Trivial Extension of π-Regular Rings

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ABSTRACT

In this paper we investigate if it is possible that the trivial extension ring $T(R,R)$ inherit the properties of the ring R and present the relationship between the trivial extension $T(R,M)$ of a ring R by an R-module M and the π-regularity of R by taking new concepts as π-coherent rings and C-π-regular rings which introduced as extensions of the concept of π-regularrings. Moreover we studied the possibility of being the trivial extension $T(R,M)$ itself π-regular ring according to specific conditions. Thus we proved that if R is an Artinian ring, then the trivial extension $T(R,R)$ is a π-regular ring. As well as if the trivial extension $T(R,R)$ is a Noetherian π-regular ring, then R is a π-regular ring. On the other hand we showed that if F is a field and M is an F-vector space with infinite dimension, then the trivial extension ring $T(F,M)$ of F by M is C-π-regular ring.
Keywords: Trivial extension ring, π-regular ring, Artinian ring, Noetherian ring, C-π-regular ring.

التمديد التافه للحلقات المنتظمة من النمطπ

الخلاصة

في هذا البحث نتحقق فيما اذا كان من الممكن ان ترث حلقة التمديد التافه$T(R,R)$ صفات الحلقة R ونعرض العلاقة بين حلقة التمديد التافه$T(R,M)$ للحلقة R بواسطة الموديول M والانتظامية من النمط π للحلقة R من خلال اتخاذ مفاهيم جديدة كالحلقات المتماسكة من النمط π والحلقات المنتظمة من النمط C-π التي قدمت لاول مرة كتوسيع لمفهوم الحلقات المنتظمة من النمط π. بالاضافة الى ذلك درسنا امكانية كون حلقة التمديد التافه $T(R,M)$ نفسها منتظمة من النمط π وفقا لشروط محددة. وهكذا اثبتنا انه اذا كانت R حلقة ارتينية فان حلقة التمديد التافه $T(R,R)$ تكون منتظمة من النمط π. وكذلك اذا كانت حلقة التمديد التافه $T(R,R)$ نويثيرية ومنتظمة من النمط π فان R حلقة منتظمة من النمط π. من ناحية اخرى فقد بينا انه اذا كان F حقل و M هو فضاء متجهات لا نهائي على F فان حلقة التمديد التافه $T(F,M)$ للحقل F بواسطة M تكون منتظمة من النمط C-π .

INTRODUCTION

Throughout this paper all rings are commutative and all modules are unitary, unless otherwise stated. Recall that a ring R is π-regular if for each $r \in R$ there exist a positive integer n and $s \in R$ such that $r^n s r^n = r^n$. It is known that a ring R is π-regular if and only if for every element r in R there exists a positive integer n such that Rr^n is a direct summand of R if and only if every prime ideal of R is maximal [1]. A ring R is said to be coherent if every finitely generated ideal of R is finitely presented [2]. We introduced the concept of π-coherent rings (C-π-regular ring) as a generalization of coherent ring (π-regular ring) such that every π-regular ring is π-coherent (everyπ-regular ring is C-π-regular).

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The main idea is to put M (may be $M = R$) inside a commutative ring $T = T(R, M) = R \oplus M$ (if R is commutative) so that the structure of Mas an R-module is essentially the same as that of Mas an T-module, that is, as an ideal of T [3,4,5]. The advantage of the trivial extension is:
(a) Transfer results relating to modules to the ideal case including the R-module$R$.
(b) Extending results from rings to modules.
(c) It is easier to find counterexamples of rings especially those with zero divisors.

Generally, we studied the relationship between π-regular rings and some related concepts with trivial extension ring $T(R, R)$. Moreover we extended the property of π-regularity to the trivial extension $T(R, R)$. We proved that if the trivial extension $T(R, R)$ is Artinian ring, and then R is a π-regular ring. In addition if the trivial extension $T(R, R)$ is a Noetherian π-regular ring, then R is a π-regular ring. Also we showed that if M is a finitely generated module over a ring R and the trivial extension $T(R, M)$ of R by M is a C-π-regular ring, then R is C-π-regular. On the other hand we found that the trivial extension ring $T(D, F)$ of the domain D by F is not π-regular ring in case that $D$is not a field and$F$ is the quotient field of D. However we showed that if F is a field and M is an F-vector space with infinite dimension, then the trivial extension ring $T(F, M)$ of F by M is C-π-regular ring.

The structure of this paper is as follows. Section 2 is devoted to recall previous known definitions and information about the trivial extension. In section 3 we present our main work, we study the relationship between the trivial extension rings and π-regular rings and showed that a homomorphic image of a C-π-regular ring is C-π-regular. Several examples are given to clarify the ideas used within the section. Section 4 involves the study of the trivial extension ring $T(R, N)$ of a ring R by an R-module N in case R is a local domain. So we have the following: If M is a maximal ideal of a local domain R and N is an R-module with $MN = 0$, then the trivial extension ring $T(R, N)$ of R by N has no any proper projective ideal. Finally, in section 5 we give an inspiration for future works depending on the related existing results.

It is worth to mention that there are many generalizations of the concept of π-regular rings to modules such as GF-reguar modules [6,7] and GZ-regular modules [8] and there is an equivalent concept of π-regular rings studied in [8] named P-semiregular rings. And certainly there are many related concepts to π-regular rings most notably π-McCoy rings [9] and other related concepts as in [10].

Preliminaries

In this section we survey known results concerning $T = T(R, M) = R \oplus M$. The theme throughout is how properties of $T = T(R, M) = R \oplus M$ are related to those of R and M [3,4,5].
Let R be a ring and M be an (R, R)-bimodule. Recall that the trivial extension of R by M (also called the idealization of M over R) is defined to be the set $T = T(R, M)$ of all pairs (r, m) where $r \in R$ and $m \in M$, that is:

$$T = T(R, M) = R \oplus M = \{(r, m) | r \in R, m \in M\}$$

With addition defined componentwise as

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$$

And multiplication defined according to the rule

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$$

For all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Clearly $T = T(R, M)$ forms a ring (even an R-algebra) and it is commutative if and only if R is commutative.
Note that R naturally embeds into $T(R, M)$ via $r \rightarrow (r, 0)$, that is $T(R, 0) \cong R$ and that the module M identified with $T(0, M) = \{(0, m) | m \in M\}$ becomes a nonzero nilpotent ideal of $T(R, M)$ of index 2, which explains the term idealization. If N is a submodule of M, then $T(0, N)$ is an ideal of

$T(R,M)$, and that $(T(R,M)/T(0,M)) \cong R$. The ring $T = T(R,M)$ has identity element $(1,0)$ and any idempotent element of the trivial extension ring $T(R,R)$ is of the form $(e,0)$ where $e^2 = e \in R$.

In fact there is another realization of the trivial extension. Let $T = T(R,M) \cong \left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \middle| r \in R, m \in M\right\}$, then T is a subring of the ring of 2×2 matrices over R with the usual matrix operations and T is a commutative ring with identity. If $M = R$, then $T = T(R,R) \cong R[x]/\langle x^2 \rangle$ where $R[x]$ denote the ring of all polynomials over R and $\langle x^2 \rangle$ is the ideal generated by x^2. For convenience, let

$$T(I,N) = \{(s,n) | s \in I, n \in N\}$$

Where

I is a subset of R and N is a subset of M. Moreover, if R be a commutative ring and M an R-module, then

(1) The prime ideals of $T(R,M)$ have the form $T(P,M) = P \oplus M$ where P is a prime ideal of R.

(2) The maximal ideals of $T(R,M)$ have the form $T(W,M) = W \oplus M$ where W is a maximal ideal of R. The Jacobson radical of $T(R,M)$ is $J(T(R,M)) = T(J(R),M) = T(J(R) \oplus M)$.

Let M be an R-module, and let $\{m_\alpha\} \subseteq M$. It is obvious that M is generated by $\{m_\alpha\}$ if and only if $T(0,M)$ is generated by $\{(0,m_\alpha)\}$. Thus, M is finitely generated as an R-module if and only if $T(0,M)$ is finitely generated as an ideal. If I is an ideal of R, $IT(R,M) = I \oplus IM$. Thus, if $I$is finitely generated, so is $I \oplus IM$. However, $I \oplus IM$ can be finitely generated without IM being finitely generated.

We next determine when $T(R,M)$ is Noetherian or Artinian. Let R be a commutative ring and M be an R-module. Then $T(R,M)$is Noetherian, respectively Artinian, if and only if $R$is Noetherian, respectively Artinian, and $M$is finitely generated.

Finally, for an endomorphism α of a ring R and the trivial extension ring $T(R,R)$ of R, α can be extended to an endomorphism of the ring $T(R,R)$ as $\bar{\alpha}: T(R,R) \to T(R,R)$ defined by $\bar{\alpha}\left((a_{ij})\right) = \left(\alpha(a_{ij})\right)$, that is:

$$\bar{\alpha}\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix}$$

Since $T(R,0)$ is isomorphic to R, we can identify the restriction of $\bar{\alpha}$ on $T(R,0)$ to α.

The Trivial Extension and π-Regular Rings

In this section we explore the relationship between the trivial extension rings and π-regular rings.

It is well known that a ring R Artinian if and only if R is Noetherian π-regular [1]. On the other hand, according to [5, Theorem 4.8], it's easy to find finitely generated R-modules that have submodules that are not finitely generated. For example, R considered as a module for itself, every ring with identity is generated as an R-module by 1_R, hence is a finitely generated R-module. However, if R is not left Noetherian, R will have non-finitely generated left ideals, hence non-finitely generated submodules. Therefore according to the information stated in Section 2 above we have that $T(R,R)$is Noetherian, respectively Artinian, if and only if R is Noetherian, respectively Artinian.

Proposition 3.1: If R is an Artinian ring, then the trivial extension $T(R,R)$ is a π-regular ring.

Proof: Let R be an Artinian ring, then $T(R,R)$ is an Artinian ring [5, Theorem 4.8], hence $T(R,R)$ is a π-regular ring [1, Corollary 1.1.23].

The following is immediately by [1, Corollary 1.1.23].

Corollary 3.2: If R is a Noetherian π-regular ring, then the trivial extension $T(R,R)$ is a π-regular ring.

Proposition 3.3: If the trivial extension $T(R,R)$ is Artinian ring, then R is a π-regular ring.

Proof: Since if $T(R,R)$ is Artinian then R is Artinian [5, Theorem 4.8], therefore by [1, Corollary 1.1.23] R is π-regular ring.

The following is also immediately by [1, Corollary 1.1.23].

Corollary 3.4: If the trivial extension $T(R,R)$ is a Noetherian π-regular ring, then R is a π-regular ring.

Now we introduce the concept of π-coherent ring as a generalization of coherent rings and π-regular rings simultaneously.

Definition 3.5: A ring R is said to be π-coherent if for every $x \in R$, there exists a positive integer n such that $Rx^n$is a non-trivial finitely presented ideal.

It is clear that every coherent ring is π-coherent.

Remark 3.6: Every π-regular ring is π-coherent.

Proof: Since R is a π-regular ring, then for each $x \in R$ there exists a positive integer n such that Rx^n is direct summand [1]. So Rx^n is projective R-module, hence we have that Rx^n is finitely presented [2]. Therefore R is π-coherent ring.

Examples 3.7:

(1) Let $R = \mathbb{Z}_4$, the trivial extension ring $T(R,R) = T(\mathbb{Z}_4,\mathbb{Z}_4) = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is π-regular by [1].

(2) Let $\widehat{\mathbb{Z}}_{(p)}$ be the completion of the localization $\mathbb{Z}_{(p)}$ where $\mathbb{Z}$ is the integers and p is a prime number. That is, $\widehat{\mathbb{Z}}_{(p)}$ is the ring of p-adic integers. Put $\mathbb{Z}_{p^\infty}$ to be the $Prüferp$-group and set S be the trivial extension of $\widehat{\mathbb{Z}}_{(p)}$. The trivial extension ring $T(S,S) = S \oplus S$ is not π-regular [11].

Now we consider the following properties:

Property A: For each finitely generated prime ideal P of R there exists a positive integer n such that $r^n \in P$and Rr^n is direct summand for all $r \in R$.

We denote each ideal satisfies Property (A) by the Container.

Property B: Every Container ideal P of R is a direct summand of R.

Definition 3.8: A ring R is said to be C-π-regular if it satisfies Property B.

It is clear that every π-regular ring is C-π-regular. Note that the Container ideal is generated by an idempotent.

Theorem 3.9: Let D be a domain which is not a field and let F be the quotient field of D. The following are equivalent:

(1) The trivial extension ring $T(D,F)$ of D by F is a C-π-regular ring

(2) D has no any nonzero Container ideal.

Proof:

(1) implies (2) Suppose that $T = T(D,F) = D \oplus F$ is a C-π-regular ring and D has a proper nonzero Container ideal. Put $I \coloneqq \sum_{i=1}^{k} Db_i$ be a nonzero Container ideal of D where $b_i \in D$ for each $i = 1, \dots, k$. Let $P \coloneqq T(I,F)$, then $P \in Spec(R)$ by [4] and $P = \sum_{i=1}^{k} T(b_i, 0)$ because $bF = F$ for each $b \in D\{0\}$. Therefore $P = T(t,e)$ where (t,e) is an idempotent element of D because that D is a C-π-regular ring, that is $(t,e) = (t,e)(t,e) = (t^2, 2te)$. So $t^2 = t$ and hence $t = 0$ or $t = 1$ because D is a domain which is a contradiction since $P(\coloneqq Dt)$ is a proper ideal of D. Consequently, D has no any nonzero Container ideal.

(2) implies (1) Suppose that D has no any nonzero Container ideal and let P be a Container ideal of $T(D,F)$. So $P = T(I,F)$ where I is a nonzero Container ideal of D (because $T(0,F)$ is not a finitely generated ideal of $T(D,F)$). Therefore I is a nonzero prime ideal which implies that I is not a

Container ideal of D which is a contradiction. Hence $T(D,F)$ has no any nonzero Container ideal which means that $T(D,F)$ is a C-π-regular ring.

Theorem 3.10: Let D be a domain which is not a field and let F be the quotient field of D, then the trivial extension ring $T(D,F)$ of D by F is not π-regular ring. In fact R is not π-coherent.

Proof: To prove that $T = T(D,F)$ is not π-regular, it is enough to show that $T = T(D,F)$ is not π-coherent using Remark 3.6. Actually, there exists a nonzero element $(0,1) \in T = T(D,F)$ such that for each positive integer n the ideal $T(0,1)^n = T(o,1)$ in case $n = 1$ and $T(0,1)^n = (0)$ in case $n > 1$ (which is not in our concerned). The ideal $T(0,1)^n = T(0,1)$ is not finitely presented by the exact sequence of T-modules:

$$0 \to T(0,F) \to T \overset{u}{\to} T(0,1) \to 0$$

where $u(d,e) = (d,e)(0,1) = (0,d)$ this is because $T(0,F)$ is not a finitely generated ideal of $T(d,F)$. Therefore $T(D,F)$ is not π-coherent and so is not π-regular.

The following example shows that C-π-regular may not be π-regular.

Example 3.11: Let $I \subseteq J$ be an extension of fields such that $[J:I] = \infty$, $S = J[[X]] = J + M$ where X is an indeterminate over J and $M = XS$ is the maximal ideal of S. Put $R := I + M$ and the trivial extension ring of R by F is $T(R,F) = R \oplus F$ where F is a quotient field of A. Then R is C-π-regular ring because the only nonzero prime ideal of R is M which is not finitely generated of R [12]. Moreover $T(R,F)$ is a C-π-regular ring by Theorem 3.9 and [12] and also $T(R,F)$ is not a π-regular ring by Theorem 3.10 and because the only nonzero Container ideal of R is M which is not a finitely generated ideal of R.

It is known that every π-regular domain is a field [1]. The above example shows that this is not true for C-π-regular rings because R is a C-π-regular domain but not a field.

Example 3.12: Let F be a field and let n be a positive integer. The trivial extension ring $T(F,F^n) = F \oplus F^n$ of F by F^n is not a C-π-regular ring.

Proof: Put $P := T(0,F^n)$ to be a container ideal of $T(F,F^n)$. P is the only prime ideal of $T(F,F^n)$ because any prime ideal of R has always the form $T(I,F^n)$where I is prime ideal of F [4]. And since F is a field, then the zero ideal is the only prime ideal of F. Hence $P = T(0,F^n)$ is unique prime ideal of $T(F,F^n)$. But $T(F,F^n)$ is local [12, Example 2.4], so $T(F,F^n)$ is also local ring and has unique prime ideal. Therefore $T(F,F^n)$ is π-regular ring [1, Corollary 1.1.16].

Remark 3.13: The above example gives the following information of the trivial extension ring $T(F,F^n)$:

(1)$T(F,F^n)$ is π-regular ring.

(2)$T(F,F^n)$ is C-π-regular ring.

(3)$T(F,F^n)$ is not P-von Neumann regular ring.

(4)$T(F,F^n)$ is not von Neumann regular ring.

Now it is appropriate to ask : when we can consider D in Theorem 3.9 to be a field?

Proposition 3.14: If F is a field and M is an F-vector space with infinite dimension, then the trivial extension ring $T(F,M)$ of F by M is C-π-regular ring.

Proof: The ideal $T(0,M)$ of $T(F,M)$ is the only proper prime ideal and it is not finitely generated of $T(F,M)$ because M is an F-vector space with infinite dimension. Thus for each $r \in R$ and for each positive integer n there is no Container ideal of $T(F,M)$. Hence $T(F,M)$ is C-π-regular ring.

It is known that a homomorphic image of a π-regular ring is also π-regular [1]. The following is an analogous result for C-π-regular rings.

Proposition 3.15: A homomorphic image of a C-π-regular ring is C-π-regular.

Proof: Suppose that R is a C-π-regular ring and I is a Container ideal of R such that G is a Container ideal of R/I. Thus $G = P/I$ for some container ideal P of R. Hence P is generated by an idempotent element e of R, so G is generated by $e + I$ which is an idempotent element of R/I. This means that the Container ideal G of R/I is direct summand. Therefore R/I is a C-π-regular ring.

Corollary 3.16: Let M be a finitely generated module over a ring R. If the trivial extension $T(R, M)$ of R by M is a C-π-regular ring, then R is C-π-regular.

Proof: The trivial extension ring of R by M is $T = T(R, M)$, so $R \cong T/T(0, M)$. Since M is finitely generated R-module, then $M \cong T(0, M)$ is finitely generated ideal of R. Therefore R is C-π-regular ring by Proposition 3.15.

Trivial Extension Ring via Local Domain

In this section we study the trivial extension ring of a local domain R with its unique maximal ideal M.

Lemma 4.1: [12, Lemma 2.7]

If M is a maximal ideal of a local domain R and N is an R-module with $MN = 0$, then the trivial extension ring $T(R, N)$ of R by N has no any proper projective ideal.

Theorem 4.2: Let M be the unique maximal ideal of the local domain R and let N be an R-module such that $MN = 0$. If N is an R/M-vector space of infinite dimension, then the trivial extension $T(R, N)$ of R by N is a C-π-regular ring.

Proof: Suppose that N is an R/M-vector space of infinite dimension. By Lemma 4.1 it is obvious that any ring R is C-π-regular ring if and only if there is no proper Container ideal of R. So it is enough to prove that $T = T(R, N)$ has no proper Container ideal. If not, let $P := T(I, N) = \sum_{i=1}^{k} T(s_i, a_i)$ be a proper Container ideal of $T = T(R, N)$ where I is a Container ideal of R and $s_i \in I$, $a_i \in N$ for each $i = 1, \dots, k$. Since $s_i N = 0$ for each $i = 1, \dots, k$ hence $N \subseteq \sum_{i=1}^{k} (R/M) a_i$ which implies that N is an R/M-vector space of finite dimension which is a contradiction. Thus, there is no proper Container ideal of $T = T(R, N)$ and therefore $T = T(R, N)$ is a C-π-regular ring.

Recall that a discrete valuation ring is a principle ideal domain that has exactly one nonzero prime ideal.

Example 4.3: Let D be a discrete valuation domain which is not a field with its unique maximal ideal M and let N be a D/M-vector space with infinite dimension. Then the trivial extension ring $T = T(D, N)$ of D by N is a C-π-regular ring by Theorem 4.2 and $D = T/T(0, N)$ is not a C-π-regular ring because M is a Container ideal which is not generated by an idempotent and because D is a domain.

Future Suggestions:

It is convenient to ask about the possibility to transmit the concepts of GF-regular and GZ-regular modules to the concept of the trivial extension. We have a conjecture that there is a relationship between the trivial extension $T(R, R)$ of a π-regular rings R and the trivial extension $T(R, M)$ of a ring R and a GF-reguar module (GZ-regular module) M. Moreover we claim that we can get promising results when we relocate the concept of P-semiregular rings, the unique maximal GF-regular submodule of a moduleand related concepts to π-regular rings like π-McCoy rings to the concept of the trivial extension.

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