Commutativity Results on Prime Rings With Generalized Derivations

Dr. A. H. Majeed
Science College, University of Baghdad / Baghdad.
Email:ahmajeed6@yahoo.com
Shaima'a B. Yass
Science College, University of Baghdad / Baghdad

Received on: 29/12/2014 & Accepted on: 20/1/2016

ABSTRACT
Let R be a prime ring. For nonzero generalized derivations F and G associated with the same derivation d, we prove that if $d\neq 0$, then R is commutative, if any one of the following conditions hold: (1) $[F(x), G(y)] = 0$, (2) $F(x)oG(y) = 0$, (3) $F(x)oG(y) = \pm xoy$, (4) $[F(x), G(y)] = \pm[x, y]$, (5) $[F(x), G(y)] = \pm xoy$, (6) $F(x)oG(y) = \pm[x, y]$, for all $x, y \in R$, where F will always denote onto map.
Keywords: prime rings, derivations, generalized derivations.

INTRODUCTION
Throughout, R will denote an associative ring with $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol xoy denotes for anti-commutator xy + yx. Recall that a ring R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$. An additive mapping $d: R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

Brešar [3], introduced the notation of generalized derivation in rings as an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$, such that:

$F(xy) = F(x)y + xd(y)$, for all $x, y \in R$

It is well known that the concept of generalized derivation includes the concept of derivation and left multiplier (i.e. an additive mapping $T : R \longrightarrow R$ such that $T(xy) = T(x)$y, for all $x,y\in R$).

The study of the commutativity of prime rings with derivation was initiated by E. C. Posner [4]. And several authors Ashraf, Ali, Quadric, Rehman, and others extended the mention results for a generalized derivation ([1, 2, 5, and 6]).

In [1], Ashraf et al., studied the commutativity of a prime ring admitting a generalized derivation satisfying some conditions. In this paper, we will extend the notation of a generalized derivation F associated with derivation d, to, two generalized derivations F and G associated with the same derivation d, as a new idea, to obtain the commutativity of prime rings under certain conditions, where F will always denote onto map.

The Results
Theorem 1:
Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $[F(x), G(y)] = 0$, for all $x, y \in R$, then either d = 0 or $F(R) \subseteq Z(R)$.

Proof:
Replace y by yz in our hypotheses holds
$G(y)[F(x), z] + y[F(x), d(z)] + [F(x), y]d(z) = 0$,
for all $x, y, z \in R$ …(1)

https://doi.org/10.30684/etj.34.2B.16
2412-0758/University of Technology-Iraq, Baghdad, Iraq
This is an open access article under the CC BY 4.0 license http://creativecommons.org/licenses/by/4.0

Replace z by $zF(x)$ in (1), we get:
$G(y)[F(x), zF(x)] + y[F(x), d(zF(x))] + [F(x), y]d(zF(x)) = 0,$
for all $x, y, z \in R$ …(2)
This can be rewritten as:
$G(y)[F(x), z]F(x) + y[F(x), d(z)]F(x) + yz[F(x), d(F(x))] + y[F(x), z]d(F(x)) + [F(x), y]d(z)F(x) + [F(x), y]zd(F(x)) = 0,$
for all $x, y, z \in R$ …(3)
Right multiplication of (1) by $F(x)$, to get:
$G(y)[F(x), z]F(x) + y[F(x), d(z)]F(x) + [F(x), y]d(z)F(x) = 0,$
for all $x, y, z \in R$ ….(4)
From (3) and (4) one obtains:
$yz[F(x), d(F(x))] + y[F(x), z]d(F(x)) + [F(x), y]zd(F(x)) = 0,$
for all $x, y, z \in R$ ….(5)
Now, replace y by ty in (5), to get:
$tyz[F(x), d(F(x))] + ty[F(x), z]d(F(x)) + t[F(x), y]zd(F(x)) + [F(x), t]yzd(F(x)) = 0,$
for all $x, y, t, z \in R$ ...(6)
Left multiplication of (5) by t, gives:
$tyz[F(x), d(F(x))] + ty[F(x), z]d(F(x)) + t[F(x), y]zd(F(x)) =0,$
for all $x, y, t, z \in R$ …(7)
From (6) and (7), we obtain:
$[F(x), t]yzd(F(x)) = 0$, for all $x, y, t, z \in R$ ….(8)
Since R is prime, then we get:
either,
$[F(x), t] = 0$, for all $x, t \in R$
or,
$zd(F(x)) = 0$, for all $x, z \in R$
In the first case, we get $F(R) \subseteq Z(R)$.
And in the second case, we get $Rd(F(x)) = 0$, for all $x \in R$, then R is prime and since F is onto, we get $d = 0$.

One immediately sees that:

Corollary 2:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $[F(x), G(y)] = 0$, for all $x, y \in R$, and if $d \neq 0$, then R is commutative.

Proof:

Using Theorem (1), we get $F(R) \subseteq Z(R)$ and since F is onto, we get R is commutative.

Theorem 3:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $F(x)\text{o}G(y) = 0$, for all $x, y \in R$, then either $d = 0$ or $F(R) \subseteq Z(R)$.

Proof:

By hypotheses, we have:
$F(x)\text{o}G(y) = 0$, for all $x, y \in R$ ….(1)
Replacing y by yz in (1), w get:
$(F(x)\text{o}G(y))z - G(y)[F(x), z] + (F(x)\text{o}y)d(z) - y[F(x), d(z)] = 0,$
for all $x, y, z \in R$ …(2)
From (1) and (2), we obtain:

$(F(x)oy)d(z) - y[F(x), d(z)] - G(y)[F(x), z] = 0$, for all $x, y, z \in R$ ….(3)

Replacing z by $F(x)$ in (3), to get:

$(F(x)oy)d(F(x)) - y[F(x), d(F(x))] = 0$, for all $x, y \in R$ ….(4)

Again, replacing y by zy in (4), gives:

$z(F(x)oy)d(F(x)) + [F(x), z]yd(F(x)) - zy[F(x), d(F(x))] = 0$,

for all $x, y, z \in R$ ….(5)

Left multiplication of (4) by z, to get:

$z(F(x)oy)d(F(x)) - zy[F(x), d(F(x))] = 0$, for all $x, y, z \in R$ …(6)

From (5) and (6), one obtains:

$[F(x), z]yd(F(x)) = 0$, for all $x, y, z \in R$ ….(7)

By primeness of R, we obtain:

either, $[F(x), z] = 0$, for all $x, z \in R$ and hence $F(R) \subseteq Z(R)$.

or, $d(F(x)) = 0$, for all $x \in R$, and since F is onto, we get $d = 0$.

Following a very easy argument by Theorem (3):

Corollary 4:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $F(x)oG(y) = 0$, for all $x, y \in R$, and if $d \neq 0$, then R is commutative.

Another certain conditions with two generalized derivations associated with the same derivation, as follows:

Theorem 5:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that F(x)oG(y) = ±xoy, for all x, y ∈ R, then either $d = 0$ or $F(R) \subseteq Z(R)$.

Proof:

We have:

$F(x)oG(y) = xoy$, for all $x, y \in R$ ….(1)

Replacing y by yz in (1), we get:

$(F(x)oG(y))z - G(y)[F(x), z] + (F(x)oy)d(z) - y[F(x), d(z)] = (xoy)z - y[x, z]$, for all $x, y, z \in R$ ….(2)

Combining (1) and (2), we get:

$-G(y)[F(x), z] + (F(x)oy)d(z) - y[F(x), d(z)] + y[x, z] = 0$,

for all $x, y, z \in R$ ….(3)

Replacing z by $F(x)$ in (3), we get:

$(F(x)oy)d(F(x)) - y[F(x), d(F(x))] + y[x, F(x)] = 0$,

for all $x, y \in R$ ….(4)

Again, replacing y by ry in (4), we obtain:

$$(r(F(x)oy) + [F(x), r]y)d(F(x)) - ry[F(x), d(F(x))] + ry[x, F(x)] = 0,$$

for all $x, y, r \in R$ …(5)

Left multiplication of (4) by r, to get:

$r(F(x)oy)d(F(x)) - ry[F(x), d(F(x))] + ry[x, F(x)] = 0$,

for all $x, y, r \in R$ ….(6)

From (5) and (6), we obtain:

$[F(x), r]yd(F(x)) = 0$, for all $x, y, r \in R$ ….(7)

Since R is prime, we get:

either, $F(R) \subseteq Z(R)$

or, $d(F(x)) = 0$, for all $x \in R$, and since F is onto, we get $d = 0$.

Using the similar techniques, when $F(x)oG(y) = -xoy$, for all $x, y \in R$. Theorem (5), gives a consequence as following:

Corollary 6:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that F(x)oG(y) = ±xoy, for all x, y ∈ R, and if $d \neq 0$, then R is commutative.

Theorem 7:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $[F(x), G(y)] = \pm[x, y]$, for all $x, y \in R$, then either $d = 0$ or $F(R) \subseteq Z(R)$.

Proof:

We have:

$[F(x), G(y)] = [x, y]$, for all $x, y \in R$ …(1)

Replacing y by yz in (1), we get:

$G(y)[F(x), z] + [F(x), G(y)]z + y[F(x), d(z)] + [F(x), y]d(z) = [x, y]z + y[x, z]$, for all $x, y, z \in R$ …(2)

Combining (1) with (2), we get:

$G(y)[F(x), z] + y[F(x), d(z)] + [F(x), y]d(z) - y[x, z] = 0$, for all $x, y, z \in R$ …(3)

Replacing z by $F(x)$ in (3), reduces to:

$y[F(x), d(F(x))] + [F(x), y]d(F(x)) - y[x, F(x)] = 0$, for all $x, y \in R$ …(4)

Again, replace y by ty in (4), to get:

$ty[F(x), d(F(x))] + t[F(x), y]d(F(x)) + [F(x), t]yd(F(x)) - ty[x, F(x)] = 0$, for all $x, y, t \in R$ …(5)

Left multiplication of (4) by t, to get:

$ty[F(x), d(F(x))] + t[F(x), y]d(F(x)) - ty[x, F(x)] = 0$, for all $x, y, t \in R$ (6)

From (5) and (6), one obtains:

$[F(x), t]yd(F(x)) = 0$, for all $x, y, t \in R$ ….(7)

By primeness of R, (7) gives for all $x \in R$

either $F(x) \in Z(R)$

or $d(F(x)) = 0$, and since F is onto, we get $d = 0$.

A slight modification in the proof of the above theorem, if $[F(x), G(y)] = -[x, y]$, for all $x, y \in R$.

The following corollary immediately yields:

Corollary 8:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $[F(x), G(y)] = \pm[x, y]$, for all $x, y \in R$ and if $d \neq 0$, then R is commutative.

We will go on proving the main results as follows:

Theorem 9:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $[F(x), G(y)] = \pm x o y$, for all $x, y \in R$, then either $d = 0$ or $F(R) \subseteq Z(R)$.

Proof:

We have:

$[F(x), G(y)] = x o y$, for all $x, y \in R$ …(1)

Replacing y by yz in (1), we get:

$[F(x), G(y)]z + G(y)[F(x), z] + [F(x), y]d(z) + y[F(x), d(z)] = (xoy)z - y[x, z]$, for all $x, y, z \in R$ …(2)

Combining (1) with (2), we get:

$G(y)[F(x), z] + [F(x), y]d(z) + y[F(x), d(z)] + y[x, z] = 0$, for all $x, y, z \in R$ ….(3)

Replace z by $zF(x)$ in (3), we get:

$G(y)[F(x), z]F(x) + [F(x), y]d(z)F(x) + [F(x), y]zd(F(x)) + y[F(x), d(z)]F(x) + yz[F(x), d(F(x))] + y[F(x), z]d(F(x)) + yz[x, F(x)] + y[x, z]F(x) = 0$, for all $x, y, z \in R$ …(4)

Right multiplication of (3) by $F(x)$, to get:

$G(y)[F(x), z]F(x) + [F(x), y]d(z)F(x) + y[F(x), d(z)]F(x) + y[x, z]F(x) = 0$, for all $x, y, z \in R$ …(5)

From (4) and (5), one obtains:

$[F(x), y]zd(F(x)) + yz[F(x), d(F(x))] + y[F(x), z]d(F(x)) + yz[x, F(x)] = 0$, for all $x, y, z \in R$ …(6)

Now, replace y by ry in (6), we get:

$r[F(x), y]zd(F(x)) + [F(x), r]yzd(F(x)) + ryz[F(x), d(F(x))] + ry[F(x), z]d(F(x)) + ryz[x, F(x)] = 0$, for all $x, y, r, z \in R$ …(7)

Left multiplication of (6) by r, to get:

$r[F(x), y]zd(F(x)) + ryz[F(x), d(F(x))] + ry[F(x), z]d(F(x)) + ryz[x, F(x)] = 0$, for all $x, y, r, z \in R$ ….(8)

From (7) and (8), we get:

$[F(x), r]yzd(F(x)) = 0$, for all $x, y, r, z \in R$ ….(9)

By prime of R, (9) gives

either, $[F(x), r] = 0$, for all $x, r \in R$, and thus $F(R) \subseteq Z(R)$

or $d(F(x)) = 0$, for all $x \in R$ and since F is onto, we get $d = 0$.

In the same way, if $[F(x), G(y)] = -xoy$, for all $x, y \in R$, then also the result holds.

As an immediate consequence of Theorem (9), we obtain the following corollary:

Corollary 10:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $[F(x), G(y)] = \pm xoy$, for all $x, y \in R$ and if $d \neq 0$, then R is commutative.

Now, we will prove the next result with necessary variations as follows:

Theorem 11:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $F(x)oG(y) = \pm[x, y]$, for all $x, y \in R$, then either $d = 0$ or $F(R) \subseteq Z(R)$.

Proof:

We have:

$F(x)oG(y) = [x, y]$, for all $x, y \in R$ …(1)

Replacing y by yz in (1), we get:

$(F(x)oG(y))z - G(y)[F(x), z] + (F(x)oy)d(z) - y[F(x), d(z)] = [x, y]z + y[x, z]$, for all $x, y, z \in R$ …(2)

Combining (1) and (2), we get:

$-G(y)[F(x), z] + (F(x)oy)d(z) - y[F(x), d(z)] - y[x, z] = 0$,

for all $x, y, z \in R$ …(3)

Replacing z by $F(x)$ in (3), reduces to:

$(F(x)oy)d(F(x)) - y[F(x), d(F(x))] - y[x, F(x)] = 0$,

for all $x, y \in R$ ….(4)

Replacing y by ty in (4), we get:

$t(F(x)oy)d(F(x)) + [F(x), t]yd(F(x)) - ty[F(x), d(F(x))] - ty[x, F(x)] = 0$,

for all $x, y, t \in R$ …(5)
Left multiplication of (4) by t, to get:
$t(F(x)oy)d(F(x)) - ty[F(x), d(F(x))] - ty[x, F(x)] = 0$,
for all $x, y, t \in R$ ….(6)
From (5) and (6), we obtain:
$[F(x), t]yd(F(x)) = 0$, for all $x, y, t \in R$ ….(7)
By primeness of R, we get:
either, $[F(x), t] = 0$, for all $x, t \in R$, hence $F(R) \subseteq Z(R)$
or, $d(F(x)) = 0$, for all $x \in R$, and since F is onto, we get $d = 0$.
And similarly, if $F(x)oG(y) = -[x, y]$, for all $x, y \in R$, then either $d = 0$ or $F(R) \subseteq Z(R)$.
This establishes the following corollary:

Corollary 12:

Let R be a prime ring. If R admits nonzero generalized derivations F and G associated with the same derivation d, such that $F(x)oG(y) = \pm[x, y]$, for all $x, y \in R$ and if $d \neq 0$, then R is commutative.

REFERENCES

[1] Ashraf, M., Ali, A. and Rekha, R., "On generalized derivations of prime rings", South-East Bull. Math. 29 (4) (2005), 669-675.

[2] Ashraf, M. Ali , A. and Ali, S., "Some commutativity theorems for rings with generalized derivations", South-East Asian Bull. Math. 31 (2007), 415-421.

[3] Brešar, M., "On the distance of the composition of two derivations to the generalized derivations", Glasgow Math. J. 33 (1991), 89-93.

[4] Posner, E. C.,"Derivations in prime rings", Proc. Amer. Math. Soc. 8 (1957), 1093-1100.

[5] Quadri , M. A., Khan , M. S. and Rehman, N., "Generalized derivations and commutativity of prime rings", Indian J. Pure and Appl. Math. 34 (9) (2003), 1393-1396.

[6] Rehman, N., "On commutativity of rings with generalized derivations", Math. J. Okayama Univ. 44 (2002), 43-49.