3D Surface Reconstruction of Mathematical modelling Used for Controlling the Generation of Different Bi-cubic B-Spline in Matrix Form without Changing the Control Points

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ABSTRACT

This paper introduces a 3D surface reconstruction of mathematical modelling by using blossoming, dependent on a parameter in the coefficient of the control points, the bi-cubic B-spline surface in matrix form scheme can be utilized to obtain a better quality of reconstructed surface, by using different values of parameter of coefficients used for controlling generating 3D complex surface, which is not easy to recreate. The de-Boor method is used to upgrade the matrix form of 2D to bicubic 3D- b-spline surface depends on parameter value. The model can be seen more efficient of surface in comparison with that needed in conventional methods. The effectiveness of the proposed algorithm is illustrated through several comparative examples of 3D matrices surface. The change in surface is made without any change in the control points. Applications of the modeling in both the curve and surface can be used in many fields, such as banknote design, shape design, decorations, governmental document, and other documents, all of which are of high impotence. And it is used to find the appropriate solutions that prevent or reduce the forgery and counterfeiting of the important and vital documents

اعادة بناء سطح ثلاثي الابعاد باستخدام نموذج رياضي للسيطرة على توليد نماذج مختلفة من الدرجة الثالثة بصيغة المصفوفات من دون تغيير نقاط السيطرة (B-Spline)

الخلاصة:

هذا البحث ينتج سطوح ثلاثية الأبعاد من اعادة بناء نموذج رياضي باستخدام صيغة الاستقطاب معتمدا المتغيرات في معاملات نقاط السيطرة، لـ(bi-b-Spline) من الدرجة الثالثة بصيغة المصفوفة بالإمكان استخدام قيم مختلفة لمتغير المعاملات تستخدم للسيطرة على توليد سطوح معقد من الدرجة الثالثة ثلاثي الأبعادوالذي ليس من السهل تقليده. تم استخدام اسلوب(De-Boor) لتغيير السطح من ثنائي الأبعاد الى ثلاثي الأبعاد بألأعتماد على المتغيرات من خلال تغيير درجة المصفوفة. النماذج ولأساليب التي توصلنا اليها ملموسة الكفاءة عند التأثيرات في السطح اكثر من الطرق التقليدية. كفاءة النموذج وضحت من خلات مجموعة من التطبيقات على المصفوفات والتي وضحت بالأمثلة والشواهد الفاعلة للوصول الى حالة الرسم ثلاثي الأبعاد، التاثيرات على السطح تجري من دون تغيير نقاط السيطرة للسطح. يمكن الاستفادة من استخدام كلا النموذجين المنحنيات والسطوح في كثير من التطبيقات

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كتصميم العملة النقدية، شكل التصميم، الزخرفة، السجلات الحكومية، وغيرها من الوثائق التي تكون لها اهمية وايجاد الحلول المناسبة التي تمنع او تقلل من تزوير الوثائق المهمة والحيوية.

INTRODUCTION

3D surface reconstruction from mathematical modelling is used in the field of computer graphic. Parameters (t, s) coefficient of the control points are used widely to express computer 3D modelling. It can be used for controlling the generating of different modelling 3D complex surfaces, by using blossoming [1], [2], [3]. And, it can find the appropriate solutions that prevent or reduce the forgery and counterfeiting of the important and vital documents. Mathematical technique is used to generate blossoming of cubic B-spline surfaces by using de-Boor algorithm in matrix form. (Gallier [1] is used arithmetical technique to generate B-spline curve by using de Boor algorithm in blossoming). Recently, the authors [1]-[3] developed the cubic B-spline scheme, by using the blossoming. The surface involved contains two variables, natural way to polarize polynomial surface. The approach yields surface {also called tensor product two matrices} [4]. Finally, our work gives mathematical technique dependent on the method of blossoming that is used to find bi-cubic b-spline surfaces

Coefficient of (16) control points. One can see many case studies that made the designer control the generation of the different modeling surface. The result of this work is organized as follows: Section 2 defines the cubic B-spline curve and Bicubic B-spline surface in matrix form. Section 3 introduces surface in the blossoming in both the curve and surface in matrix form. Section 4 introduces blossoming method for generating cubic b-spline curves in matrix form. Section 5 introduces blossoming modified modelling for generating Bicubic spline surface in linear form. This paper provides several comparative examples of controlling the generating of a bicubic B-spline surface by using difference cases dependent on parameters t_i and s_j (for i, j= 1,2 and3), and finally the contributions of this paper are summarized in Section 6, and 7.

Backgrounds

De-Boor algorithm is an algorithm that uses a sequence of four control points to construct a well-defined curve *F(t)* at each value of *t* from 0 to 1. This provides a way to generate a curve from a set of points. Changes in the points will change the curve. *F (t)* defined as: [5], [6].

$$F(t)=\frac{1}{6}(1-t)^3 p_0+\frac{1}{6}\{3t^3-6t^2+4\}p_1+\frac{1}{6}\{-3t^3+3t^2+3t+1\}p_2+\frac{1}{6}t^3 p_3$$

Putting this equation in matrix form yields:-

$$F(t)=\frac{1}{6}\mathrm{TMP} \qquad \ldots (1)$$

Where

$$T=\begin{pmatrix}t^3 & t^2 & t & 1\end{pmatrix},\quad M=\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 0 & 3 & 0\\ 1 & 4 & 1 & 0\end{bmatrix},\quad P=\begin{bmatrix}P_0\\ P_1\\ P_2\\ P_3\end{bmatrix}$$

Eq. (1) is the original (classical) cubic B-spline curve that is dependent on interval in *[0, 1]* in matrix form,. Now a sequence of 16 control points are used to define surfaces (Mathematically said to be generated from the Cartesian product of two matrices) bi-cubic B-spline surface is defined as: [7], [8].

$$\boldsymbol{F(t,s)} = \frac{1}{36}\mathrm{TMP_{ij}M^{T}S^{T}} \qquad \ldots (2)$$

Where M^T is the transpose of matrix M.

$$P_{ij} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix},\ S = \begin{pmatrix} s^3 & s^2 & s & 1 \end{pmatrix},\quad \text{where } t \in [0, 1],\ s \in [0, 1],$$

Eq. (2) is called the original (classical) Bi-cubic B-Spline surfaces in matrix form.

Bi-polynomial Surface in Blossoming

A method of specifying polynomial construction of the curves is based on blossoming form. We are now ready to see the actual blossom associated with degree -3 polynomial, the blossom of $F(t)$ is a function of $f(t_1, t_2, t_3)$ that satisfies the following: [1], [10].

1-f is linear in each variable t_i, for $i=1, 2, 3$.

2-f is symmetric, order of variables is irrelevant. Thus $f(t_1, t_2, t_3)=f(t_2, t_1, t_3)=f(t_1, t_3, t_2)=f(t_2, t_3, t_1)=f(t_3, t_1, t_2)=f(t_3, t_2, t_1)$.

3-The diagonal $f(t, t, t)$ of f = $F(t)$. Requirement 1 suggests the name "multilinear function." Given $F(t)$, such a multiafine function is easy to derive and is also unique. Here are the properties of blossoming (polar form) to cubic polynomial:

Degree: 0 =f = a_0

Degree: 1 = $f(t_1) = a_0 + a_1 t_1$.

Degree: 2 = $f(t_1, t_2) = a_0 + a_1 \dfrac{t_1 + t_2}{2} + a_2 t_1 t_2$

Degree: 3 = $f(t_1,t_2,t_3) = a_0 + a_1 \dfrac{t_1 + t_2 + t_3}{3} + a_2 \dfrac{t_1 t_2 + t_1 t_3 + t_2 t_3}{3} + a_3 t_1,t_2,t_3$

It is easily generalized to be bi-parametric in a bipolynomial surface [1] [2]. A point on the surface is given by bi-parametric function and a set of blending or bas is the function used for each parameter. Now the study investigates the possibility of defining bi-polynomial surface in terms of polar forms. Note that they are intentionally denoting (t×t×t×s×s×s), instead of $t^3 \times s^3$, to avoid the confusion between the affine space A^3 and the Cartesian product A×A×A. Linearity is used to explain how to polarize a monomial $F(t, s)$ of the form $t^3 s^3$ with respect to the bidegree $\langle 3, 3 \rangle$. In order to find the polar form $f(t, t, t; s, s, s)$ of F viewed as a bi polynomial surface of degree, $\langle 3, 3 \rangle$ polarizes each of the $F(t,s)$ separately in t and s, it is quite obvious that the same result is obtained if you first polarize with respect to t, and then with respect to s, or conversely.

Method for Generating Cubic Spline Curves

To find the spline in Blossoming, one can find control points from f($u_{k+1}, \ldots, u_{k+m}$), where u_i is real numbers, k integer number taken from sequence of 2m knots $\{u_1, u_2 \ldots u_{2m}\}$, of such length that satisfies certain inequality conditions[1], [10], [11]. Using de-Boor algorithm points to calculate the cubic curve $f(t, t, t)$, degree three (m=3), can be used to find 4 control points in the form $f(u_{k-2+i}, u_{k-1+i}, u_{k+i})$, $0 \le i \le 3$. Taken from the sequence $\{u_{k-2+i}, u_{k-1+i}, u_{k+i}, u_{k+1+i}, u_{k+2+i}\ u_{k+3+i}\}$, it is said to be progressive if it yields 4 sequences of consecutive knots ($u_{k-2+i}, u_{k-1+i}, u_{k+i}$), each of length 3 where $0 \le i \le 3$, and these sequences turn out to define 4 de-Boor control points for the curve segment f_k associated with the middle interval $[u_k, u_{k+1}]$. f_k is the polar form of segment F_k. Then the de Boor control points are obtained from the given sequence $\{u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3}\}$, one can say the sequence is progressive iff the inequalities in the following array hold:

$$\begin{array}{llll} u_{k-2} & \neq & & \\ u_{k-1} & \neq & \neq & \\ u_k & \neq & \neq & \neq \\ & u_{k+1} & u_{k+2} & u_{k+3} \end{array}$$

This is explained as follows:-

At stage 1 $u_{k-2} \neq u_{k+1}$, $u_{k-1} \neq u_{k+2}$, and $u_k \neq u_{k+3}$, this corresponds to the inequalities on main descending diagonal of the array of inequality conditions.

At stage 2 $u_{k-1} \neq u_{k+1}$, $u_k \neq u_{k+2}$, this corresponds to the inequalities on second descending diagonal of the array of inequality conditions. At stage 3 $u_k \neq u_{k+1}$, this corresponds to third-lowest descending diagonal of the array of inequality conditions. The new de Boor algorithm for calculates f (t_1,t_2,t_3) at each value of t is $u_k \leq t \leq u_{k+1}$. See Figs.1, and 2. From stage 3, $f(t_1,t_2,t_3)$ is given by : [1], [10].

$$f(t_1,t_2,t_3)= (1-\lambda_6)f(t_1,t_2,u_k)+\lambda_6 f(t_1,t_2, u_{k+1}) \quad \ldots(3)$$

From stage 2, $f(t_1, t_2, u_k)$ and $f(t_1, t_2, u_{k+1})$ are given by:

$$f(t_1,t_2, u_k)=(1-\lambda_4)f(t_1, u_{k-1}, u_k) +\lambda_4 f(t_1, u_k, u_{k+1}), \quad \ldots(4)$$

$$f(t_1,t_2,u_{k+1})= (1-\lambda_5)f(t_1, u_k, u_{k+1})+ \lambda_5 f(t_1, u_{k+1}, u_{k+2}). \quad \ldots(5)$$

From stage 1, $f(t_1, u_{k-1},u_k)$, $f(t_1, u_k.u_{k+1})$, and $f(t_1, u_{k+1},u_{k+2})$ are given by:

$$f(t_1, u_{k-1}, u_k)= (1-\lambda_1) f(u_{k-2}, u_{k-1}, u_k) + \lambda_1 f(u_{k-1}, u_k, u_{k+1}), \quad \ldots(6)$$

$$f(t_1, u_k,u_{k+1}) = (1-\lambda_2)f(u_{k-1}, u_k, u_{k+1}) +\lambda_2 f(u_k, u_{k+1}, u_{k+2}) \quad \ldots(7)$$

$$f(t_1, u_{k+1},u_{k+2}) =(1-\lambda_3) f(u_k, u_{k+1}, u_{k+2})+\lambda_3 f(u_{k+1}, u_{k+2}, u_{k+3}). \quad \ldots(8)$$

Substitution of equations {4, 5, 6, 7, and 8} in (3) gives

$$\begin{aligned} F_k=f_k(t_1,t_2,t_3)=&(1-\lambda_6)(1-\lambda_4)(1-\lambda_1)[f(u_{k-2}, u_{k-1}, u_k)]+\{(1-\lambda_4)(1-\lambda_6)\lambda_1+(1-\lambda_2)(1-\lambda_6)\lambda_4+(1-\lambda_2)(1-\lambda_5)\lambda_6\} \\ &f(u_{k-1}, u_k, u_{k+1}) +\{\lambda_2\lambda_4(1-\lambda_6)+\lambda_6(1-\lambda_5)\lambda_2 \\ &+(1\lambda_3)\lambda_5\lambda_6\}f(u_k,u_{k+1}, u_{k+2}) \\ &+ \lambda_3\lambda_5\lambda_6 f(u_{k+1}, u_{k+2}, u_{k+3}). \end{aligned} \quad \ldots(9)$$

Suppose $f(u_{k-2}, u_{k-1}, u_k)]$, $f(u_{k-1}, u_k, u_{k+1})$, $f(u_k,u_{k+1}, u_{k+2})$, $f(u_k,u_{k+1}, u_{k+2})= p_0,p_1 ,p_2 ,p_3$ are control points and Eq (9) becomes. [1], [10].

$$f_k(t_1,t_2,t_3)=(1-\lambda_6)(1-\lambda_4)(1-\lambda_1)p_0+\{(1-\lambda_4)(1-\lambda_6)\lambda_1+(1-\lambda_2)(1-\lambda_6)\lambda_4+(1-\lambda_2)(1-\lambda_5)\lambda_6\}p_1+\{\lambda_2\lambda_4(1-\lambda_6)+\lambda_6(1-\lambda_5)\lambda_2+(1-\lambda_3)\lambda_5\lambda_6\}p_2+\lambda_3\lambda_5\lambda_6 p_3. \quad \ldots(10)$$

Eq (10) is formula of a modified cubic spline curve.

Method for Generating Cubic Spline Curves in Matrices Form

It is easy to show that See Figs.1 and 2.

$$\lambda_1=\frac{t_1-u_{k-2}}{u_{k+1}-u_{k-2}},\ \lambda_2=\frac{t_1-u_{k-1}}{u_{k+2}-u_{k-1}}\ \ \lambda_3=\frac{t_1-u_k}{u_{k+3}-u_k},$$

$$\lambda_4=\frac{t_2-u_{k-1}}{u_{k+1}-u_{k-1}},\ \lambda_5=\frac{t_2-u_k}{u_{k+2}-u_k},\ \lambda_6=\frac{t_3-u_k}{u_{k+1}-u_k}.$$

By substituting λ_i (for i= 1 to 6), in Eq (10) from the first point you can see

$$(1-\lambda_6)(1-\lambda_4)(1-\lambda_1)\,P_0=\left\{\frac{(u_{k+1}-t_1)(u_{k+1}-t_2)(u_{k+1}-t_3)}{(u_{k+1}-u_{k-2})(u_{k+1}-u_{k-1})(u_{k+1}-u_{k-2})}\right\}P_0$$

$$\left\{\frac{-t_1t_2t_3+u_{k+1}(t_1t_2+t_1t_3+t_2t_3)\ -u^2{}_{k+1}(t_1+t_2+t_3)+u^3{}_{k+1}}{(u_{k+1}-u_{k-2})(u_{k+1}-u_{k-1})(u_{k+1}-u_k)}\right\}P_0$$

$$=\frac{1}{u_{k+1}-u_k}\left\{\frac{-t_1t_2t_3+u_{k+1}(t_1t_2+t_1t_3+t_2t_3)\ -u^2{}_{k+1}(t_1+t_2+t_3)+u^3{}_{k+1}}{(u_{k+1}-u_{k-2})(u_{k+1}-u_{k-1})}\right\}P_0.$$

In the same way find other points as follows:-

$$f_k(t_1,t_2,t_3)=\frac{1}{u_{k+1}-u_k}\{[\frac{-t_1t_2t_3+u_{k+1}(t_1t_2+t_1t_3+t_2t_3)\ -u^2{}_{k+1}(t_1+t_2+t_3)+u^3{}_{k+1}}{(u_{k+1}-u_{k-2})(u_{k+1}-u_{k-1})}]P_0$$

$$+[\frac{t_1t_2t_3-t_1t_2u_{k+1}-t_1t_3u_{k+1}-t_2t_3u_{k-2}+t_1u^2{}_{k+1}+t_2u_{k-2}u_{k+1}+t_3u_{k-2}u_{k+1}-u_{k-2}u^2{}_{k+1}}{(u_{k+1}-u_{k-2})(u_{k+1}-u_{k-1})}]$$

$$+[\frac{t_1t_2t_3-t_1t_2u_{k+1}-t_1t_3u_{k-1}-t_2t_3u_{k+2}+t_1u_{k+1}u_{k-1}+t_2u_{k+1}u_{k+2}+t_3u_{k-1}u_{k+2})-u_{k-1}u_{k+1}u_{k+2}}{(u_{k+1}-u_{k-1})(u_{k+2}-u_{k-1})}]$$

$$+[\frac{t_1t_2t_3-t_1t_2u_k-t_1t_3u_{k+2}-t_2t_3u_{k+2}+t_1u_ku_{k+2}+t_2u_ku_{k+2}+t_3u^2{}_{k+2})-u_ku^2{}_{k+2}}{(u_{k+2}-u_{k-1})(u_{k+2}-u_k)}]P_1$$

$$+[\frac{-t_1t_2t_3+t_1t_2u_{k+1}+t_1t_3u_{k-1}+t_2t_3u_{k-1}-t_1u_{k-1}u_{k+1}-t_2u_{k-1}u_{k+1}-t_3u^2{}_{k-1}+u^2{}_{k-1}u_{k+1}}{(u_{k+2}-u_{k-1})(u_{k+1}-u_{k-1})}]$$

$$[\frac{-t_1t_2t_3+t_1t_2u_k+t_1t_3u_{k+2}+t_2t_3u_{k-1}-t_1u_ku_{k+2}-t_2u_{k-1}u_k-t_3u_{k-1}u_{k+2}+u_{k+2}u_{k-1}u_k}{(u_{k+2}-u_k)(u_{k+2}-u_{k-1})}]$$

$$[\frac{-t_1t_2t_3+t_1t_2u_k+t_1t_3u_k+t_2t_3u_{k+3}-t_1u^2{}_k-t_2u_{k+3}u_k-t_3u_ku_{k+3}+u_{k+3}u^2{}_k}{(u_{k+2}-u_k)(u_{k+3}-u_k)}]P_2+\{$$

$$+\{[\frac{t_1t_2t_3-u_k(t_1t_2+t_1t_3+t_2t_3)\ +u^2{}_k(t_1+t_2+t_3)-u^3{}_k}{(u_{k+3}-u_k)(u_{k+2}-u_k)}]P_3\}. \qquad \text{.... (11)}$$

The above system (Eq (11)) simplified to the following

$$\boldsymbol{f_k}(t_1,t_2,t_3)=\frac{1}{u_{k+1}-u_k}\{$$

$$[-A_1t_1t_2t_3+A_1u_{k+1}(t_1t_2+t_1t_3+t_2t_3)\ -A_1u^2{}_{k+1}(t_1+t_2+t_3)+A_1u^3{}_{k+1}]P_0+$$
$$[t_1t_2t_3(A_1+A_2+A_3)+t_1t_2(-u_{k+1}A_1-u_{k+1}A_2-u_kA_3)+t_1t_3(-u_{k+1}A_1-u_{k-1}A_2-u_{k+2}A_3)$$
$$+t_2t_3(-u_{k-2}A_1-u_{k+2}A_2-u_{k+2}A_3)+t_1(u^2{}_{k+1}A_1+u_{k+1}u_{k-1}A_2+u_ku_{k+2}A_3)+t_2(u_{k-2}u_{k+1}A_1$$
$$+u_{k+1}u_{k+2}A_2+u_ku_{k+2}A_3)+t_3(u_{k-2}u_{k+1}A_1+u_{k-1}u_{k+2}A_2+u^2{}_{k+2}A_3)$$
$$-(u_{k-2}u^2{}_{k+1}A_1+u_{k-1}u_{k+1}u_{k+2}A_2+u_ku^2{}_{k+2}A_3)]P_1+$$

$$[t_1t_2t_3(-A_4-A_5-A_6)+t_1t_2(u_{k+1}A_4+u_kA_5+u_kA_6)+t_1t_3(u_{k-1}A_4+u_{k+2}A_5+u_kA_6)$$
$$+t_2t_3(u_{k-1}A_4+u_{k-1}A_5+u_{k+3}A_6)+t_1(-u_{k-1}u_{k+1}A_4-u_{k+2}u_kA_5-u^2{}_kA_6)$$
$$+t_2(-u_{k-1}u_{k+1}A_4-u_ku_{k-1}A_5-u_ku_{k+3}A_6)+t_3(-u^2{}_{k-1}A_4-u_{k-1}u_{k+2}A_5+u_ku_{k+3}A_6)$$
$$+(u_{k+1}u^2{}_{k-1}A_4+u_{k-1}u_ku_{k+2}A_5+u_{k+3}u^2{}_kA_6)]P_2+$$

$$[A_6t_1t_2t_3-A_6u_k(t_1t_2+t_1t_3+t_2t_3)\ +A_6u^2{}_k(t_1+t_2+t_3)-A_6u^3{}_k]P_3\} \quad \ldots (12)$$

Where

$$A_1=\frac{1}{(u_{k+1}-u_{k-2})(u_{k+1}-u_{k-1})} \qquad A_2=\frac{1}{(u_{k+1}-u_{k-1})(u_{k+2}-u_{k-1})}$$
$$A_3=\frac{1}{(u_{k+2}-u_{k-1})(u_{k+2}-u_k)} \qquad A_4=\frac{1}{(u_{k+2}-u_k)(u_{k+1}-u_{k-1})}$$

$$A_5=\frac{1}{(u_{k+2}-u_k)(u_{k+2}-u_{k-1})} \qquad A_6=\frac{1}{(u_{k+2}-u_k)(u_{k+3}-u_k)}$$

It can also be seen that the above system (Eq (12)) can be put in the following system of equations:-

$$B_0(u_k)=-A_1t_1t_2t_3,\ B_1(u_k)=A_1u_{k+1}(t_1t_2+t_1t_3+t_2t_3),$$
$$B_2(u_k)=-A_1u^2{}_{k+1}(t_1+t_2+t_3),\ B_3(u_k)=A_1u^3{}_{k+1}$$

$$B_4(u_k)=t_1t_2t_3(A_1+A_2+A_3),$$
$$B_5(u_k)=(-1)\{t_1t_2(A_1u_{k+1}+A_2u_{k+1}+A_3u_k)+t_2t_3(A_1u_{k+1}+A_2u_{k-1}+A_3u_{k+2})$$
$$+t_1t_3(A_1u_{k-2}+A_2u_{k+2}+A_3u_{k+2})\},$$
$$B_6(u_k)=\{t_1(A_1u^2{}_{k+1}+A_2u_{k+1}u_{k-1}+A_3u_{k+1}u_k)+t_2(A_1u_{k+1}u_{k-2}+A_2u_{k+1}u_{k+2}+A_3u_{k+2}u_k)$$
$$+t_3(A_1u_{k+1}u_{k-2}+A_2u_{k+2}u_{k-1}+A_3u^2{}_{k+2})\},$$
$$B_7(u_k)=(-1)(A_1u^2{}_{k+1}u_{k-2}+A_2u_{k+1}u_{k+2}u_{k-1}+A_3u_ku^2{}_{k+2}\},$$

$$B_8(u_k)=(-1)t_1t_2t_3(A_4+A_5+A_6),$$
$$B_9(u_k)=\{t_1t_2(A_4u_{k+1}+A_5u_k+A_6u_k)+t_2t_3(A_4u_{k-1}+A_5u_{k+2}+A_6u_k)$$
$$+t_2t_3(A_4u_{k-1}+A_5u_{k-1}+A_6u_{k+3})$$

$$B_{10}(u_k)=(-1)\{t_1(A_4u_{k-1}u_{k+1}+A_5u_{k+2}u_k+A_6u^2{}_k)+t_2(A_4u_{k+1}u_{k-1}+A_5u_ku_{k-1}+A_6u_{k+3}u_k)$$
$$+t_3(A_4u^2{}_{k-1}+A_5u_{k+2}u_{k-1}+A_6u_ku_{k+3})\},$$

$$B_{11}(u_k)=+A_4u^2{}_{k-1}u_{k+1}+A_5u_{k+2}u_ku_{k-1}+A_6u^2{}_ku_{k+3}$$

$$B_{12}(u_k) = A_6 t_1 t_2 t_3, \quad B_{13}(u_k) = (-1)A_6 u_k (t_1 t_2 + t_1 t_3 + t_2 t_3)$$
$$B_{14}(u_k) = A_6 u^2 (t_1 + t_2 + t_3), \quad B_{15}(u_k) = (-1)(A_6 u^3{}_k).$$

Now simplify the above equations in matrices form as follows:-

$$B_0(u_k) = (-1)A_1 \begin{pmatrix} t_1 t_2 t_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, B_1(u_k) = A_1 \begin{pmatrix} t_1 t_2 & t_1 t_3 & t_2 t_3 \end{pmatrix} u_{k+1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$B_2(u_k) = (-1)A_1 \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} u^2{}_{k+1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, B_3(u_k) = A_1 \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} u^3{}_{k+1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$B_4(u_k) = \begin{pmatrix} t_1 t_2 t_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 + A_2 + A_3 \\ 0 \\ 0 \end{pmatrix},$$

$$B_5(u_k) = (-1)\begin{pmatrix} t_1 t_2 & t_1 t_3 & t_2 t_3 \end{pmatrix} \begin{pmatrix} A_1 u_{k+1} + A_2 u_{k+1} + A_3 u_k \\ A_1 u_{k+1} + A_2 u_{k-1} + A_3 u_{k+2} \\ A_1 u_{k-2} + A_2 u_{k+2} + A_3 u_{k+2} \end{pmatrix},$$

$$B_6(u_k) = \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} A_1 u^2{}_{k+1} + A_2 u_{k+1} u_{k-1} + A_3 u_{k+1} u_k \\ A_1 u_{k+1} u_{k-2} + A_2 u_{k+1} u_{k+2} + A_3 u_{k+2} u_k \\ A_1 u_{k+1} u_{k-2} + A_2 u_{k+2} u_{k-1} + A_3 u^2{}_{k+2} \end{pmatrix},$$

$$B_7(u_k) = (-1)\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_1 u^2{}_{k+1} u_{k-2} \\ A_2 u_{k+2} u_{k+1} u_{k-1} \\ A_3 u^2{}_{k+2} u_k \end{pmatrix}$$

$$B_8(u_k) = (-1)\begin{pmatrix} t_1 t_2 t_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_4 + A_5 + A_6 \\ 0 \\ 0 \end{pmatrix},$$

$$B_9(u_k) = \begin{pmatrix} t_1 t_2 & t_1 t_3 & t_2 t_3 \end{pmatrix} \begin{pmatrix} + A_4 u_{k+1} + A_5 u_k + A_6 u_k \\ A_4 u_{k-1} + A_5 u_{k+2} + A_6 u_k \\ A_4 u_{k-1} + A_5 u_{k-1} + A_6 u_{k+3} \end{pmatrix}$$

$$B_{10}(u_k) = (-1)\begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} A_4 u_{k-1} u_{k+1} + A_5 u_{k+2} u_k + A_6 u^2{}_k \\ A_4 u_{k+1} u_{k-1} + A_5 u_k u_{k-1} + A_6 u_{k+3} u_k \\ A_4 u^2{}_{k-1} + A_5 u_{k+2} u_{k-1} + A_6 u_k u_{k+3} \end{pmatrix}$$

$$B_{11}(u_k) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} + A_4 u^2{}_{k-1} u_{k+1} + A_5 u_{k+2} u_k u_{k-1} + A_6 u^2{}_k u_{k+3} \\ 0 \\ 0 \end{pmatrix}$$

$$B_{12}(u_k)=\begin{pmatrix}t_1t_2t_3 & 0 & 0\end{pmatrix}A_6\begin{pmatrix}1\\0\\0\end{pmatrix},\ B_{13}(u_k)=(-1)(A_6u_k)\begin{pmatrix}t_1t_2 & t_1t_3 & t_2t_3\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$B_{14}(u_k)=\begin{pmatrix}t_1 & t_2 & t_3\end{pmatrix}(A_6u^2{}_k)\begin{pmatrix}1\\1\\1\end{pmatrix},\ B_{15}(u_k)=(-1)(A_6u^3{}_k)\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Now put Eq (12) in the following simple form:-

$$\boldsymbol{f_k}(t_1,t_2,t_3)=\frac{1}{u_{k+1}-u_k}\{\,[B_0+B_1+B_2+B_3]\,p_0+[B_4+B_5+B_6+B_7]\,p_1$$
$$+[B_8+B_9+B_{10}+B_{11}]\}\,p_2\ +[B_{12}+B_{13}+B_{14}+B_{15}]\,p_3. \qquad \dots(13)$$

Eq (13) can be put in the following matrices form:-

$$f_k(t_1,t_2,t_3)=\frac{1}{u_{k+1}-u_k}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\begin{bmatrix}B_0(u_k) & B_4(u_k) & B_8(u_k) & B_{12}(u_k)\\ B_1(u_k) & B_5(u_k) & B_9(u_k) & B_{13}(u_k)\\ B_2(u_k) & B_6(u_k) & B_{10}(u_k) & B_{14}(u_k)\\ B_3(u_k) & B_7(u_k) & B_{11}(u_k) & B_{15}(u_k)\end{bmatrix}\begin{bmatrix}P_0\\P_1\\P_2\\P_3\end{bmatrix}, \qquad \dots(14)$$

It is an easy method used to upgrade of above equation to 2D B-spline curve by using the set of points as, [12].

$p_i=(x_i,\ y_i)$ for $i=0,\ 1,\ 2,\ 3.$

The coordinates of each point are treated as a two-component vector. That is

$$P_i=\begin{pmatrix}x_i\\ \\ y_i\end{pmatrix}.$$

The set of points, in parametric form is

$$\boldsymbol{F}\,(t)=\begin{pmatrix}F_1(t)\\ \\ F_2(t)\end{pmatrix}=\begin{pmatrix}x(t)\\ \\ y(t)\end{pmatrix}, \qquad u_k\le t\le u_{k+1} \qquad \dots(15)$$

Eq (14) is a formula of a modified mathematical modelling 2D B-spline curve in matrices form of degree three .The curve is defined by de-Boor algorithm for the curve segment f$(t_1,\ t_2,\ t_3)$ associated with the middle intervals $[u_k,\ u_{k+1}]$.

Comparison with Original Cubic B- Spline in Matrices Form.

Now a comparison between Eq(14), and Eq(1) can easily be made by using $t_1=t_2=t_3=t$ and $k=0$, the sequence $\{u_{k-2},\ u_{k-1}\ ,\ u_k,\ u_{k+1},\ u_{k+2},\ u_{k+3}\}$becomes $(u_{-2},\ u_{-1}\ u_0,\ u_1\ u_2,\ u_3)$, and using uniform knot sequence. $(u_{-2},\ u_{-1}\ u_0,\ u_1\ u_2,\ u_3)=(-2,\ -1,\ 0,\ 1,\ 2,\ 3)$, implies $u_k=u_0=0$, $u_{k+1}=u_1=1$, and $t\in[0,\ 1]$, $(0\le t\le 1)$, Eq. (14) becomes:

$$\text{F (t)} = \frac{1}{1}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\begin{bmatrix}\frac{-t^3}{6} & \frac{3t^3}{6} & \frac{-3t^3}{6} & \frac{t^3}{6} \\ \frac{3t^2}{6} & \frac{-6t^2}{6} & \frac{3t^2}{6} & 0 \\ \frac{-3t}{6} & 0 & \frac{3t}{6} & 0 \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0\end{bmatrix}\begin{bmatrix}P_0 \\ P_1 \\ P_2 \\ P_3\end{bmatrix}, \text{ or}$$

$$\text{F (t)} = \begin{pmatrix}t^3 & t^2 & t & 1\end{pmatrix}\frac{1}{6}\begin{bmatrix}-1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0\end{bmatrix}\begin{bmatrix}P_0 \\ P_1 \\ P_2 \\ P_3\end{bmatrix}. \qquad \ldots (16)$$

Or in simple form

$$\text{F (t)} = \frac{1}{6}\text{TMP}_i \qquad \ldots (16)$$

Eq (16) is identical to Eq (1).

Bi-cubic Spline in Matrices Form

Consider points in the form $f(u_{k-2+i}, u_{k-1+i}, u_{k+i}; v_{l-2+j}, v_{l-1+j}, v_{l+j})$, $(0 \le i \le 3)(0 \le j \le 3)$, k and l are integers, suppose the bidegree is $\langle 3, 3 \rangle$ where u_i and v_j are real numbers taken from the knot sequence, $\{u_{k-2+i}, u_{k-1+i}, u_{k+i}, u_{k+1+i}, u_{k+2+i}\ u_{k+3+i}; v_{l-2+j}, v_{l-1+j}, v_{l+j}, v_{l+1+j}, v_{l+2+j}\ v_{l+3+j}\}$, of length (4m= 12) satisfying certain inequality conditions. The same natural way can be used to polarize polynomial surfaces to find second variable s [1], [2], [3]. Now the de-Boor algorithm can be discussed in some detail in bipolynomial surfaces. De-Boor algorithm can be generalized very easily to bipolynomial surfaces by using the following inequalities indicated in the following array:

$$\begin{array}{llll} v_{l-2+j} & \neq & & \\ v_{l-1+j} & \neq & \neq & \\ v_{l+j} & \neq & \neq & \neq \\ & v_{l+1+j} & v_{l+2+j} & v_{l+3+j} \end{array}$$

That is explained as follows:-

At stage 1 $v_{l-2+j} \neq v_{l+1+j}$, $v_{l-1+j} \neq v_{l+2+j}$ and $v_{l+j} \neq v_{l+3+j}$.

This corresponds to the inequalities on main descending diagonal of the array of inequality conditions. At stage 2 $v_{l-1+j} \neq v_{l+1+j}$, $v_{l+j} \neq v_{l+2+j}$. This corresponds to the inequalities on second descending diagonal of the array of inequality conditions. At stage 3 $v_{l+j} \neq v_{l+1+j}$. This corresponds to third lowest descending diagonal of the array of inequality conditions. The same way is used to find second parameters $f(s_1, s_2, s_3)$ in matrices form. It is easy to show that:

$$E_1 = \frac{1}{(v_{l+1} - v_{l-2})(v_{l+1} - v_{l-1})} \quad E_2 = \frac{1}{(v_{l+1} - v_{l-1})(v_{l+2} - v_{l-1})}$$

$$E_3 = \frac{1}{(v_{l+2} - v_{l-1})(v_{l+2} - v_l)} \quad E_4 = \frac{1}{(v_{l+2} - v_l)(v_{l+1} - v_{l-1})}$$

$$E_5 = \frac{1}{(v_{l+2} - v_l)(v_{l+2} - v_{l-1})} \quad E_6 = \frac{1}{(v_{l+2} - v_l)(v_{l+3} - v_l)}$$

$$Z_0(v_l) = (-1)E_1\begin{pmatrix} s_1s_2s_3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, Z_1(u_k) = E_1\begin{pmatrix} s_1s_2 & s_1s_3 & s_2s_3 \end{pmatrix} v_{l+1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$Z_2(v_l) = (-1)E_1\begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix} v^2{}_{l+1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, Z_3(v_k) = E_1\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} v^3{}_{l+1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$Z_4(v_k) = \begin{pmatrix} s_1s_2s_3 & 0 & 0 \end{pmatrix}\begin{pmatrix} E_1 + E_2 + E_3 \\ 0 \\ 0 \end{pmatrix},$$

$$Z_5(v_k) = (-1)\begin{pmatrix} s_1s_2 & s_1s_3 & s_2s_3 \end{pmatrix}\begin{pmatrix} E_1v_{l+1} + E_2v_{l+1} + E_3v_l \\ E_1v_{l+1} + E_2v_{l-1} + E_3v_{l+2} \\ E_1v_{l-2} + E_2v_{l+2} + E_3v_{l+2} \end{pmatrix},$$

$$Z_6(v_k) = \begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix}\begin{pmatrix} E_1v^2{}_{l+1} + E_2v_{l+1}v_{l-1} + E_3v_{l+1}v_l \\ E_1v_{l+1}v_{l-2} + E_2v_{l+1}v_{l+2} + E_3v_{l+2}v_l \\ E_1v_{l+1}v_{l-2} + E_2v_{l+2}v_{l-1} + E_3v^2{}_{l+2} \end{pmatrix},$$

$$Z_7(v_k) = (1-)\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} E_1v^2{}_{l+1}v_{l-2} \\ E_2v_{l+2}v_{l+1}v_{l-1} \\ E_3v^2{}_{l+2}v_l \end{pmatrix}$$

$$Z_8(v_l) = (-1)\begin{pmatrix} s_1s_2s_3 & 0 & 0 \end{pmatrix}\begin{pmatrix} E_4 + E_5 + E_6 \\ 0 \\ 0 \end{pmatrix},$$

$$Z_9(v_l) = \begin{pmatrix} s_1s_2 & s_1s_3 & s_2s_3 \end{pmatrix}\begin{pmatrix} E_4v_{l+1} + E_5v_l + E_6v_l \\ E_4v_{l-1} + E_5v_{l+2} + E_6v_l \\ E_4v_{l-1} + E_5v_{l-1} + E_6v_{l+3} \end{pmatrix},$$

$$Z_{10}(v_l) = (-1)\begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix}\begin{pmatrix} E_4v_{l-1}v_{l+1} + E_5v_{l+1}v_l + E_6v^2{}_l \\ +E_4v_{l+1}v_{l-1} + E_5v_lv_{l-1} + E_6v_{l+3}v_l \\ +E_4v^2{}_{l-1} + E_5v_{l+2}v_{l-1} + E_6v_lv_{l+3} \end{pmatrix},$$

$$Z_{11}(v_l) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} +E_4v^2{}_{l-1}v_{l+1} \\ E_5v_{l+2}v_lv_{l-1} \\ E_6v_lv_{l+3} \end{pmatrix}$$

$$Z_{12}(v_l)=E_6\begin{pmatrix}s_1s_2s_3 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}, Z_{13}(v_l)=(-1)E_6\begin{pmatrix}s_1s_2 & s_1s_3 & s_2s_3\end{pmatrix}v_l\begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$Z_{14}(v_l)=(-1)E_6\begin{pmatrix}s_1 & s_2 & s_3\end{pmatrix}v^2{}_l\begin{pmatrix}1\\1\\1\end{pmatrix}, Z_{15}(v_l)=E_6\begin{pmatrix}1 & 0 & 0\end{pmatrix}v^3{}_l\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Put the above in the following matrix form:-

$$f_l(s_1, s_2, s_3) = \frac{1}{(v_{l+1}-v_l)}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\begin{bmatrix}Z_0(v_l) & Z_4(v_l) & Z_8(v_l) & Z_{12}(v_l)\\ Z_1(v_l) & Z_5(v_l) & Z_9(v_l) & Z_{13}(v_l)\\ Z_2(v_l) & Z_6(v_l) & Z_{10}(v_l) & Z_{14}(v_l)\\ Z_3(v_l) & Z_7(v_l) & Z_{11}(v_l) & Z_{15}(v_l)\end{bmatrix}\begin{bmatrix}P_0\\P_1\\P_2\\P_3\end{bmatrix}.$$

It is easy to show that the above equation is a formula of a modified cubic spline curve in matrix form of second variable *s.* Now from the fact that modified mathematical bipolynomial surface, of bidegree $\langle 3, 3\rangle$ mathematically comes from product of two matrices defined as:

$$f(t_1, t_2, t_3; s_1, s_2, s_3) = \frac{1}{(u_{k+1}-u_k)(v_{l+1}-v_l)}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}M_tP_{ij}M_s{}^T\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad \ldots (17)$$

Where

$$M_t=\begin{bmatrix}B_0(u_k) & B_4(u_k) & B_8(u_k) & B_{12}(u_k)\\ B_1(u_k) & B_5(u_k) & B_9(u_k) & B_{13}(u_k)\\ B_2(u_k) & B_6(u_k) & B_{10}(u_k) & B_{14}(u_k)\\ B_3(u_k) & B_7(u_k) & B_{11}(u_k) & B_{15}(u_k)\end{bmatrix}, \text{ and } M_s=\begin{bmatrix}Z_0(v_l) & Z_4(v_l) & Z_8(v_l) & Z_{12}(v_l)\\ Z_1(v_l) & Z_5(v_l) & Z_9(v_l) & Z_{13}(v_l)\\ Z_2(v_l) & Z_6(v_l) & Z_{10}(v_l) & Z_{14}(v_l)\\ Z_3(v_l) & Z_7(v_l) & Z_{11}(v_l) & Z_{15}(v_l)\end{bmatrix}$$

$$P_{ij}=\begin{bmatrix}p_{00} & p_{01} & p_{02} & p_{03}\\ p_{10} & p_{11} & p_{12} & p_{13}\\ p_{20} & p_{21} & p_{22} & p_{23}\\ p_{30} & p_{31} & p_{32} & p_{33}\end{bmatrix}$$

It is an easy method used to upgrade the above equation to 3D of bidegree $\langle 3, 3\rangle$ by using the set of points as,, [12]. [13].

$p_i=(x_i, y_i, z_i)$ for $i=0, 2,.,15$.

The coordinates of each point are treated as a three-component vector. That is

$$P_i=\begin{pmatrix}x_i\\y_i\\z_i\end{pmatrix}$$

The set of points, in parametric form is

$$F(t,s) = \begin{pmatrix}F_1(t,s)\\F_2(t,s)\\F_3(t,s)\end{pmatrix}=\begin{pmatrix}x(t,s)\\y(t,s)\\z(t,s)\end{pmatrix} \qquad u_k \le t \le u_{k+1} \text{ and } v_l \le s \le v_{l+1}.$$

Eq(17) i.e. is modified mathematical modelling of bi-cubic 3D B-spline surface in matrix form obtained by using blossoming. The surface is defined by de Boor control points for the surface segment f$(t_1, t_2, t_3; s_1, s_2, s_3)$ associated with the middle intervals $[u_k. u_{k+1}]$and$[v_l, v_{l+1}]$.

Comparison with Original Bi-cubic B-Spline Surfaces.

Now a Comparison between Eq(17), and Eq(2) can easily be made by using $(t_1=t_2=t_3=t, s_1=s_2=s_3=s)$, and $k=0,l=0$ the sequence $\{u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3}; v_{l-2}, v_{l-1}, v_l, v_{l+1}, v_{l+2} v_{l+3}\}$becomes $(u_{-2}, u_{-1} u_0, u_1 u_2, u_3; v_{-2}, v_{-1} v_0, v_1 v_2, v_3)$, and using uniform knot sequence. $(u_{-2}, u_{-1} u_0, u_1 u_2, u_3)=(-2, -1, 0, 1, 2, 3)$, implies $u_k = u_0 = 0$, and $u_{k+1} = u_1 = 1$, then $t \in [0,1]$, $(v_{-2}, v_{-1} v_0, v_1 v_2, v_3) = (-2, -1, 0, 1, 2, 3)$, implies $v_l = v_0 = 0$, and $v_{l+1} = v_1 = 1$, then $s \in [0,1]$, and (17) becomes:

$$f(t,t,t;s,s,s) = F(t,s) = \frac{1}{36} TMP_{ij}M^TS^T \quad \dots (18)$$

Eq (18), is original bi-cubic spline surface dependent on frame $[0, 1]^2$. It is identical to Eq (2) which is bi-cubic B-spline surfaces in matrix form.

Discussion and Conclusions

In this paper, we use a mathematical technique dependent on the method of Blossom used to find 3D bi-cubic b-spline surfaces in matrix form in Eq (17). This equation is dependent on parameters t_i, and s_j (for i,j= 1, 2, 3), coefficient of (16) control points. Now we can study in the following many different cases of the control of generation of the surface by using Eq (17) dependent on parameters t_i, and s_j:

A. First technique, one can see that controlling the generation of the surface and of change is dependent on the change in value of the parameters t_i or s_j (for i, j = 1or 2 or 3), at all parts of Eq (17), as seen in the following different properties:

1. Using the parameter t_i increases values, of (*i.e.* using $+t_i$) and decreases values of t_i (*i.e.* by using $- t_i$) the surface to be changed moves to the exterior at increases, and to interior at decrease. See Figs. 4, 5, 7, and 8.
2. Using the parameter s_j increases values, (*i.e.* by using $+ s_j$) and decreases values of s_j (*i.e.* by using $- s_j$), the surface to be changed moves to the exterior at increase, and to interior at decrease. See Figs. 6, 7, and 8.
3. Using the parameters t_i is taken to increase (decrease) the moving in exterior (interior). One can see that controlling the generation of change is dependent on change in the value of $+ t_1$ ($-t_1$). The changes took place without change in the control points. See Fig 9.

B. Second technique, one can see that control on the generation of change is dependent on change in the value t_i and s_j at the same time (for i, j = 1or 2 or 3), at all parts of Eq (17) as seen in the following different properties:-

1. The parameters t_i and s_j are taken to increase (i.e,. by using $+ t_i$, and $+ s_j$) at the same time, or by using the parameters t_i and s_j, are taken to decrease (i.e., by using $- t_i$, and $- s_j$) at the same time. One can see the following:

The parameters t_i and s_j increase (i.e., using $+ t_1$, and $+s_1$) and are taken at the same time. The surface to be changed moves to the exterior in symmetric direction, but not same shape, depending on the values of t_i and s_j ($t_i = s_j$ or $t_i > s_j$ or $t_i < s_j$). All these effects are both similar and different in direction from them, and by using ($+t_i$) only or ($+ s_j$) only . See Figs. 10, 11, 12, 13, and 14.

C. The parameters, t_i, and s_j are taken to be different values (one increases, and other deceases) at the same time. (*i.e.*, using $+ t_i$ and $- s_j$) at same time in one case, and $- t_i$ and $+ s_j$ at same time in another case). The surface to be changed moves to the interior (dependent on the values of t_i and s_j). See Figs.15, 16, and 17.

D. All above effects on the surface take place without change in the control points.

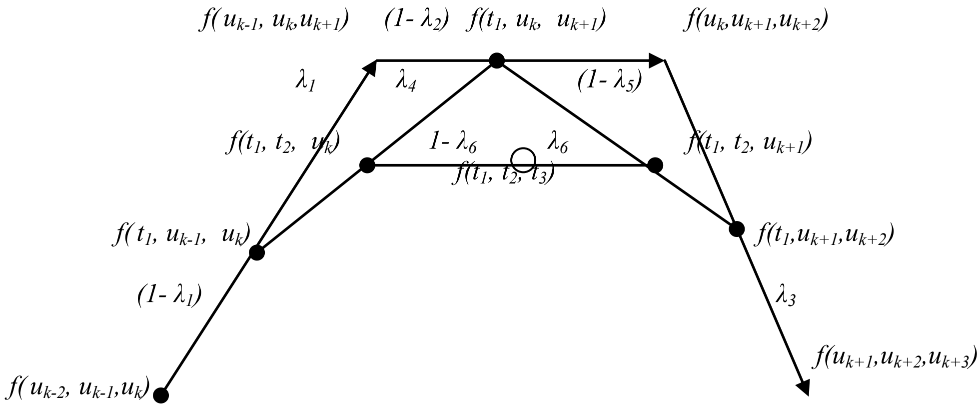
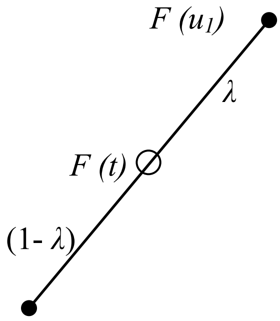


Figure.1. the de Boor Algorithm on cubic case of degree three



Figure(2). Linear Curve

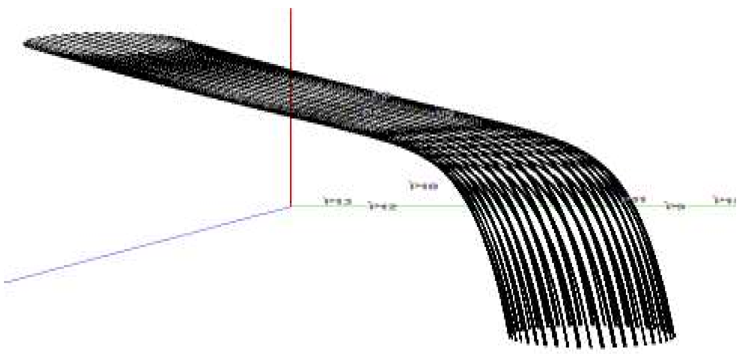
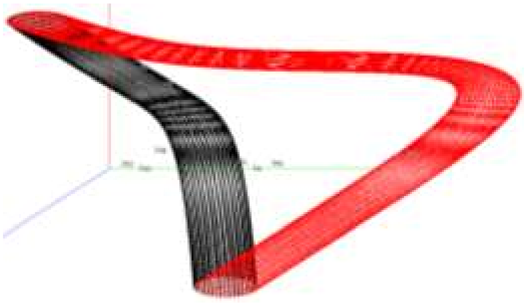
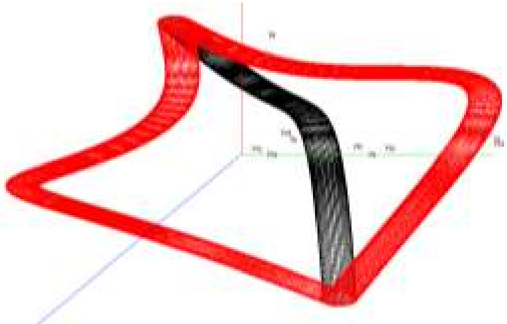


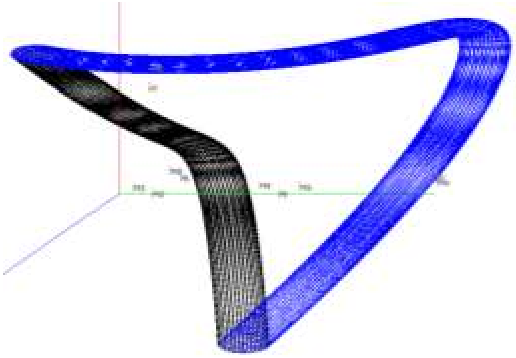
Figure 3 By using *($t_1=t_2=t_3=t$, $s_1=s_2=s_3=s$)*, and *$k=0,l=0$* the sequence {*$u_{k-2}, u_{k-1}, u_k, u_{k+1}, u_{k+2}, u_{k+3}$*; *$v_{l-2}, v_{l-1}, v_l, v_{l+1}, v_{l+2}\ v_{l+3}$*}becomes (*$u_{-2}, u_{-1}\ u_0, u_1\ u_2, u_3$*; *$v_{-2}, v_{-1}\ v_0, v_1\ v_2, v_3$*) , and by using uniform knot sequence. (*$u_{-2}, u_{-1}\ u_0, u_1\ u_2, u_3$*)=*(-2, -1, 0, 1, 2, 3)*, implies $u_k = u_0 = 0$, and *$u_{k+1} = u_1 = 1$, then $t \in [0,1]$*, (*$v_{-2}, v_{-1}\ v_0, v_1\ v_2, v_3$*) =*(-2, -1, 0, 1, 2, 3)*, implies $v_l = v_0 = 0$, and *$v_{l+1} = v_1 = 1$*, then *$s \in [0,1]$*}. Changing in the surface is dependent on on change in the control points of the surface.



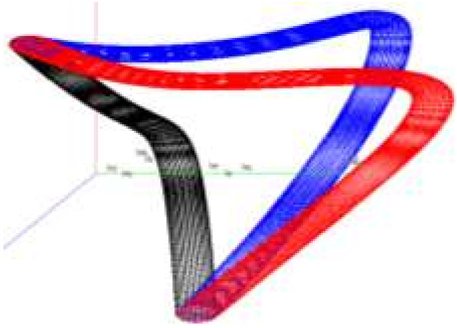
Figure(4) The parameter t_i is taken to be increased *(i.e.* by using *+t_i)*. The surface to be changed moves to the exterior, without change in the control points.



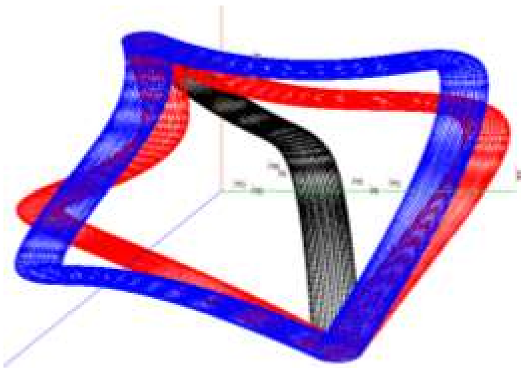
Figure(5) The parameter t_i is taken to be increased in first, and decreases in second, the surface to be changed moves to the opposite direction.



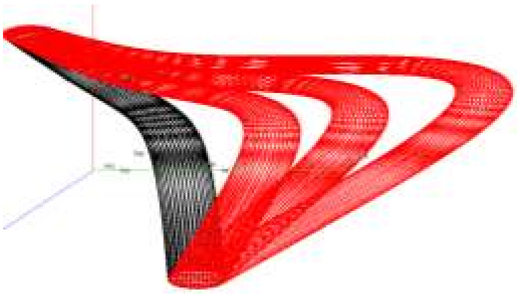
Figure(6) The parameter s_j *is* taken to be increased *(i.e.by* using *+s_j), t*he surface to be changed moves to the exterior, without change in the control points.



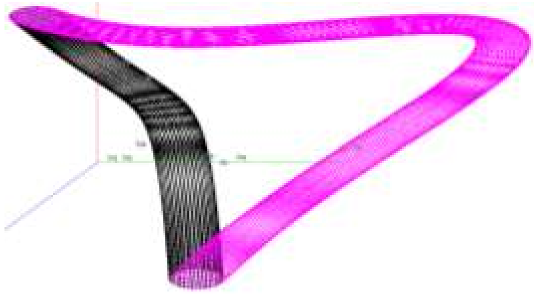
Figure(7) The parameters are taken to be increases t_i in the first, and s_j in the second, the surface to be changed moves to the exterior, thus the effect is both similar and different direction from them.



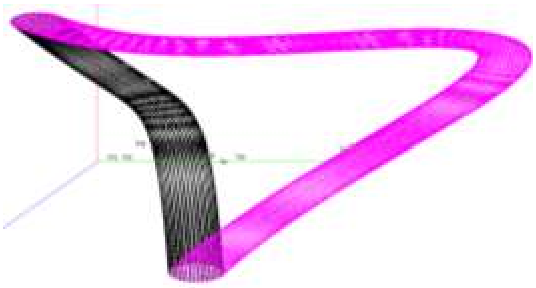
Figure(8) The parameters t_i, are taken to be increases(i.e. $+t_i$) in the first, decrease(i.e. $-t_i$) in the second and s_j is taken to be increases(i.e. $+s_j$) in the first, decrease (i.e. $-s_j$) in the second. The surface to be changed moves to the opposite between using $+t_i$, and $-t_i$, or $+s_j$, and $-s_j$, symmetric, between using $+t_i$, and $+s_j$, or $-t_i$, and $-s_j$ effected.



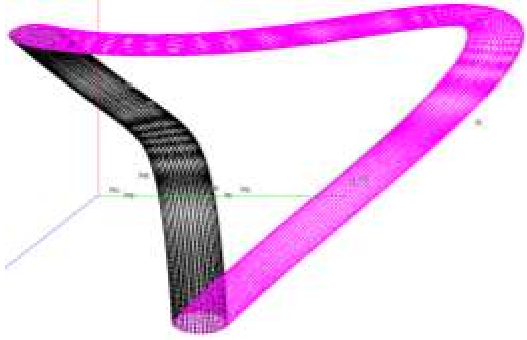
Figure(9). the parameters t_i is taken to be increased (i.e. $+t_i$), the surface to be changed moves to the exterior. Can see the controlling on changing of the surface dependent on changing the value of ($+t_i$).



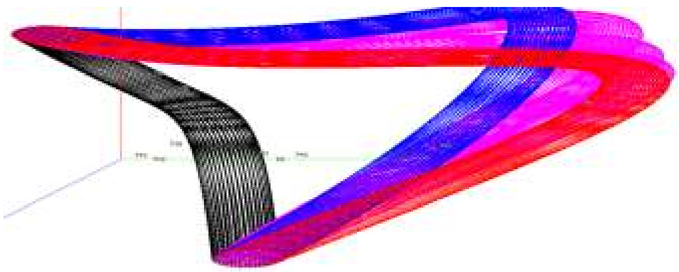
Figure(10) The parameters t_i and s_i (where $t_i = s_j$) are taken to be increase at same time.



Figure(11) The parameters t_i and s_j (where $t_i > s_j$) are taken to be increase at same time.



Figure(12).The parameters t_i and s_j (where $t_i < s_j$) are taken to be increase at same time.



Figure(13) Comparison between three effects by using ($+t_i$) only or ($+s_j$) only, and from using t_i and s_j are taken to increase(i.e. $t_i = s_j$ or $t_i > s_j$ or $t_i < s_j$) at the same time for example. Note that the surface to be changed in exterior, thus the effect is both similar and different direction from them.

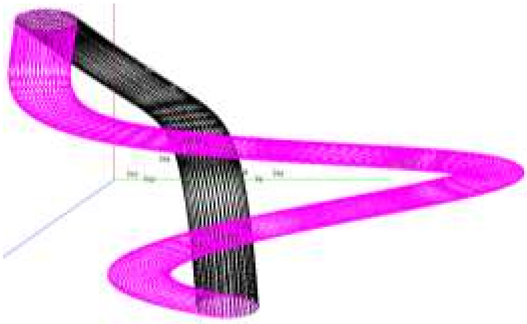


Figure. 14 The parameters t_i and s_j are taken to decrease (i.e. $-t_i$ and $-s_j$) at the same time, the surface to be changed moves to the interior, and then to exterior.

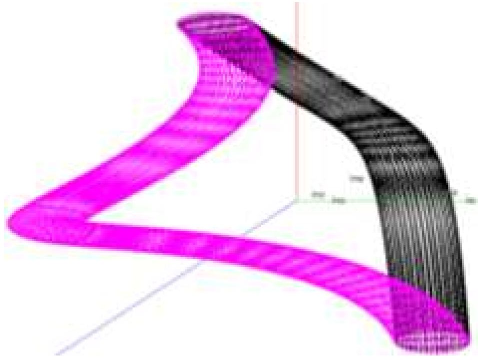
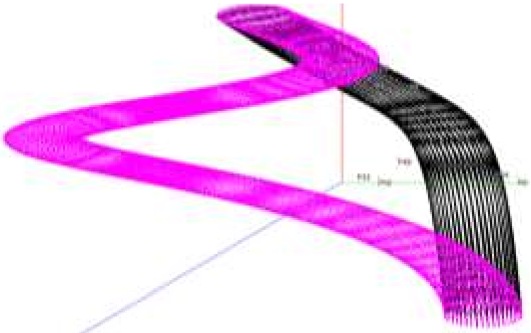
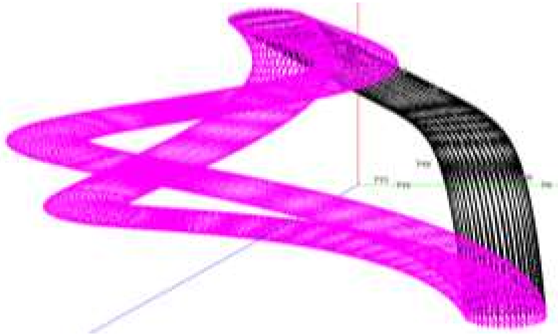


Figure (15) The parameter t_i is taken to be increased,] (i.e. $+t_i$) and s_j is taken to be decreased (i.e. $-s_j$) at same time, the surface to be changed moves to the interior.



Figure(16) The parameter t_i is taken to be decrease (i.e. $-t_i$) and s_j is taken to be increase (i.e. $+s_j$) at same time, the surface to be changed moves to the interior.



Figure(17) Comparison between figures (15 and 16) the change in interior in symmetric direction.

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