The Solutions of Certain Stochastic Differential Equations Heun's Numerical Method in 2-wiener process

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Abstract:

In this paper, we studied the numerical method to solve certain stochastic differential equation because the difficulty of finding solution analytical of many stochastic differential equations. Heun's numerical method in 2-wiener process was used to perform numerical simulations for a number of applied examples by finding the difference between the numerical solution and the controlled solution of the stochastic differential equation.

الملخص:

في هذا البحث ، درسنا الطريقة العددية لحل معادلة تفاضلية عشوائية معينة بسبب صعوبة إيجاد حل تحليلي للعديد من المعادلات التفاضلية العشوائية .تم استخدام الطريقة العددية لهيون في عملية 2-wiener لإجراء المحاكاة العددية لعدد من الأمثلة التطبيقية من خلال إيجاد الفرق بين الحل العددي والحل الدقيق للمعادلة التفاضلية العشوائية.

I. Introduction.

The numerical analysis of stochastic differential equations differs significantly from that of ordinary differential equations due to peculiarities of stochastic calculus. Stochastic differential equations (SDEs) driven by Brownian motions or Lévy processes are important tools in a wide range of applications, including biology, chemistry, mechanics, economics, physics and finance [2, 3 , 4, 5, 6,11,12]. Those equations are interpreted in the framework of Itô calculus [2, 45] and examples are like, the geometric Brownian motion,

$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$, $X(0) = X0$, … (a)

which plays a very important role in the Black-Sholes-Merton option pricing model, or, the Feller's branching diffusion in biology,

$dX(t)=\alpha X(t)dt+\sigma pX(t)dW(t), X(0)=X0>0$, …(b)

where W(t) is the Brownian motion in both examples. Another example of SDE driven by a Lévy process is the following jump-diffusion process [8,14.15]

$dS(t)=a(t,S(t))dt+b(t,S(t))dW(t)+c(t,S(t))\ dJ(t), 0\leq t\leq T$, …(c)

where the jump term J(t) is a compound Poisson process PN(t) i=1 Yi , the jump magnitude Yi has a prescribed distribution and N (t) is a Poisson process with intensity λ, independent of the Brownian motion W(t). This equation is used to model the stock price which may be discontinuous and is a generalization of equation (1). Usually, the SDEs we encounter do not have analytical solutions and developing efficient numerical methods to simulate those SDEs is an important research topic. The goal of this thesis is to introduce the recent development of those numerical methods.

Early attempts are made in the area of numerical methods for stochastic differential equations in 2-wiener process using Heun's method[1,8]. provides an early account for constructing a numerical method for solving stochastic differential equations in 2-wiener process . This method is known as the Milstein method[1,4,8]. proved an application of the central difference and predictor methods for finding a solution of differential equations with stochastic. Numerical methods for SDE's constructed by translating a deterministic numerical method (like the Heun's method or Runge-Kutta method[6,16]. and applying it to a stochastic ordinary differential equation. However, merely translating a deterministic numerical method and applying it to an SDE will generally not provide accurate methods [**7,**14,15**]**. Suitably appropriate numerical methods for SDE's should take into account a detailed analysis of the order of convergence as well as stability of the numerical scheme and the behavior of the errors. The Heun's method for SDE's is the simplest method which is a direct translation of the deterministic Heun's method, but according to [4,9]this method is not very accurate. However, this method is useful in that it provides a starting point for more advanced numerical methods for SDE's.

Definition .1[6]. Let $x(t) \in (0 \leq t \leq T)$ be a stochastic process such that for any $0 \leq t_1 < t_2 \leq T$

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} a(t)\, dt + \int_{t_1}^{t_2} b(t)\, dw(t)$$

where $a \in L_{\omega}^{1}[0, T]$, $b \in L_{\omega}^{2}[0, T]$. Then we say that dw(t) has stochastic differential dx, on [0, T], given by:

$dx(t) = a(t)dt + b(t)dw(t)$

Observe that x(t) is a no anticipative function. It is also a continuous process. Hence, in particular, it belongs to $L_{\omega}^{\infty}[0, T]$.

Theorem1[7]: Let $d\xi(t) = adt + bdw(t)$, and let f(x, t) be a continuous function in $(x, t) \in R^1 \times [0, \infty)$ with partial derivatives f_x, f_{xx}, f_t. Then the process $f(\xi(t), t)$ has a stochastic differential, given by:

$$df(\xi(t), t) = [f_t(\xi(t), t) + f_x(\xi(t), t)a(t) + \frac{1}{2} f_{xx}(\xi(t), t)b^2(t)]dt + f_x(\xi(t), t)b(t)dw(t) \quad \ldots(1)$$

This is called the Itô formula. Notice that if w(t) were continuously differentiable in t, then (by the standard calculus formula for total derivatives) the term $\frac{1}{2} f_{xx}b^2dt$ will not appear.

Our work is solving stochastic differential equation in 2-wiener process , by using Heun's (modified Euler's method) .Moreover we apply some examples to show that the numerical solutions of different examples are implemented properly.

II.Main Results:

In this section we can state and prove the Lemma by using theorem 1[7].

Lemma1. : Suppose that

$$dy_t = a(t, x_t)dt + b(t, x_t)dw_1 + k(t, x_t)dw_2 \quad \ldots(2)$$

where $a(t, x_t)$, $b(t, x_t)$ and $k(t, x_t)$ are continuous functions which is defined on interval $[t_0, T]$,

provided that dw_1, dw_2 are wiener process with components W_t^1, W_t^2, …, W_t^m .

<u>Proof.</u> We integrate the SDE the equation (2) , we get

$y(x_1) = y(x_0) + \int_{x_0}^{x_1} a(s, x_s)ds + \int_{t_0}^{t} b(s, x_{s_1})dw_{s_1} + \int_{t_0}^{t} k(s, x_{s_2})dw_{s_2} \ldots(3)$

Also let $I_1 = \int_{t_0=x_0}^{t=x_1} a(s, x_s)ds = \int_{t_0=x_0}^{t=x_1} f(x)\, dx$

where trapezoidal rule $x_0 = t_0$, $x_1 = t$,

$h = x_0 - x_1 = t_0 - t.$

And $f(x) = P_n(x) + \frac{f^{n+1}(\delta)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)$ ...(4)

where $P_n(x) = \sum_{\substack{i=0\\ i\neq k}}^{n} f(x_i) * L_i(x)$,

where $L_i(x) = \frac{x - x_i}{x_k - x_i}$

And putting n=1

$\rightarrow P_1(x) = \sum_{\substack{i=0\\ i\neq k}}^{1} f(x_i) * \frac{x - x_i}{x_k - x_i} = \frac{x - x_1}{x_0 - x_1} * f(x_0) + \frac{x - x_0}{x_1 - x_0} * f(x_1)$ Hence

$f(x) = \frac{x - x_1}{x_0 - x_1} * f(x_0) + \frac{x - x_0}{x_1 - x_0} * f(x_1) + \frac{f^{(n+1)}(\varepsilon)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)$ Thus if $I_1 = \int_{t_0=x_0}^{t=x_1} f(x)\, dx = \int_{t_0=x_0}^{t=x_1}\left(\frac{x - x_1}{x_0 - x_1} * f(x_0) + \frac{x - x_0}{x_1 - x_0} * f(x_1)\right)dx + \int_{t_0=x_0}^{t=x_1}\frac{f^{(n+1)}(\varepsilon)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)dx.$

Where

$\int_{t_0=x_0}^{t=x_1}\frac{f^{(n+1)}(\varepsilon)}{(n+1)!}\prod_{i=0}^{n}(x - x_i)dx$ is called error

Then

$I_1 = \int_{t_0=x_0}^{t=x_1} f(x)\, dx = \int_{t_0=x_0}^{t=x_1}\left\{\frac{(x - x_1)^2}{2(x_0 - x_1)} * f(x_0) + \frac{(x - x_0)^2}{2(x_1 - x_0)} * f(x_1)\right\}dx$

$= \frac{x_1 - x_0}{2}[f(x_0) + f(x_1)] = \frac{h}{2}[f(x_0) + f(x_1)]$...(5)

And if

$$I_2 = \int_{t_0}^{t} b(x_{s_1})dw_{s_1} = \int_{t_0=x_0}^{t=x_1} b(s, x_{s_1})dw_{s_1} = \int_{t_0=x_0}^{t=x_1} b(x)\, dw_{s_1} = \frac{X_1 - X_0}{2}[b(x_0) + b(x_1)]$$

...(6)

and

$I_3 = \int_{t_0}^{t} k(x_{s_2})dw_{s_2} = \int_{t_0=x_0}^{t=x_1} k(s, x_{s_1})dw_{s_2} = \int_{t_0=x_0}^{t=x_1} k(x)\, dw_{s_2} = \frac{x_1 - x_0}{2}[k(x_0) + k(x_1)] \ldots(7)$

Then Euler Scheme method which is $x_t = x_{t_0} + a\Delta n + b\Delta w_n$

Or

$y_{n+1} = y_n + a\,\Delta n + b\Delta w_n;\ n = 0,1,2,..,N - 1 \ldots(8)$

The Equation (8) in equations (5) , (6) and (7) we get

$I_1 = \frac{h}{2}[f(y_n) + f(y_n + a\,\Delta n + b\Delta w_n)]$

$I_2 = \frac{h}{2}[b(y_n) + b(y_n + a\,\Delta n +$

$b\Delta w_n)]$And $I_3 = \frac{h}{2}[k(y_n) + k(y_n + a\,\Delta n + b\Delta w_n)]$ ….(9)

where $x = \{x_t : t_0 \le t \le T\}$ is an Itô process with initial value $x_{t_0} = x_0$. Subdivide the interval $[t_0, T]$ into N-subintervals according to the following discretization: $t_0 = \tau_0 < \tau_1 < \ldots < \tau_n < \ldots < \tau_N = T$.

The Heun′s approx. is defined as cont. time stochastics process $y = \{y(T);\ t_0 \le t < T\}$ satisfying the iterative scheme:

$y_{n+1} = y_n + \frac{h}{2}[f(y_n) + f(y_n + a\,\Delta n + b\Delta w_n)]\Delta n + \frac{h}{2}[b(y_n) + b(y_n + a\,\Delta n + b\Delta w_n)]\Delta w_{n_1} + \cdots \ldots + \frac{h}{2}[k(y_n) + k(y_n + a\,\Delta n + b\Delta w_n)]\Delta w_{n_2}$ ...(10)

for $n = 0, 1, \ldots, N-1$; with initial value $y_0 = x_0$.

Equation (10) is the Itô-Taylor expansion of $x_t(\omega)$ in equation (3). The Itô-Taylor expansion is useful in approximating a sufficiently smooth function in a neighborhood of a given point to a desired order of accuracy. Thus, considering the first three terms of equation (10) provides the Heun's scheme in (10) where each term in the right hand side of equation (10) approximates the corresponding term on the right hand side of equation (3). For brevity, equation (10) is written as:

$Y_{n+1} = Y_n + a\Delta n + b\Delta w_{n_1} + k\Delta w_{n_2}$

where

$p_{i+1} = y_n + a\,\Delta n + b\Delta w_n + k\Delta w_s$ And $\Delta n = t - t_0 = \int_{t_0}^{t} ds$, $\Delta w_n = x_{t_{n+1}} - x_{t_n} = \int_{t_0}^{t} dw_s$, $a = a(\tau_n, y(\tau_n))$, $b = b(\tau_n, y(\tau_n))$, and $y_n = y(\tau_n)$.

III.Illustration.(WithAbsolute Error Test):

The stochastic differential equations Considered by:

with initial condition

$$\begin{cases} dx = fxdt + g_1 x dw_1 + g_2 x\, dw_2 \\ x(0) = 1 \end{cases}$$

The unique solution has the form

$x(t) = e^{\int_0^t f ds - \frac{1}{2}({g_1}^2 + {g_2}^2)ds + \int_0^t (g_1 dw_1 + g_2 dw_2)}$, for $0 \le t \le 1$.

Where $f(t) = \int_0^t \cos(\int_0^s \sin s\, ds)dx$; $g_1(t) = \int_0^t \sin(\int_0^s \tan s\, ds)dx$; $g_2(t) = \exp t$; $X_0 = 1$ and $Y_0 = 0$;

The absolute error at the final time interval for different sample space numbers, where $\Delta t = \delta t$; $R = 1$; the step time

for discretization of Brownian motion equals to the step time of Euler scheme, are shown in the following (table (A) and Figure (1.1)). As one can see, increasing the number of sample (N) leads to improving the absolute error at the different time steps, where $\Delta t = \delta_t$.

Table (A) Error generated by the Heun's scheme.

R	N	Error at final time
1	2^7	1.2957
	2^8	0.6259
	2^9	1.0524
2	2^7	1.3076
	2^8	0.6949
	2^9	1.0555

On using R = 1, N = 2^8, the following numerical solution is obtained and presented in the following figure (1.1)

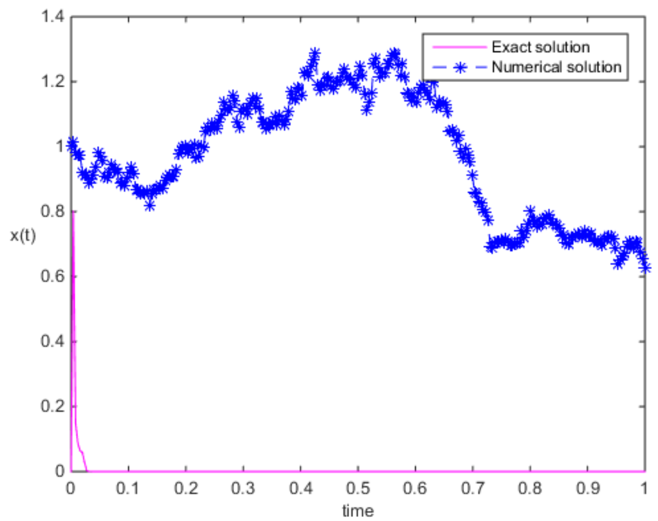

Figure (1) Exact solution and the numerical solution by Heun's scheme with N = 2^8; R = 1

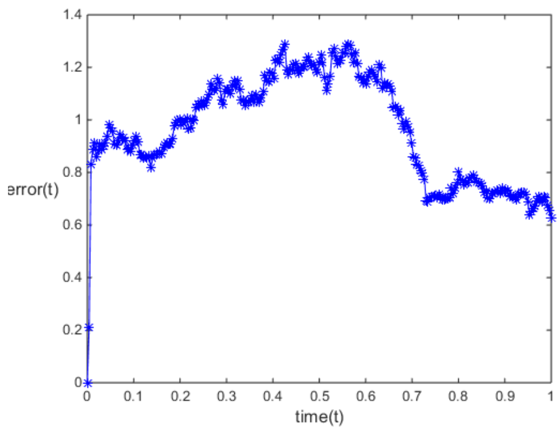

Figure (1.1) Absolute error between the Heun's scheme and exact

IV.Illustration. (With Absolute Error Test):

The stochastic differential equations Considered by:

with initial condition $\begin{cases} dx = fxdt + gxdw_1 + kxdw_2 \\ \quad x(0) = 1 \end{cases}$

The unique solution has the form

$x(t) = e^{\int_0^t f ds - \frac{1}{2}(g_1{}^2 + g_2{}^2) ds + \int_0^t (g_1 dw_1 + g_2 dw_2)}$, for $0 \le t \le 1$.

Where $f(t) = \int_0^t \sec(\int_0^s \sin s\, ds) dx$; $g(t) = \int_0^t \csc(\int_0^s \cos s\, ds) dx$; $k(t) = \tan t$;$X_0 = 0$ and $Y_0 = 1$;

As discussed previously in illustration (II), the following table (B) is needed for error analysis and as follows

Table (B) Error generated by the Heun's scheme.

R	N	Error at final time
1	2^5	0.7114
	2^6	1.0874
	2^8	0.7075
	2^9	0.9826
	2^{10}	0.9775
	2^{11}	0.8826

On using R = 1, N = 2^8, the following numerical solution is obtained and presented in the following figure (2.1)

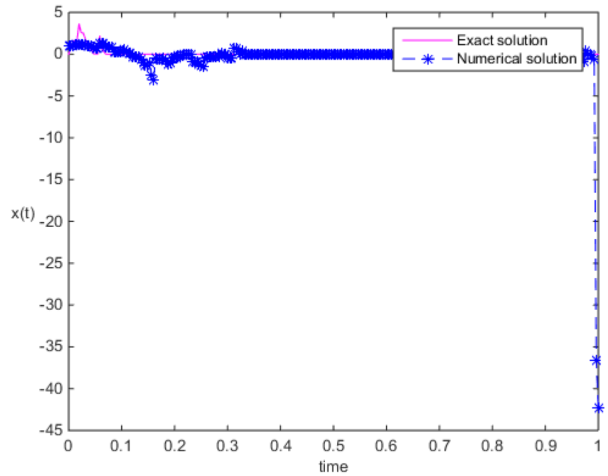

Figure (2) Exact solution and the numerical solution by Heun's scheme with N = 2^8; R = 1

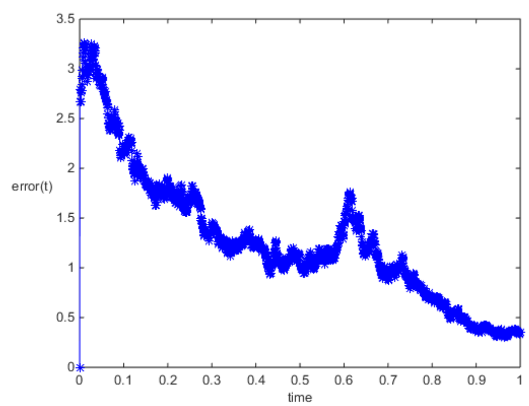

Figure (2.1) Absolute error between the Heun's scheme and exact

V.Illustration. (With Absolute Error Test):

Consider the SDE is:

$dx = fxdt + gxdw_1 + kxdw_2$

$x(0) = 0$

Where $f(t) = \int_0^t \exp(\int_0^s \sin s\, ds)dx$; $g(t) = \int_0^t \sin(\int_0^s \cos s\, ds)dx$; $k(t) = \tan t$; $X_0 = 0$; $Y_0 = 1$.

The error at final time interval for R = 1 and different number of sample N is discussed in the following table (C)

Table (C) Error generated by the Heun's scheme

R	N	Error at final time
1	2^5	0.4138
	2^8	0.1032
	2^{11}	0.1297

One can select R =1, N = 2^8 for accuracy, the following numerical solution is then obtained and presented in the following figure (3).

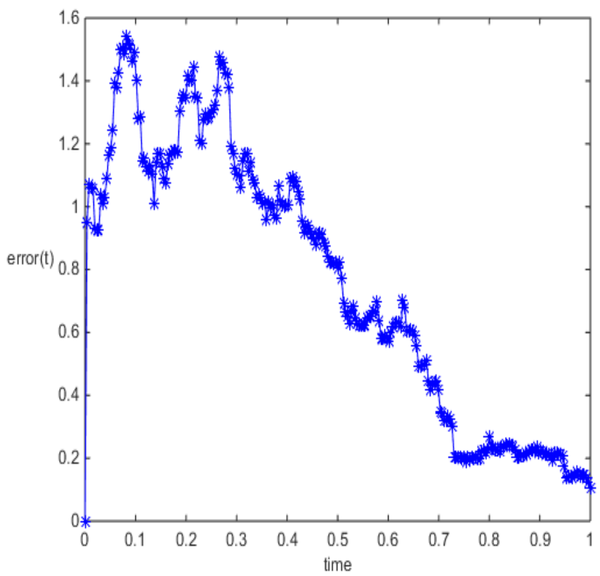

Figure (3) Absolute error between the Heun's scheme and exact.

Summary.

Numerical methods for the solution of stochastic differential equations in2-wiener process are essential for the analysis of random phenomena. Strong solvers are necessary when exploring characteristics of systems that depend on trajectory-level properties. Several approaches exist for strong solvers, in particular Heun's type methods , although both increase greatly in complication for orders greater than one. In many _ financial applications, major emphasis is placed on the probability distribution of solutions, and in particular mean and variance of the distribution. In such cases, weak solvers may sauce. Independent of the choice of stochastic differential equation in 2-wiener process solver, methods of variance reduction exist that may increase computational efficiency. The replacement of pseudorandom numbers with quasi random analogues from low-discrepancy sequences is applicable as long as statistical independence along trajectories is maintained. In addition, control variates offer an alternate means of variance reduction and increases inefficiency simulation of stochastic differential equations trajectories.

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