Shape Preserving Approximation of 3–Monotone Multivariate Function

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Abstract

In this article we estimate the degree of 3-monotone uniform multiapproximation to get a Jackson type estimation as a direct consequence.

Keywords: 3-monotone multiapproximation by piecewise multipolynomials, Degree of multiapproximation.

الخلاصة

برهنا في هذا البحث نظرية حول درجة التقريب المنتظم للدوال 3− رتيبة متعددة المتغيرات للحصول على مبرهنة مباشرة من نوع مبرهنة جاكسون.

الكلمات المفتاحية: التقريب المتعدد للدوال 3−رتيبة باستخدام متعددات الحدود بأكثر من متغير, درجة التقريب المتعدد.

1. Introduction and main results

Let f be function defined on $I := [a_1\,, b_1] \times \dots \times [a_d\,, b_d]$ and ν a natural number. Denoted by

$$f[(x_{01}, \dots x_{0d}), \dots, (x_{\nu 1}, \dots x_{\nu d})] := \sum_{i=0}^{\nu} \frac{f\big((x_{i1}, \dots, x_{id})\big)}{\prod_{j=0, j\neq i}^{\nu} \big(x_{i1} - x_{j1}\big) \dots \big(x_{id} - x_{jd}\big)}$$

the νth order multi divided difference of f at the distinct points $x_0 = (x_{01}, \dots x_{0d})$, …, $x_\nu = (x_{\nu 1}, \dots x_{\nu d})$. The function f is called ν- monotone on I ,if $f[(x_{01}, \dots x_{0d}), \dots, (x_{\nu 1}, \dots x_{\nu d})] \geq 0$ for all choices of $\nu + 1$ distinct points $x_0 = (x_{01}, \dots, x_{0d})$, …, $x_\nu = (x_{\nu 1}, \dots x_{\nu d}) \in I$. Let us define Δ_I^ν the set of all ν- monotone function on I , and let Δ_I^1 and Δ_I^2 are the sets of non – decreasing and convex multifunctions on I. It is well known that Δ_I^3 is the set of all bounded functions, having a convex derivative on$(a_1\,, b_1) \times \dots \times (a_d\,, b_d)$ see [A.Guntuboyina and B.Sen (2012)]. Note that when $f \in \Delta_I^\nu$, $\nu \geq 2$, then f is continuous on $(a_1\,, b_1) \times \dots \times (a_d\,, b_d)$ and $f\big((a_1+, \dots, a_d+)\big)$, $f\big((b_1-, \dots, b_d-)\big)$ exist. Let $C(I)$ be the space of all continuous multivariate functions defined on I and equipped with the uniform norm $\|f\|_\infty = \sup_{x=(x_1,\dots,x_d)} \big|f\big((x_1, \dots, x_d)\big)\big|$.

Many authors studied the monotone and convex approximations such us [DeVore, 1977; Beatson, 1981; Hu, 1993; Kopotun, 1994; Shevchuk, 1997; Leviatan and Prymak, 2005] they used functions of one variable defined on finite interval. Little is known for the shape preserving approximation of ν-monotone functions for $\nu \geq 3$. In [Shvedov, 1981] , Shvedov prove positive and negative inequality for the best approximation of $\nu -$ monotone univariate function for $\nu \geq 4$.

In this paper we shall prove the following theorem

Theorem. Let $F \in \Delta_I^3$ and $f\big((x_1, \dots, x_d)\big) = F'\big((x_1, \dots, x_d)\big), x = (x_1, \dots, x_d) \in (a_1\,, b_1) \times \dots \times (a_d\,, b_d)$, and let k be an integer greater than or equal 2 a partition $a_1 := x_{01} < x_{11} < \dots < x_{n1} := b_1, \dots, a_d := x_{0d} < x_{1d} < \dots < x_{nd} := b_d$, and a piecewise multipolynomial $s \in \Delta_I^2$ of degree not exceeding or equal $k - 1$, with knots $x_i = (x_{i1}, \dots, x_{id})$, $i = 1, \dots, n-1$, such that

$$s((x_{i1},\dots,x_{id})) = f((x_{i1},\dots,x_{id})), i = 1,\dots,n-1, \qquad (1)$$

there exists a piecewise multipolynomial $S \in \Delta_I^3$ of degree not exceeding or equal k with knots $x_i = (x_{i1},\dots,x_{id})$ $i = 1,\dots,n-1$, for which

$$\|F - S\| \le c \max_{1\le i\le n} \|f - s\|_{L_1(J)}, \qquad (2)$$

where $J = [x_{i1-1}, x_{i1}] \times \dots \times [x_{id-1}, x_{id}]$, c is positive constant and $\|.\|_{L_1(J)}$ denotes the L_1- norm on J, defined by $\|f\|_{L_1(J)} = \left(\int_{x_{i1-1}}^{x_{i1}} \dots \int_{x_{id-1}}^{x_{id}} |f((x_1,\dots,x_d))|\, dx_1 \dots dx_d\right)$

Note that we can consider the above theorem as a direct estimate for convex multiapproximation since it is in terms of a derivative of a 3-monotone multifunction.

2. Auxiliary results and the proof of the main results.

Let f be a function defined on $I := [a_1, b_1] \times \dots \times [a_d, b_d]$, let $L((x_1,\dots,x_d); f; a, b)$denote the linear Lagrange interpolating multipolynomials of f at the points $a = (a_1,\dots,a_d)$ and $b = (b_1,\dots,b_d)$. Assume $k \ge 2$.

To prove our main result we need the following lemmas

Lemma 1. Let $f \in \Delta_I^2$, and let $q \in \Delta_I^2$ is amultipolynomial of degree not exceeding or equal $k-1$, with $f((a_1,\dots,a_d)) = q((a_1,\dots,a_d))$ and $f((b_1,\dots,b_d)) = q((b_1,\dots,b_d))$. Then there is a multipolynomial p $\in \Delta_I^2$ of degree not exceeding or equal $k-1$, satisfying

$$f((a_1,\dots,a_d)) = p((a_1,\dots,a_d)),$$
$$f((b_1,\dots,b_d)) = p((b_1,\dots,b_d)), \qquad (3)$$
$$q'((a_1,\dots,a_d)) \le p'((a_1,\dots,a_d)), p'((b_1,\dots,b_d)) \le q'((b_1,\dots,b_d)), \qquad (4)$$
$$\left\|\int_{a_1}^{(\cdot)} \dots \int_{a_d}^{(\cdot)} \left(p((t_1,\dots,t_d)) - f((t_1,\dots,t_d))\right) dt_1 \dots dt_d\right\|_I \le 2\left\|\int_{a_1}^{(\cdot)} \dots \int_{a_d}^{(\cdot)} \left(q((t_1,\dots,t_d)) - f((t_1,\dots,t_d))\right) dt_1 \dots dt_d\right\|_I, \qquad (5)$$

and

$$\int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} p((t_1,\dots,t_d))\, dt_1 \dots dt_d \ge \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} f((t_1,\dots,t_d))\, dt_1 \dots dt_d. \qquad (6)$$

Proof . If

$$\int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} q((t_1,\dots,t_d))\, dt_1 \dots dt_d \ge \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} f((t_1,\dots,t_d))\, dt_1 \dots dt_d.$$

then we take $p := q$ and (3) through (6) are self evident. Otherwise,

$$\int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} f((t_1,\dots,t_d))\, dt_1 \dots dt_d - \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} q((t_1,\dots,t_d))\, dt_1 \dots dt_d := A > 0.$$

Clearly

$$A \leq \left\| \int_{a_1}^{(\cdot)} \dots \int_{a_d}^{(\cdot)} \Big(q\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big) \, dt_1 \dots dt_d \right\|_I . \qquad (7)$$

Let $l\big((x_1, \dots, x_d)\big) \coloneqq L\big((x_1, \dots, x_d); f; (a_1, \dots, a_d), (b_1, \dots, b_d)\big)$ Then since f is convex, $l\big((x_1, \dots, x_d)\big) \geq f\big((x_1, \dots, x_d)\big)$, $(x_1, \dots, x_d) \in I$. Hence

$$\int_{a_1}^{x_1} \dots \int_{a_d}^{x_d} l\big((t_1, \dots, t_d)\big) \, dt_1 \dots dt_d - \int_{a_1}^{x_1} \dots \int_{a_d}^{x_d} f\big((t_1, \dots, t_d)\big) \, dt_1 \dots dt_d \qquad (8)$$

$$\leq \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} l\big((t_1, \dots, t_d)\big) \, dt_1 \dots dt_d - \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} f\big((t_1, \dots, t_d)\big) \, dt_1 \dots dt_d \quad \coloneqq B \geq 0,$$

$$x = (x_1, \dots, x_d) \in I.$$

Let

$$p\big((x_1, \dots, x_d)\big) = \frac{A l\big((x_1, \dots, x_d)\big) + B q\big((x_1, \dots, x_d)\big)}{A + B}, \qquad (x_1, \dots, x_d) \in I.$$

(3) and (4) imply p is a convex . using (8) and (7) to get

$$\left| \int_{a_1}^{x_1} \dots \int_{a_d}^{x_d} p\big((t_1, \dots, t_d)\big) \, dt_1 \dots dt_d - \int_{a_1}^{x_1} \dots \int_{a_d}^{x_d} f\big((t_1, \dots, t_d)\big) \, dt_1 \dots dt_d \right|$$

$$= \left| \frac{A}{A+B} \int_{a_1}^{x_1} \dots \int_{a_d}^{x_d} \Big(l\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big) \, dt_1 \dots dt_d + \frac{B}{A+B} \int_{a_1}^{x_1} \dots \int_{a_d}^{x_d} \Big(q\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big) \, dt_1 \dots dt_d \right|$$

$$\leq \frac{A}{A+B} \left| \int_{a_1}^{x_1} \dots \int_{a_d}^{x_d} \Big(l\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big) \, dt_1 \dots dt_d \right| + \frac{B}{A+B} \left| \int_{a_1}^{x_1} \dots \int_{a_d}^{x_d} \Big(q\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big) \, dt_1 \dots dt_d \right|$$

$$\leq \frac{A}{A+B} B + \frac{B}{A+B} \left\| \int_{a_1}^{(\cdot)} \dots \int_{a_d}^{(\cdot)} \Big(q\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big) \, dt_1 \dots dt_d \right\|_I$$

$$\leq \frac{2B}{A+B} \left\| \int_{a_1}^{(\cdot)} \dots \int_{a_d}^{(\cdot)} \Big(q\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big) \, dt_1 \dots dt_d \right\|_I$$

$$\leq 2 \left\| \int_{a_1}^{(\cdot)} \dots \int_{a_d}^{(\cdot)} \Big(q\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big) \, dt_1 \dots dt_d \right\|_I ,$$

that is, (5). Finally,

$$\int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} p\big((t_1, \dots, t_d)\big)\, dt_1 \dots dt_d$$
$$= \frac{A}{A+B} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} l\big((t_1, \dots, t_d)\big)\, dt_1 \dots dt_d$$
$$+ \frac{B}{A+B} \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} q\big((t_1, \dots, t_d)\big)\, dt_1 \dots dt_d$$
$$\geq \frac{A}{A+B} \left(-B + \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} l\big((t_1, \dots, t_d)\big)\, dt_1 \dots dt_d \right)$$
$$+ \frac{B}{A+B} \left(\int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} q\big((t_1, \dots, t_d)\big)\, dt_1 \dots dt_d + A \right)$$
$$= \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} f\big((t_1, \dots, t_d)\big)\, dt_1 \dots dt_d,$$

Which prove (6) , and that completes our proof.

Lemma 2. Let $q \in \Delta_I^2$ be a multipolynomial of degree $\leq k-1$, and let α and β be arbitrary nonnegative real numbers. Suppose that d_{a_ℓ} and d_{b_ℓ} , $\ell = 1, \dots, d$ are real numbers satisfying,

$$d_{a_1} \dots d_{a_d} \leq \frac{\big(q\big((b_1, \dots, b_d)\big) - \beta\big) - \big(q\big((a_1, \dots, a_d)\big) - \alpha\big)}{(b_1 - a_1) \dots (b_d - a_d)} \leq d_{b_1} \dots d_{b_d}, \qquad (9)$$

and

$$d_{a_1} \dots d_{a_d} \leq q'\big((a_1, \dots, a_d)\big) \leq q'\big((b_1, \dots, b_d)\big) \leq d_{b_1} \dots d_{b_d}.$$

Then there is a multipolynomial $p \in \Delta_I^2$ of degree $\leq k-1$,such that

$$p\big((a_1, \dots, a_d)\big) = q\big((a_1, \dots, a_d)\big) - \alpha\,, \qquad (10)$$
$$p\big((b_1, \dots, b_d)\big) = q\big((b_1, \dots, b_d)\big) - \beta \quad ,$$
$$d_{a_1} \dots d_{a_d} \leq p'\big((a_1, \dots, a_d)\big) \leq p'\big((b_1, \dots, b_d)\big) \leq d_{b_1} \dots d_{b_d} \qquad (11)$$

and

$$p\big((x_1, \dots, x_d)\big) \leq q\big((x_1, \dots, x_d)\big)\,, \quad (x_1, \dots, x_d) \in I \coloneqq [a_1\,, b_1] \times \dots \times [a_d\,, b_d] \qquad (12)$$

Proof. If $\alpha = \beta$, and $p\big((x_1, \dots, x_d)\big) = q\big((x_1, \dots, x_d)\big) - \alpha$ **,** $a_\ell \leq x_\ell \leq b_\ell,\ \ell = 1, \dots, d$, and the proof of (10), (11) and (12) is clear. then, assume that $\alpha > \beta$ (the other results are similar). Let

$$\lambda = \frac{\big((b_1 - a_1) \dots (b_d - a_d)\big)\big(d_{b_1} \dots d_{b_d}\big) + q\big((a_1, \dots, a_d)\big) - q\big((b_1, \dots, b_d)\big)}{\alpha - \beta}$$

we have that the right hand side of (11) is equivalent to $\lambda \geq 1$. Put

$$l\big((x_1, \dots, x_d)\big) = \big(d_{b_1} \dots d_{b_d}\big)\big((x_1 - b_1) \dots (x_d - b_d)\big) + q\big((b_1, \dots, b_d)\big) - \lambda\beta.$$

Then ,

$$l\big((x_1, \dots, x_d)\big) \leq \big(d_{b_1} \dots d_{b_d}\big)\big((x_1 - b_1) \dots (x_d - b_d)\big) + q\big((b_1, \dots, b_d)\big) \leq q\big((x_1, \dots, x_d)\big)\,, \qquad (x_1, \dots, x_d) \in I \qquad (13)$$

Now let

$$p\big((x_1, \dots, x_d)\big) = \lambda^{-1} \Big((\lambda - 1) q\big((x_1, \dots, x_d)\big) + l\big((x_1, \dots, x_d)\big) \Big), \quad (x_1, \dots, x_d) \in I$$

Then the multipolynomial p is convex since it is written in terms of l and q, with nonnegative coefficients. Also the proof of (10) and (11) is clear (11) since q' is monotone. Now for (13),

$$p\big((x_1,\dots,x_d)\big) \le \lambda^{-1}\Big((\lambda-1)q\big((x_1,\dots,x_d)\big)+q\big((x_1,\dots,x_d)\big)\Big)$$
$$= q\big((x_1,\dots,x_d)\big), \qquad x=(x_1,\dots,x_d)\in I,$$

thus (12) is proved and the proof is completed.

Lemma 3. Let $f\in\Delta^2_{I_1}$ and $g\in\Delta^2_{I_1}\cap C^{(1)}(I_1)$, satisfy $f\big((z_{i1},\dots,z_{id})\big)=g\big((z_{i1},\dots,z_{id})\big)$, $i=1,2$. Let $l_i\big((x_1,\dots,x_d)\big)=\big((x_1-z_{i1})\dots(x_d-z_{id})\big)\ g'\big((z_{i1},\dots,z_{id})\big)+g\big((z_{i1},\dots,z_{id})\big)$, and denote

$$\Delta_i=\int_{z_{11}}^{z_{21}}\dots\int_{z_{1d}}^{z_{2d}}\Big(l_i\big((t_1,\dots,t_d)\big)-f\big((t_1,\dots,t_d)\big)\Big)_+\ dt_1\dots dt_d\ ,\quad i=1,2.$$

Then

$$\Delta_i\le\left\|\int_{z_{i1}}^{(\cdot)}\dots\int_{z_{id}}^{(\cdot)}\Big(f\big((t_1,\dots,t_d)\big)-g\big((t_1,\dots,t_d)\big)\Big)dt_1\dots dt_d\right\|_{I_1} \qquad (14)$$

Proof. Let $i=1$. The convexity of g implies that $l_1\big((x_1,\dots,x_d)\big)\le g\big((x_1,\dots,x_d)\big)$, $(x_1,\dots,x_d)\in I_1$, $I_1=[z_{11},z_{21}]\times\dots\times[z_{1d},z_{2d}]$. The convexity of f and the linearity of $l_1$we can find a $\theta=(\theta_1,\dots,\theta_d)\in I_1$, satisfy $f\big((x_1,\dots,x_d)\big)\le l_1\big((x_1,\dots,x_d)\big)$, $x\in[z_{11},\theta_1]\times\dots\times[z_{1d},\theta_d]$and $l_1\big((x_1,\dots,x_d)\big)\le f\big((x_1,\dots,x_d)\big)$, $x\in[\theta_1,z_{21}]\times\dots\times[\theta_d,z_{2d}]$. Hence

$$\Delta_1=\int_{z_{11}}^{\theta_1}\dots\int_{z_{1d}}^{\theta_d}\Big(l_1\big((t_1,\dots,t_d)\big)-f\big((t_1,\dots,t_d)\big)\Big)\ dt_1\dots dt_d$$
$$\le\int_{z_{11}}^{\theta_1}\dots\int_{z_{1d}}^{\theta_d}\Big(g\big((t_1,\dots,t_d)\big)-f\big((t_1,\dots,t_d)\big)\Big)\ dt_1\dots dt_d$$
$$\le\left\|\int_{z_{11}}^{(\cdot)}\dots\int_{z_{1d}}^{(\cdot)}\Big(f\big((t_1,\dots,t_d)\big)-g\big((t_1,\dots,t_d)\big)\Big)\ dt_1\dots dt_d\right\|_{I_1},$$

This completes the proof of (14) when $i=1$. And that implies:

$$\Delta_2\le\left\|\int_{(\cdot)}^{z_{21}}\dots\int_{(\cdot)}^{z_{2d}}\Big(f\big((t_1,\dots,t_d)\big)-g\big((t_1,\dots,t_d)\big)\Big)\ dt_1\dots dt_d\right\|_{I_1},$$

Which yield the proof of (14) when $i=2$.

Lemma 4. Let $f,g\in\Delta^2_I$, be a multi real functions with

$$f\big((b_1,\dots,b_d)\big)-f\big((a_1,\dots,a_d)\big)$$
$$=g\big((b_1,\dots,b_d)\big)-g\big((a_1,\dots,a_d)\big). \qquad (15)$$

Then

$$f'\big((a_1+,\dots,a_d+)\big)\le g'\big((b_1-,\dots,b_d-)\big).$$

Proof. We have $f'$and g' are monotone on $(a_1,b_1)\times\dots\times(a_d,b_d)$. Suppose to the contrary that $f'\big((a_1+,\dots,a_d+)\big)>g'\big((b_1-,\dots,b_d-)\big)$. Therefore

$$f\big((b_1,\dots,b_d)\big)-f\big((a_1,\dots,a_d)\big)=\int_{a_1}^{b_1}\dots\int_{a_d}^{b_d}f'\big((x_1,\dots,x_d)\big)dx_1\dots dx_d$$
$$\ge f'\big((a_1+,\dots,a_d+)\big)\big((b_1-a_1)\dots(b_d-a_d)\big)$$
$$> g'\big((b_1-,\dots,b_d-)\big)\big((b_1-a_1)\dots(b_d-a_d)\big)$$

$$\geq \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} g'\big((x_1, \dots, x_d)\big)\, dx_1 \dots dx_d = g\big((b_1, \dots, b_d)\big) - g\big((a_1, \dots, a_d)\big),$$

contradicting (15).

Corollary 1. Let $f \in \Delta_I^2$and let $s \in \Delta_I^2$ be a piecewise multipolynomial of degree not exceeding or equal $k-1$ with knots partition $a_1 := x_{01} < x_{11} < \dots < x_{n1} := b_1, \dots$, $a_d := x_{0d} < x_{1d} < \dots < x_{nd} := b_d$, satisfying (1). now if $i = 2, \dots, n-1$,

$$f'\big((x_{i1-1}+, \dots, x_{id-1}+)\big) \leq s'\big((x_{i1}-, \dots, x_{id}-)\big), \qquad (16)$$

$$s'\big((x_{i1-1}+, \dots, x_{id-1}+)\big) \leq f'\big((x_{i1}-, \dots, x_{id}-)\big). \qquad (17)$$

Given f and $s \in \Delta_I^2$ denote

$$Z = \max_{1\leq i\leq n} \|f - s\|_{L_1(J)}, \qquad J = [x_{i1-1}, x_{i1}] \times \dots \times [x_{id-1}, x_{id}] \qquad (18)$$

Let us write $g \in A_{i,j}$, $1 \leq i < j \leq n-1$, when g is a convex piecewise multipolynomial of degree not exceeding or equal $k-1$, on $[x_{i1}, x_{j1}] \times \dots \times [x_{id}, x_{jd}]$, with knots $(x_{i1+1}, \dots, x_{id+1}), \dots, (x_{j1-1}, \dots, x_{jd-1})$ and satisfies $s'\big((x_{i1}+, \dots, x_{id}+)\big) \leq g'\big((x_{i1}+, \dots, x_{id}+)\big)$ and $g'\Big((x_{j1}-, \dots, x_{jd}-)\Big) \leq s'\Big((x_{j1}-, \dots, x_{jd}-)\Big)$, and $g\big((x_{i1}, \dots, x_{id})\big) = s\big((x_{i1}, \dots, x_{id})\big)$, $g\Big((x_{j1}, \dots, x_{jd})\Big) = s\Big((x_{j1}, \dots, x_{jd})\Big)$. For $r = 1, \dots, n-1$ and $t = (t_1, \dots, t_d)$ let

$$h_r(t) = \begin{cases} f'\big((x_{i1}-, \dots, x_{id}-)\big), & \text{if } t \in (x_{i1-1}, x_{i1}] \times \dots \times (x_{id-1}, x_{id}], \\ & \quad i = 1, \dots, r-1, \\ s'\big((x_{r1}-, \dots, x_{rd}-)\big), & \text{if } t \in (x_{r1-1}, x_{r1}] \times \dots \times (x_{rd-1}, x_{rd}] \\ s'\big((x_{r1}+, \dots, x_{rd}+)\big), & \text{if } t \in (x_{r1}, x_{r1+1}) \times \dots \times (x_{rd}, x_{rd+1}) \\ f'\big((x_{i1-1}+, \dots, x_{id-1}+)\big), & \text{if } t \in [x_{i1-1}, x_{i1}) \times \dots \times [x_{id-1}, x_{id}), \\ & \quad i = r+2, \dots, n, \end{cases}$$

and set

$$g_r\big((x_1, \dots, x_d)\big) = f\big((x_{r1}, \dots, x_{rd})\big) + \int_{x_{r1}}^{x_1} \dots \int_{x_{rd}}^{x_d} h_r\big((t_1, \dots, t_d)\big) dt_1 \dots dt_d.$$

using of Corollary 1,we get h_r is non-decreasing on $(a_1, b_1) \times \dots \times (a_d, b_d)$, also we have g_r is convex on $(a_1, b_1) \times \dots \times (a_d, b_d)$. It follows by (1) we have $g_r\big((x_{r1+1}, \dots, x_{rd+1})\big) \leq f\big((x_{r1+1}, \dots, x_{rd+1})\big)$ and $g_r\big((x_{r1-1}, \dots, x_{rd-1})\big) \leq f\big((x_{r1-1}, \dots, x_{rd-1})\big)$. Therefore ,

$$g_r\big((x_1, \dots, x_d)\big) \leq f\big((x_1, \dots, x_d)\big), \qquad (19)$$

$x = (x_1, \dots, x_d) \in I \setminus (x_{r1-1}, x_{r1+1}) \times \dots \times (x_{rd-1}, x_{rd+1})$.

By lemma 3,

$$\int_{x_{r1}}^{x_{r1+1}} \dots \int_{x_{rd}}^{x_{rd+1}} \Big(g_r\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big)_+ dt_1 \dots dt_d \leq Z, \qquad (20)$$

and

$$\int_{x_{r1-1}}^{x_{r1}} \dots \int_{x_{rd-1}}^{x_{rd}} \Big(g_r\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big)_+ dt_1 \dots dt_d \leq Z. \qquad (21)$$

If $1 \leq i < j \leq n-1$, let us construct a function $g_{i,j} \in A_{i,j}$. Then, if $j = i + 1$, then let $g_{i,i+1} = s\,|_{[x_{i1},x_{i1+1}]\times\ldots\times[x_{id},x_{id+1}]}$, which belongs to $A_{i,i+1}$. From the other ,we have by (19), that $g_j\big((x_{i1},\ldots,x_{id})\big) \leq g_i\big((x_{i1},\ldots,x_{id})\big)$, $g_i\Big(\big(x_{j1},\ldots,x_{jd}\big)\Big) \leq g_j \Big(\big(x_{j1},\ldots,x_{jd}\big)\Big)$.

$g_j - g_i$ is continuous on $I_2 = [x_{i1}, x_{j1}] \times \ldots \times [x_{id}, x_{jd}]$, therefore there exists a $\theta \in (x_{i1}, x_{j1}) \times \ldots \times (x_{id}, x_{jd})$ such that $g_i\big((\theta_1,\ldots,\theta_d)\big) = g_j\big((\theta_1,\ldots,\theta_d)\big)$. In addition , by (16) and (17) $h_i\big((t_1,\ldots,t_d)\big) \leq h_j\big((t_1,\ldots,t_d)\big)$, $t \in (x_{i1}, x_{j1}) \times \ldots \times (x_{id}, x_{jd})$, whence $h_j - h_i$ is nonnegative on $(x_{i1}, x_{j1}) \times \ldots \times (x_{id}, x_{jd})$ leads to $g_j - g_i$ is non-decreasing on $(x_{i1}, x_{j1}) \times \ldots \times (x_{id}, x_{jd})$. Therefore

$$\max\{g_i\big((x_1,\ldots,x_d)\big), g_j\big((x_1,\ldots,x_d)\big)\} = \begin{cases} g_i\big((x_1,\ldots,x_d)\big), & \text{if } x_\ell \leq \theta_\ell, \ell = 1,\ldots d \\ g_j\big((x_1,\ldots,x_d)\big), & \text{if } x_\ell > \theta_\ell, \ell = 1,\ldots d, \end{cases}$$

and we set $\bar{g}_{i,j}\big((x_1,\ldots,x_d)\big) = \max\{g_i\big((x_1,\ldots,x_d)\big), g_j\big((x_1,\ldots,x_d)\big)\}$, $x \in I_2 = [x_{i1}, x_{j1}] \times \ldots \times [x_{id}, x_{jd}]$. Note that $\bar{g}_{i,j}$ convex on I_2. For the integers m, $i+1 \leq m \leq j$, $\theta = (\theta_1,\ldots,\theta_d)$ satisfies $\theta \in [x_{m1-1}, x_{m1}] \times \ldots \times [x_{md-1}, x_{md}]$. And for the integers $\zeta, \zeta \neq m$, $i+1 \leq \zeta \leq j$, the function $\bar{g}_{i,j}$ is linear on $[x_{\zeta 1-1}, x_{\zeta 1}] \times \ldots \times [x_{\zeta d-1}, x_{\zeta d}]$, but it may not be so on the interval $[x_{m1-1}, x_{m1}] \times \ldots \times [x_{md-1}, x_{md}]$. Then we shall replace it by amultipolynomial of degree$\leq k-1$. Then by the convexity of $\bar{g}_{i,j}$ we get

$$\bar{g}'_{i,j}\big((x_{m1-1}+,\ldots,x_{md-1}+)\big) \leq \frac{\bar{g}_{i,j}\big((x_{m1},\ldots,x_{md})\big) - \bar{g}_{i,j}\big((x_{m1-1},\ldots,x_{md-1})\big)}{(x_{m1} - x_{m1-1})\ldots(x_{md} - x_{md-1})} \leq \bar{g}'_{i,j}\big((x_{m1}-,\ldots,x_{md}-)\big).$$

Put

$$d_{a_1}\ldots d_{a_d} = \begin{cases} s'\big((x_{m1-1}+,\ldots,x_{md-1}+)\big), & \text{if } m-1 = i \\ f'\big((x_{m1-2}+,\ldots,x_{md-2}+)\big), & \text{o.w} \end{cases},$$

and

$$d_{b_1}\ldots d_{b_d} = \begin{cases} s'\big((x_{m1}-,\ldots,x_{md}-)\big), & \text{if } m = j, \\ f'\big((x_{m1+1}-,\ldots,x_{md+1}-)\big), & \text{o.w} \end{cases},$$

Therefore

$$d_{a_1}\ldots d_{a_d} \leq \bar{g}'_{i,j}\big((x_{m1-1}+,\ldots,x_{md-1}+)\big) \leq \bar{g}'_{i,j}\big((x_{m1}-,\ldots,x_{md}-)\big) \leq d_{b_1}\ldots d_{b_d}.$$

Also, in view of (16) and (17),

$d_{a_1}\ldots d_{a_d} \leq s'\big((x_{m1-1}+,\ldots,x_{md-1}+)\big)$, $s'\big((x_{m1}-,\ldots,x_{md}-)\big) \leq d_{b_1}\ldots d_{b_d}$.

Applying Lemma 2 with $a_\ell = x_{m\ell-1}$ and $b_\ell = x_{m\ell}$, $\ell = 1,\ldots,d$,

$q = s\,|_{[x_{m1-1},x_{m1}]\times\ldots\times[x_{md-1},x_{md}]}$ and

$\alpha = f\big((x_{m1-1},\ldots,x_{md-1})\big) - \bar{g}_{i,j}\big((x_{m1-1},\ldots,x_{md-1})\big)$ and $\beta = f\big((x_{m1},\ldots,x_{md})\big) - \bar{g}_{i,j}\big((x_{m1},\ldots,x_{md})\big)$, so we get the required polynomial p. Let

$$g_{i,j}\big((x_1,\ldots,x_d)\big) = \begin{cases} \bar{g}_{i,j}\big((x_1,\ldots,x_d)\big), & \text{if } x \notin [x_{m1-1}, x_{m1}] \times \ldots \times [x_{md-1}, x_{md}], \\ p\big((x_1,\ldots,x_d)\big), & \text{if } x \in [x_{m1-1}, x_{m1}] \times \ldots \times [x_{md-1}, x_{md}]. \end{cases}$$

Then (10) and (11) yield $g_{i,j} \in A_{i,j}$ and (12) gives

$$\int_{x_{m1-1}}^{x_{m1}} \dots \int_{x_{md-1}}^{x_{md}} \Big(g_{i,j}\big((t_1,\dots,t_d)\big) - f\big((t_1,\dots,t_d)\big)\Big)_+ dt_1 \dots dt_d \qquad (22)$$
$$\le \int_{x_{m1-1}}^{x_{m1}} \dots \int_{x_{md-1}}^{x_{md}} \Big(s\big((t_1,\dots,t_d)\big) - f\big((t_1,\dots,t_d)\big)\Big)_+ dt_1 \dots dt_d \le Z.$$

By virtue of (20) and (21) we have

$$\int_{x_{i1}}^{x_{i1+1}} \dots \int_{x_{id}}^{x_{id+1}} \Big(g_{i,j}\big((t_1,\dots,t_d)\big) - f\big((t_1,\dots,t_d)\big)\Big)_+ dt_1 \dots dt_d \le Z, \qquad (23)$$

and

$$\int_{x_{j1-1}}^{x_{j1}} \dots \int_{x_{jd-1}}^{x_{jd}} \Big(g_{i,j}\big((t_1,\dots,t_d)\big) - f\big((t_1,\dots,t_d)\big)\Big)_+ dt_1 \dots dt_d \le Z. \qquad (24)$$

Since (21) implies that $g_{i,j}\big((x_1,\dots,x_d)\big) \le f\big((x_1,\dots,x_d)\big)$ for all $(x_1,\dots,x_d) \in [x_{\zeta 1-1}, x_{\zeta 1}] \times \dots \times [x_{\zeta d-1}, x_{\zeta d}]$, $i+1 < \zeta < j$, $\zeta \ne m$, we conclude from (21), (23) and (24) that

$$\int_{x_{i1}}^{x_{j1}} \dots \int_{x_{id}}^{x_{jd}} \Big(g_{i,j}\big((t_1,\dots,t_d)\big) - f\big((t_1,\dots,t_d)\big)\Big)_+ dt_1 \dots dt_d \le 3Z. \qquad (25)$$

If $\delta\big((t_1,\dots,t_d)\big)$ is a continuous function on $I_2 = [x_{i1}, x_{j1}] \times \dots \times [x_{id}, x_{jd}]$, then we have

$$\left\| \int_{x_{i1}}^{(\cdot)} \dots \int_{x_{id}}^{(\cdot)} \delta\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d \right\|_{I_2} \le \left| \int_{x_{i1}}^{x_{j1}} \dots \int_{x_{id}}^{x_{jd}} \delta\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d \right| + \int_{x_{i1}}^{x_{j1}} \dots \int_{x_{id}}^{x_{jd}} \delta\big((t_1,\dots,t_d)\big)_+ dt_1 \dots dt_d. \qquad (26)$$

Indeed , for $x_{i\ell} < x_\ell < x_{j\ell}$, $\ell = 1,\dots,d$, when $\int_{x_{i1}}^{x_1} \dots \int_{x_{id}}^{x_d} \delta\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d \ge 0$, then

$$0 \le \int_{x_{i1}}^{x_1} \dots \int_{x_{id}}^{x_d} \delta\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d \le \int_{x_{i1}}^{x_1} \dots \int_{x_{id}}^{x_d} \delta_+\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d \le \int_{x_{i1}}^{x_{j1}} \dots \int_{x_{id}}^{x_{jd}} \delta_+\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d.$$

On the other hand, if $\int_{x_{i1}}^{x_1} \dots \int_{x_{id}}^{x_d} \delta\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d < 0$, then

$$\left| \int_{x_{i1}}^{x_1} \dots \int_{x_{id}}^{x_d} \delta\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d \right| \le \int_{x_{i1}}^{x_1} \dots \int_{x_{id}}^{x_d} \delta_-\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d \le \int_{x_{i1}}^{x_{j1}} \dots \int_{x_{id}}^{x_{jd}} \delta_-\big((t_1,\dots,t_d)\big) dt_1 \dots dt_d$$

$$= -\int_{x_{i1}}^{x_{j1}} \dots \int_{x_{id}}^{x_{jd}} \delta\big((t_1, \dots, t_d)\big) dt_1 \dots dt_d$$
$$+ \int_{x_{i1}}^{x_{j1}} \dots \int_{x_{id}}^{x_{jd}} \delta\big((t_1, \dots, t_d)\big)_+ dt_1 \dots dt_d.$$

This completes the proof of (26). Then let

$$\Delta_{i,j}(\cdot) = \int_{x_{i1}}^{(\cdot)} \dots \int_{x_{id}}^{(\cdot)} \Big(g_{i,j}\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big) dt_1 \dots dt_d,$$

using (25) we get

$$\|\Delta_{i,j}\|_{I_2} \leq \left|\Delta_{i,j}\Big(\big(x_{j1}, \dots, x_{jd}\big)\Big)\right| + 3Z. \qquad (27)$$

Lemma 5. If $1 \leq i \leq n-2$ is an integer. Then, there is an integer $i+1 \leq j \leq n-1$ and a multifunction $g^*_{i,j} \in A_{i,j}$, satisfying

$$\Delta^*_{i,j}(\cdot) = \int_{x_{i1}}^{(\cdot)} \dots \int_{x_{id}}^{(\cdot)} \Big(g^*_{i,j}\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big) dt_1 \dots dt_d,$$

and

$$\|\Delta^*_{i,j}\|_{I_2} \leq 12Z. \qquad (28)$$

When $j < n-1$, we have

$$\Delta^*_{i,j}\Big(\big(x_{j1}, \dots, x_{jd}\big)\Big) < 0. \qquad (29)$$

Proof. assume $\Delta_{i,n-1}\big((x_{n1-1}, \dots, x_{nd-1})\big) \geq 0$, therefore (25), $\Delta_{i,n-1}\big((x_{n1-1}, \dots, x_{nd-1})\big) \leq 3Z$, and let $g^*_{i,n-1} = g_{i,n-1}$, we get (28) by (27). at the other side , at least one of $\Delta_{i,i+r}\big((x_{i1+r}, \dots, x_{id+r})\big)$, $1 \leq r \leq n-i-1$, is negative. When $1 \leq r \leq n-i-1$, $-6Z \leq \Delta_{i,i+r}\big((x_{i1+r}, \dots, x_{id+r})\big) < 0$, then let $j = i + r$ also $g^*_{i,j} = g_{i,j}$. And (29) is proved , also by (27), we get (28). At last , if all the negative numbers above $< -6Z$, let $1 \leq r \leq n-i-1$, be small and satisfy $\Delta_{i,i+r}\big((x_{i1+r}, \dots, x_{id+r})\big) < -6Z$. Then for , $r \geq 2$, since $g_{i,i+1}\big((x_1, \dots, x_d)\big) = s((x_1, \dots, x_d))$, $x \in [x_{i1}, x_{i1+1}] \times \dots \times [x_{id}, x_{id+1}]$, whence $\big|\Delta_{i,i+1}\big((x_{i1+1}, \dots, x_{id+1})\big)\big| \leq Z$. Let $j = i + r$, and $p = s\,|_{[x_{j1-1}, x_{j1}] \times \dots \times [x_{jd-1}, x_{jd}]}$. Then by (18) ,

$$\left\|\int_{x_{j1-1}}^{(\cdot)} \dots \int_{x_{jd-1}}^{(\cdot)} \Big(p\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big) dt_1 \dots dt_d\right\|_{I_3} \leq Z, \qquad (30)$$

$I_3 = \big[x_{j1-1}, x_{j1}\big] \times \dots \times \big[x_{jd-1}, x_{jd}\big]$.

Denote

$$\tilde{g}_{i,j}\big((x_1, \dots, x_d)\big) = \begin{cases} g_{i,j-1}\big((x_1, \dots, x_d)\big), & \text{if } x \in [x_{i1}, x_{j1-1}\big) \times \dots \times \big[x_{id}, x_{jd-1}\big), \\ p\big((x_1, \dots, x_d)\big), & \text{if } x \in \big[x_{j1-1}, x_{j1}\big] \times \dots \times \big[x_{jd-1}, x_{jd}\big]. \end{cases}$$

And by $\tilde{g}_{i,j} \in A_{i,j-1}$ and $\tilde{g}_{i,j} \in A_{j-1,j}$, we have $\tilde{g}_{i,j} \in A_{i,j}$. Assume $g^*_{i,j}\big((x_1, \dots, x_d)\big) = \lambda g_{i,j}\big((x_1, \dots, x_d)\big) + (1-\lambda)\tilde{g}_{i,j}\big((x_1, \dots, x_d)\big)$, $x = (x_1, \dots, x_d) \in I_2 = \big[x_{i1}, x_{j1}\big] \times \dots \times \big[x_{id}, x_{jd}\big]$ where $\lambda = 6Z\left|\Delta_{i,j}\Big(\big(x_{j1}, \dots, x_{jd}\big)\Big)\right|^{-1}$. Clearly

$\lambda \in (0,1) \times \ldots \times (0,1)$, therefore $g^*_{i,j} \in A_{i,j}$. The definition of r leads to $0 \leq \Delta_{i,j-1}\left((x_{j1-1}, \ldots, x_{jd-1})\right) \leq 3Z$.

Then using (27)

$$\left\|\int_{x_{i1}}^{(\cdot)} \ldots \int_{x_{id}}^{(\cdot)} \left(\tilde{g}_{i,j}\big((t_1, \ldots, t_d)\big) - f\big((t_1, \ldots, t_d)\big)\right) dt_1 \ldots dt_d\right\|_{I_4}$$

$$= \left\|\Delta_{i,j-1}\right\|_{I_4}, \quad I_4 = [x_{i1}, x_{j1-1}] \times \ldots \times [x_{id}, x_{jd-1}],$$

$$\leq \left|\Delta_{i,j-1}\left((x_{j1-1}, \ldots, x_{jd-1})\right)\right| + 3Z \leq 6Z.$$

Also by (30)

$$\left|\int_{x_{i1}}^{x_1} \ldots \int_{x_{id}}^{x_d} \left(\tilde{g}_{i,j}\big((t_1, \ldots, t_d)\big) - f\big((t_1, \ldots, t_d)\big)\right) dt_1 \ldots dt_d\right|$$

$$= \left|\Delta_{i,j-1}\left((x_{j1-1}, \ldots, x_{jd-1})\right) + \int_{x_{j1-1}}^{x_1} \ldots \int_{x_{jd-1}}^{x_d} \left(p\big((t_1, \ldots, t_d)\big) - f\big((t_1, \ldots, t_d)\big)\right) dt_1 \ldots dt_d\right|$$

$$\leq \left|\Delta_{i,j-1}\left((x_{j1-1}, \ldots, x_{jd-1})\right)\right| + Z \leq 4Z,$$

$$x = (x_1, \ldots, x_d) \in I_3 = [x_{j1-1}, x_{j1}] \times \ldots \times [x_{jd-1}, x_{jd}].$$

Therefore

$$\left\|\int_{x_{i1}}^{(\cdot)} \ldots \int_{x_{id}}^{(\cdot)} \left(\tilde{g}_{i,j}\big((t_1, \ldots, t_d)\big) - f\big((t_1, \ldots, t_d)\big)\right) dt_1 \ldots dt_d\right\|_{I_2} \leq 6Z. \qquad (31)$$

In particular ,

$$\Delta^*_{i,j}\left((x_{j1}, \ldots, x_{jd})\right) = \lambda \Delta_{i,j}\left((x_{j1}, \ldots, x_{jd})\right)$$

$$+(1-\lambda) \int_{x_{i1}}^{x_{j1}} \ldots \int_{x_{id}}^{x_{jd}} \left(\tilde{g}_{i,j}\big((t_1, \ldots, t_d)\big) - f\big((t_1, \ldots, t_d)\big)\right) dt_1 \ldots dt_d$$

$$\leq -6Z + (1-\lambda)6Z < 0,$$

therefore (29) is proved . Then using (27) and (31) ,to obtain

$$\left\|\Delta^*_{i,j}\right\|_{I_2} \leq \lambda \left\|\Delta_{i,j}\right\|_{I_2}$$

$$+(1-\lambda) \left\|\int_{x_{i1}}^{(\cdot)} \ldots \int_{x_{id}}^{(\cdot)} \left(\tilde{g}_{i,j}\big((t_1, \ldots, t_d)\big) - f\big((t_1, \ldots, t_d)\big)\right) dt_1 \ldots dt_d\right\|_{I_2}$$

$\leq \lambda\left(\left|\Delta_{i,j}\left((x_{j1}, \ldots, x_{jd})\right)\right| + 3Z\right) + 6(1-\lambda)Z = 6Z + 6Z - 3\lambda Z \leq 12Z$. Then (28) is proved , which completes the proof .

Proof of the Theorem. Note that

$$S\big((x_1, \ldots, x_d)\big) = F\big((x_{11}, \ldots, x_{1d})\big) + \int_{x_{11}}^{x_1} \ldots \int_{x_{1d}}^{x_d} \bar{g}\big((t_1, \ldots, t_d)\big) dt_1 \ldots dt_d, x \in I$$

where

$$\bar{g}\big((t_1, \ldots, t_d)\big) = \begin{cases} s\big((t_1, \ldots, t_d)\big), & \text{if } t \in [x_{01}, x_{11}) \times \ldots \times [x_{0d}, x_{1d}) \\ & \quad \cup (x_{n1-1}, x_{n1}] \times \ldots \times (x_{nd-1}, x_{nd}] \\ g\big((t_1, \ldots, t_d)\big), & \text{if } t \in [x_{11}, x_{n1-1}] \times \ldots \times [x_{1d}, x_{nd-1}], \end{cases}$$

is in $A_{1,n-1}$. This shows that S is 3-monotone . Now let us define $g(t_1, \dots, t_d)$ using the induction. Then using Lemma1 for $J = [x_{i1-1}, x_{i1}] \times \dots \times [x_{id-1}, x_{id}]$, $2 \le i \le n-1$, where $q = s\,|_J$, then $p \in A_{i-1,i}$. Also, we have if $g \in A_{i,j}$, $1 \le i < j < \zeta \le n-1$ and $g \in A_{j,\zeta}$ then $g \in A_{i,\zeta}$. We can define g using induction. And using Lemma 1 for $[x_{11}, x_{21}] \times \dots \times [x_{1d}, x_{2d}]$,with $q = s\,|_{[x_{11},x_{21}]\times\dots\times[x_{1d},x_{2d}]}$, we get a multipolynomial $p \in A_{1,2}$. and let $g\big((x_1, \dots, x_d)\big) = p\big((x_1, \dots, x_d)\big), (x_1, \dots, x_d) \in [x_{11}, x_{21}] \times \dots \times [x_{1d}, x_{2d}]$. Assume the g defined on $[x_{11}, x_{i1}] \times \dots \times [x_{1d}, x_{id}]$ for some $2 \le i \le n-2$, it belongs to $A_{1,i}$, with all $x \in [x_{11}, x_{i1}] \times \dots \times [x_{1d}, x_{id}]$, that

$$\left| \int_{x_{11}}^{x_1} \dots \int_{x_{1d}}^{x_d} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \right| \le 24Z , \qquad (32)$$

where Z is defined in (18) , then

$$\left| \int_{x_{11}}^{x_{i1}} \dots \int_{x_{1d}}^{x_{id}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \right| \le 12Z. \qquad (33)$$

Then we define g on some $I_2 = \left[x_{i1}, x_{j1}\right] \times \dots \times \left[x_{id}, x_{jd}\right], i < j \le n-1$, so that $g \in A_{i,j}$, (32) is true, on the interval $\left[x_{11}, x_{j1}\right] \times \dots \times \left[x_{1d}, x_{jd}\right]$ and when $j < n-1$, we have

$$\left| \int_{x_{11}}^{x_{j1}} \dots \int_{x_{1d}}^{x_{jd}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \right| \le 12Z. \qquad (34)$$

If

$$\int_{x_{11}}^{x_{i1}} \dots \int_{x_{1d}}^{x_{id}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \le 0, \qquad (35)$$

then if $j = i + 1$ using Lemma 1 for$I_3 = \left[x_{j1-1}, x_{j1}\right] \times \dots \times \left[x_{jd-1}, x_{jd}\right]$ and $q = s\,|_{I_3}$. let $g((x_1, \dots, x_d)) := p((x_1, \dots, x_d))$, $x = (x_1, \dots, x_d) \in I_3$ where p is the required multipolynomial. For $x = (x_1, \dots, x_d) \in I_3$, using (33) and (5) ,we get

$$\begin{aligned} &\left| \int_{x_{11}}^{x_1} \dots \int_{x_{1d}}^{x_d} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \right| \\ &\qquad \le \left| \int_{x_{11}}^{x_{i1}} \dots \int_{x_{1d}}^{x_{id}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \right| \\ &\qquad + \left| \int_{x_{i1}}^{x_1} \dots \int_{x_{id}}^{x_d} \Big(p\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \right| \\ &\qquad \le 12Z + 2Z \le 14Z. \end{aligned}$$

Hence, combining with (32) for $x = (x_1, \dots, x_d) \in [x_{11}, x_{i1}] \times \dots \times [x_{1d}, x_{id}]$, we see that (32) holds for $x = (x_1, \dots, x_d) \in \left[x_{11}, x_{j1}\right] \times \dots \times \left[x_{1d}, x_{jd}\right]$.

Moreover, (6) implies that

$$0 \le \int_{x_{j1-1}}^{x_{j1}} \dots \int_{x_{jd-1}}^{x_{jd}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \le 2Z,$$

which together with (33) and (35) yield

$$-12Z \le \int_{x_{11}}^{x_{j1}} \dots \int_{x_{1d}}^{x_{jd}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big) \Big)\, dt_1 \dots dt_d \le 12Z.$$

So (34) is proved . Otherwise,

$$\int_{x_{11}}^{x_{i1}} \dots \int_{x_{1d}}^{x_{id}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big)\, dt_1 \dots dt_d > 0. \qquad (36)$$

using Lemma 5, to get some integer $j,\ i + 1 \le j \le n-1$, and $g_{i,j}^{*}$is in$A_{i,j}$, and satisfy (28) and (29) when $j < n - 1$. Let $g((x_1, \dots, x_d)) = g_{i,j}^{*}\big((x_1, \dots, x_d)\big)$ $(x_1, \dots, x_d) \in I_2$, when $j = n-1$, using (28) to get (32) when $(x_1, \dots, x_d) \in [x_{11}, x_{n1-1}] \times \dots \times [x_{1d}, x_{nd-1}]$. Then , if $(x_1, \dots, x_d) \in I_2$, using (28) and (33), we get

$$\begin{aligned}
&\left|\int_{x_{11}}^{x_1} \dots \int_{x_{1d}}^{x_d} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big)\, dt_1 \dots dt_d\right| \\
&\qquad \le \left|\int_{x_{11}}^{x_{i1}} \dots \int_{x_{1d}}^{x_{id}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big)\, dt_1 \dots dt_d\right| \\
&\qquad + \left|\int_{x_{i1}}^{x_1} \dots \int_{x_{id}}^{x_d} \Big(g_{i,j}^{*}\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big)\, dt_1 \dots dt_d\right| \\
&\qquad \le 12Z + 12Z \le 24Z.
\end{aligned}$$

Hence, (32) holds for $(x_1, \dots, x_d) \in \big[x_{11}, x_{j1}\big] \times \dots \times \big[x_{1d}, x_{jd}\big]$. Also, by (36) and (33),

$$0 < \int_{x_{11}}^{x_{i1}} \dots \int_{x_{1d}}^{x_{id}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big)\, dt_1 \dots dt_d \le 12Z,$$

which combined with (28) and (29) give

$$-12Z < \int_{x_{11}}^{x_{j1}} \dots \int_{x_{1d}}^{x_{jd}} \Big(g\big((t_1, \dots, t_d)\big) - f\big((t_1, \dots, t_d)\big)\Big)\, dt_1 \dots dt_d \le 12Z.$$

Thus the proof of (34) is complete.

Then using (32) and definition of S to prove (2)

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