)r + 2d(x)xd(r)y = [x, ry]x$, for all $x, y, r \in R$ (8)
From (6) and (8), we obtain:
$[x, y]xr + [x, r]xy = [x, ry]x$, for all $x, y, r \in R$ (9)
This implies that:
$[[x, y]x, r] + [x, r][x, y] = 0$, for all $x, y, r \in R$ (10)
Replacing r by sr in (10), we obtain:
$s[[x, y]x, r] + [[x, y]x, s]r + s[x, r][x, y] + [x, s]r[x, y] = 0$,
for all $x, y, r, s \in R$ (11)
From (10) and (11), we obtain:
$[[x, y]x, s]r + [x, s]r[x, y] = 0$, for all $x, y, r, s \in R$ (12)
Replacing r by rt in (12), we have:
$[[x, y]x, s]rt + [x, s]rt[x, y] = 0$, for all $x, y, r, s, t \in R$ (13)
Again, right multiplying of (12) by t, we have:
$[[x, y]x, s]rt + [x, s]r[x, y]t = 0$, for all $x, y, r, s, t \in R$ (14)
Subtracting (14) form (13), we obtain:
$[x, s]r[t, [x, y]] = 0$, for all $x, y, r, s, t \in R$ (15)

Replacing y by yx in (15), we have:
$[x, s]r[t, [x, y]x] = 0$, for all $x, y, r, s, t \in R$ (16)
Now, (10) can be rewritten as:
$-[r, [x, y]x] + [x, r][x, y] = 0$, for all $x, y, r \in R$ (17)
For $r = t$ in (17), we have:
$-[t, [x, y]x] + [x, t][x, y] = 0$, for all $x, y, t \in R$ (18)
Left multiplying of (18) by $[x, s]r$, we obtain:
$-[x, s]r[t, [x, y]x] + [x, s]r[x, t][x, y] = 0$, for all $x, y, r, s, t \in R$ (19)
From (16) and (19), we obtain:
$[x, s]r[x, t][x, y] = 0$, for all $x, y, r, s, t \in R$ (20)
Replacing r by $[x, y]r$ in (20), we have:
$[x, s][x, y]r[x, t][x, y] = 0$, for all $x, y, r, s, t \in R$ (21)
For $s = t$ in (21) and since R is semiprime ring, we obtain:
$[x, t][x, y] = 0$, for all $x, y, t \in R$ (22)
Replacing t by yt in (22) and using (22), we have:
$[x, y]t[x, y] = 0$, for all $x, y, t \in R$ (23)
Invoking semiprimeness again, (23) gives:
$[x, y] = 0$, for all $x, y \in R$ (24)
Thus R is commutative.
In the same way given $d(x)oF(y) = -xoy$, for all $x, y \in R$. The argument is similar. ■

Corollary 11:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d, such that d(x)oF(y) = ±xoy, for all x, y ∈ R. Then R is commutative.

Theorem 12:

Let R be a 2-torsion free semiprime ring. If R admits a generalized left derivation F associated with Jordan left derivation d, such that $d(x)oF(y) = \pm[x, y]$, for all $x, y \in R$. Then R is commutative.

Proof:

We assume:
$d(x)oF(y) = [x, y]$, for all $x, y \in R$ (1)
By Theorem (2.1), (1) gives:
$2d(x)F(y) = [x, y]$, for all $x, y \in R$ (2)
Replacing y by yz in (2) we obtain:
$2d(x)yF(z) + 2d(x)zd(y) = y[x, z] + [x, y]z$, for all $x, y, z \in R$ (3)
Again, Theorem (2), (3) gives:
$2yd(x)F(z) + 2d(x)zd(y) = y[x, z] + [x, y]z$, for all $x, y, z \in R$ (4)
From (2) and (4), we obtain:
$2d(x)zd(y) = [x, y]z$, for all $x, y, z \in R$ (5)
In particular, (5) gives:
$2d(x)xd(y) = [x, y]x$, for all $x, y \in R$ (6)
This implies (see how relation (24) was obtained from relation (6) in the proof of Theorem (10)) that R is commutative.
Similarly, we get R is commutative in case:
$d(x)oF(y) = -[x, y]$, for all $x, y \in R$. ■

Corollary 13:

Let R be a 2-torsion free semiprime ring with unity. If R admits a generalized Jordan left derivation F associated with Jordan left derivation d, such that $d(x)oF(y) = \pm[x, y]$, for all $x, y \in R$. Then R is commutative.

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