On The Indecomposable Chains

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Abstract

In this research , We prove that the relation of covering By right or left indecomposable chains , is still an equivalence relation , and the union of both equivalence relations , is again an equivalence . After We prove that for any infinte chain, if every chain x <A satisfies $x.2 \leq A$. then A is indecomposable .

الخلاصة

في هذا البحث برهنا بأنه العلاقة المغطاة من اليمين الى اليسار بسلسلة متصلة تبقى علاقة تكافؤ . كما وأن اتحـاد هـذه العلاقات المتكافئة تبقى علاقة متكافئة ايضاً. ثم برهنا بأنه لأية علاقة السلسلة اللانهائية وسلسلة اخرى مثل بحيـث انـه x<A وتحقق $x.2 \leq A$ فأن A تكون متراصة

1. Introduction

Following (Laver, 1976) a two elements u,v of the base of a chain A are equivalent with respect to right indecomposable chains if there exists an interval of A which is right indecomposable and contains the elements u and v . On the other hand , (Dushnik &.Miller,2002) a doublet is a chain which is the union of a left indecomposable interval and a right indecomposable in decomposable interval and a right indecomposable interval both having at least on common element. We prove the following facts: the equivalence relation of covering by doublets of indecomposable chains has only a finite number of equivalence classes , each class is an indecomposable interval or a doublet.

2. Definitions :

2.1 Let A be a relation, an element Z is between x and y, if y ≥ z ≥ x or x≥ z ≥ y (mod A) , it called intermediacy (Ginsburg, 1982).

2.2 Let c be a relation . A subset X of its base which closed with respect to the intermediate(mod d) (Ginsburg, 1982).

2.3 A chain, is a partial ordering whose elements are mutually comparable the chain is said to be indecomposable if every interval is trivial (Dushnik &.Miller,2002) .

2.4 Let, v be two elements in the base of the chain covering u and v are equivalent with respect to doublets if there exists a doublet of indecomposable chain covering u and v (Farah, 1990) .

2.5 let R,S be two relations , R is embeddable in S iff there exists a restriction of S isomorphic with R (Higman, 1977).

2.6 An Z-sequence is a sequence of length Z, where Z is the set of integers (Laver, 1976).

2.7 An ordinal is a transitive set which is totally ordered by € (Farah, 1990) .

2.8 Let R , s be two relations , R is said to de equimorphic with S, iff R is embedded in S and S is embedded in R (Higman, 1977).

2.9 An aleph (the cardinal of well – orderable set) is said to be regular iff considered as an ordinal (Ginsburg, 1982).

2.10 Every chain isomorphic with a regular aleph, as well as the converse of such , is h-indecomposable (Ginsburg, 1982).

2.11 A chain is said to be scattered if the chain Q of rationales is not embeddable in it (Laver, 1976).

3. om Of Choice (Higman, 1977).

Every set, even infinite , of non-empty mutually disjoint sets admits a choice set .[which is the set A whose intersection with each element x of A is a singleton.]

Proposition 3.1 (Higman, 1977). A chain A is indecomposable iff , for every x < 1A [1A is strictly left indecomposable] with respect to emdeddability, x -2<1A.

Proof: Suppose that A is indecomposable , and that . than $x \geq A$, conversely, suppose that A is decomposable , so $A = B + C$ with $B < A$ and $C < B$. yet A satisfies the conclusion then $B - 2 \ngeq A = B + C$ so $C \nleq B$, similarly B $\nleq C$ Moreover $(C + B).2 = C + B + C + B \geq A$ So $C + B \geq A = B + C$ and thus $B \leq C$ or $C \leq B$ contradiction.

Proposition 3.2 (Dushnik &Miller,2002), let A be a chain which is the union of an initial interval and a final interval . both having at least one common element and both of which are right indecomposable then A is right indecomposable (same statement for left)

Proof: Let B be initial interval, C the finial interval and Δ their intersection . Then B is embeddable in Δ . Either Δ=C so that A =B is right indecomposable .Or C has the form D + E and so A=B+E is embeddable in C=Δ+E hence A is again right indecomposable .

Corollary :-

A chain which is the union of a right indecomposable initial interval B and a left indecomposable final interval C, both infinite and having at least one common element. Then the intersection B$\cap$ C , is both left and right indecomposable and admits an embedding of the chain of rationales .

Proposition 3.3 (Dushnik &Miller,2002)

There exist a strictly smaller (with respect to embeddebilty), restriction of A. which has continuam cardinality .

Theorem 3.4 (Aigner, 2004).

Let ω_α be an infinite regular alpha . Every chain with cardinatly ω_α , admits an embedding either of the ordinal ω_α , or it's converse ω_α' .

Proposition 3.5 (Dushnik &Miller,2002),

Let A be small a non empty chain with an initial interval and a final interval, both disjoint and in each of which A is embeddable . then the chain of rational Q is embeddable in A

Proposition 3.6 (Dushnik &Miller,2002),

Two elements x and y are equivalent with respect to doublets iff there exists a finite sequence of element from x until y , where two consecutive term are equivalent with respect to right of left indecomposable chains

Theorem 3.7 (Laver, 1976).

Let A be a scattered chain and that the h–indecomposable restrictions of A form a will–quasi– ordering with respect to embeddability then A is a finity some of h- indecompesible chains

Theorem 3.8 (Laver, 1976).

Let A be a scattered indecomposable chain. and suppose that the h–indecomposable restrictions of A form a will quasi–ordering with respect to imbecility . then A is h- indecomposable .

Theorem 3.9 (Higman , 1977).

If A is a will partial ordering , then the partial ordering of embeddiblty of words in A is a well partial ordering .

Theorem 3.10 (Laver, 1976).

A necessary sufficient condition for A to be a well partial ordering is that every ω-a sequence in A be good .

Proposition 3.11 (Laver, 1976).

Every set of h-indecomposable chains form a quasi- ordering under embeddebility.

Proposition 3.12 (Laver, 1976).

Every scattered chain is a finite some of h-indecomposable chains . We may now demonstrate he next theorems

Theorem 3.13 :-

Given a scattered chain, the equivalence relation of covering by doublets of indecomposable chains has only a finite number of equivalence classes. Each class is an indecomposable interval or a doublet

Proof:-

By using proposition 3.12 we decompose our chain ,into a finite number of right or left indecomposable intervals. Replace any two contiguous such intervals by their union, provided this union is indecomposable. When it becomes impossible to effect these replacements, then the intervals thus obtained, or the unions of two contiguous intervals, constitute the covering by doublets. The uniqueness of this decomposition follows from proposition 3.2 and Corollary.

Theorem 3.14

For a non-scattered chain, the relation of covering by right or left indecomposable chains, is still an equivalence relation. Hence the union of both equivalence relations, is again an equivalence relation. However there can be infinitely many equivalence classes for this equivalence relation

Proof :-

We Start with A_0 = the chain of the reals by proposing 3.3, we have a strictly decreasing (with respect to embeddability) ω-sequence of chains A_i (i integer) , where each A_i has cardinality of the continuum; moreover we can require that $A_i \not\leq A_i + 1 . \omega$ for each i. On the other hand, we have $Q \geq A_i$. for each i ; indeed A_i has at least ω_1 many elements, and neither the. ordinal ω_1 nor its converse is embeddable in the reals, hence in A_i :by theorem 3.4 the particular case where $\alpha = 1$. Thus $A_i \not\leq Q + A_{i+1} + Q + \ldots + Q + A_{i+h}$ for any two natural numbers i and h . Let $U = \omega_1' + \omega_1$ and consider the sum of the ω-sequence $A_0 + U + A_1 + U + \ldots + U + A_i + U + . ..$ .We shall prove that each interval isomorphic with U is one of the desired equivalence classes; hence that there exist infinitely many equivalence classes.

Indeed, take two elements x and y in two consecutive components: for example x belongs to U and y belongs to A_i following the considered component U . We must join x to y by finitely many intermediate elements, such that any two consecutive elements be either right equivalent (i.e. covered by a same right indecomposable interval) or left equivalent. We can assume that x and y are themselves consecutive elements; then it suffices to see that they are neither right nor left equivalent.

First, a non-final interval I which contains x and y is obviously decomposable into a finite sequence of disjoint sub-intervals in which I cannot be embedded. Secondly, a final interval is obviously not left indecomposable; nor is it right indecomposable; for otherwise, it would be necessary that A_i , for example, be embeddable in a sum of the form $U + A_{i+1} + U + \ldots + U + A_{i+h}$. But an interval of A_i which is a restriction of U is countable, since it is isomorphic with the union of a well-ordered set of reals and the converse of such a well-ordered set. So it must be that A_i is embeddable in $Q + A_{i+1} + Q + \ldots + Q + A_{i+h}$, contradicting the previous discussion.

Theorem 3.15:-

Let A be an infinite scattered chain. If every chain $X < A$ satisfies $X.2 \geq A$, then A is indecomposable

Proof:-

By Using Proposition 3.12, decompose A into a finite sum of either right indecomposable or left indecomposable intervals. Let I be one of these intervals, which is ≤ or | all others under embeddability. We can assume that I is infinite, and right indecomposable, to fix ideas. Let k be the number of intervals equimorphic with I , in the considered decomposition. Either I. k is equimorphic with A ; then if k = l we are finished. If $k \leq 2$, setting $X = I(k-l) + 1$, we have $X < A$ proposition 3.5 yet $X.2 \not\leq A$, contradicting our hypothesis. Or $I.k < A$; then $I(k+l) \not\leq A$; Proposition 3.5 setting $X = I.k$, this contradicts our hypothesis.

Theorem3.16 :-

Let A be an infinite scattered chain. If every proper initial interval X of A satisfies $X.2 \geq A$, then A is right indecomposable For A non-scattered, we have the counterexample $Q + \omega_1$. If the hypothesis is weakened by requiring that X be an initial interval strictly less than A under embeddability, then we have the counterexample $(\omega^2 + \omega)$- already mentioned proposion 3.1

Proof:

Wwe decompose A into a finite sum of either right or left indecomposable intervals. We can assume that there are at least two such intervals in the decomposition; denote by D the last such interval.

If D is infinite and left indecomposable, then take an element d in D and let X be the initial interval of A with last element d . Then X is equimorphic with A and so A.2 equimorphic with A , thus A is not scattered proposition 3.5 . If D is infinite and right indecomposable, then again take an element d in D and let X be the initial interval of A with last element d . Then $X.2 \geq A$ and X.2 has a last element. Thus the initial interval Y generated by X.2 satisfies $Y.2. \geq A$, and so $X.4 \geq A$. Hence either $X.2 \geq X$ and then neither X nor A is scattered. Or $X \geq D$ yet is not cofinal in D ; so A equimorphic with X + D yields $A \geq D$ and thus A itself is right indecomposable. Finally if D reduces to a singleton, then letting X = A-D : we have $X.2 \geq A$ and even $X.2 + 1 \geq A$ (distinguish the case where X has a maximum); hence $X.2 \geq X$ so that X is not scattered. We obtain the contradiction in the first and in the third cases: so that only the second case occurs, and A is right indecomposable.

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