On S – Expandable Spaces

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Abstract

In this paper, we recall the definition of expandable space which was introduced by Krajewski (1971). We also study similar definition using semi open sets which is called s–expandable space. Also we will show that if X is s–expandable, Y is locally compact and paracompact then $X \times Y$ is s–expandable.

حول الفضاءات نصف الموسعة

الخلاصة

في هذا البحث نتذكر تعريف الفضاء الموسع والذي قدم من قبل krajewski. كذلك سوف ندرس تعريف مشابه لهذا التعريف باستعمال المجموعات نصف المفتوحه والذي نسميه بالفضاء نصف الموسع بالاضافه الى ذلك سنبين اذا كان X فضاء نصف موسع وY فضاء متراص محليا ومتضام فان حاصل الضرب الديكارتي X×Y يكون فضاء نصف موسع.

Introduction

In this paper we recall the definition of s-open set which was introduced by Levine (1963) and was studied in (Caldas, 2000) (Cueva and Saraf, 2000) and (Dontchev and Ganster, 1998). Also in this paper we introduce the definition of s–expandable as a natural generalization of expandability which was introduced by Krajewski (1971). This paper consists of two sections. In section one we summarize the results concering s – expandability.

We show that every s–paracompact space is s–expandable and establish equivalence under some conditions. Also in section one we introduce various generalizations of the nation of s–expandability. In section two as proved in (Smith and Krajewski, 1971), a product of an expandable space with a locally compact and paracompact space implies expandability of that product, we obtain product theorem concerning s– expandability.

1. Preliminaries and Definitions

Let $\mathcal{U}$ be a family of subsets of a topological space X. For each $A \subset X$,the family$\{U \in \mathcal{U} : U \cap A \neq \phi\}$ is denoted by $(\mathcal{U})_A$; if $A = \{x\}$, then we write $(\mathcal{U})_x$ in place of $(\mathcal{U})_A$. For any set $A \subset X$ and collection $\mathcal{U}$, st(A, $\mathcal{U}$) (the star of $\mathcal{U}$ about A) denotes the set $\cup(\mathcal{U})_A$. If $x \in X$, st($\{x\}$,$\mathcal{U}$) is simply denoted by st(x,$\mathcal{U}$). For any set A, $[A]^n = \{B : B \subset A, |B| = n\}$ *and* $[A]^{<\omega} = \cup\{[A]^n : n < \omega\}$ is the set of all finite subsets of A.

Definition1.1: - (Levine, 1963)

A subset A of a topological space X is called semi open (s – open) set if and only if $A \subseteq Cl(Int(A))$. It is clear that every open set is s – open, but the converse is not true. Also it is clear that the intersection of two s – open sets is not necessary s – open, but the intersection of s – open set with open set is s- open.

Definition 1.2 :- (Krajewski, 1971)

A space X is called expandable if and only if every locally finite collection $\{F_\alpha : \alpha < \kappa\}$ of closed subsets of X has a locally finite collection $\{G_\alpha : \alpha < \kappa\}$ of open subsets such that $F_\alpha \subseteq G_\alpha$ for each $\alpha < \kappa$.

Definition 1.3

A space X is called s-expandable if and only if every locally finite collection $\{F_\alpha : \alpha < \kappa\}$ of subsets of X has a locally finite collection $\{G_\alpha : \alpha < \kappa\}$ of s-open subsets such that $F_\alpha \subseteq G_\alpha$ for each $\alpha < \kappa$.

Theorem 1.4

A space X is s - expandable if and only if for every locally finite collection of closed subsets $\{F_\alpha : \alpha < \kappa\}$ of X there exists a locally finite collection of s – open subsets $\{G_\alpha : \alpha < \kappa\}$ such that $F_\alpha \subseteq G_\alpha$ for every $\alpha < \kappa$.

Proof

$(\rightarrow)$ Is clear.

$(\leftarrow)$ Let $\{F_\alpha : \alpha < \kappa\}$ be a locally finite collection of subsets of X. Then $\{\overline{F}_\alpha : \alpha < \kappa\}$ is a locally finite of closed subsets of X. Therefore $\{\overline{F}_\alpha : \alpha < \kappa\}$ has a locally finite collection $\{G_\alpha : \alpha < \kappa\}$ of s – open subsets of X such that $\overline{F}_\alpha \subseteq G_\alpha$ for each $\alpha < \kappa$. Hence $F_\alpha \subseteq \overline{F_\alpha} \subseteq G_\alpha$ for each $\alpha < \kappa$.

Note 1.5

It is clear that every expandable space is s – expandable.

Theorem 1.6

A space X is s - expandable if and only if for each locally finite collection $\{F_\alpha : \alpha < \kappa\}$ of closed sets in X there exists a locally finite s – open cover $\mathcal{G}$ of X such that for each $G \in \mathcal{G}$, $\{\alpha < \kappa : G \cap F_\alpha \neq \phi\}$ is finite.

Proof

$(\rightarrow)$ Let $\{F_\alpha : \alpha < \kappa\}$ be a locally finite collection of closed sets. Let $D = [\kappa]^{<\omega}$. For each $d \in D$, let $U(d) = X \setminus \cup\{F_\alpha : \alpha \in \kappa \mid d\}$. Then $\{U(d) : d \in D\}$ is an open cover of X. Let $B(\phi) = U(\phi)\, and \quad B(d) = U(d) \cap \bigcap_{\alpha \in d} F_a \quad , \ d \in D \mid \{\phi\}$.

Then $\{B(d) : d \in D\}$ is a locally finite cover of X with $B(d) \subset U(d)$ for each $d \in D$.

There is a locally finite s– open cover $\{G(d) : d \in D\}$ of X such that $B(d) \subset G(d)$ for each $d \in D$. We may assume $G(d) \subset U(d)$ for each $d \in D$. Since for each $\alpha \in \kappa \setminus d$, $G(d) \cap F_\alpha \subset U(d) \cap F_\alpha \subset U(d) \cap \bigcup_{\alpha \in \kappa | d} F_\alpha = \phi$.

Then for each $d \in D$, $\{\alpha < \kappa : G(d) \cap F_\alpha \neq \phi\} \subset d$, so it is finite.

$(\leftarrow)$ Suppose $\{F_\alpha : \alpha < \kappa\}$ is a locally finite collection of closed sets in X and $\mathcal{G}$ is a locally finite s–open cover of X such that for each $G \in \mathcal{G}$, $A(G) = \{\alpha < \kappa : G \cap F_\alpha \neq \phi\}$ is finite. For each $\alpha < \kappa$, let $U_\alpha = st(F_\alpha, \mathcal{G})$. Then $F_\alpha \subset U_\alpha$ for each $\alpha < \kappa$. Let $x \in X$. There is a neighborhood H of x such that $|(\mathcal{G})_H| < \omega$. Let $C = \{\alpha < \kappa : H \cap U_\alpha \neq \phi\}$. Then $C \subset \cup\{A(G) : G \in (\mathcal{G})_H\}$.

Therefore $|C| < \omega$. It follows that $\{U_\alpha : \alpha < \kappa\}$ is a locally finite and X is s – expandable.

Definition 1.7 (Raad Al-Abdulla, 1992)

A space X is called s – paracompact if every open cover of X has a locally finite s - open refinement.

Theorem 1.8

Every s – paracompact space is s – expandable.

Proof:-

Suppose $\{F_\alpha : \alpha < \kappa\}$ is a locally finite collection of closed sets in s – paracompact space X. let $D = [k]^{<\omega}$. For each $d \in D$ define $G(d) = X \setminus \cup\{F_\alpha : \alpha \in \kappa \mid d\}$. For each $x \in X$, let $d(x) = \{\alpha < \kappa : x \in F_\alpha\}$, then $x \in G(d(x))$, and so $\mathcal{G} = \{G(d) : d \in D\}$ is an open cover of X . Let $\mathcal{W}$ be a locally finite s – open refinement of $\mathcal{G}$. For each $\alpha < \kappa$, let U_α= st(F_α,$\mathcal{W}$). Then $F_\alpha \subset U_\alpha$ for each $\alpha < \kappa$. To see $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ is locally finite, let $x \in X$. There is a neighborhood V of x with $(\mathcal{W})_V = \{W_1, ..., W_n\}$ is finite. For each $i \le n$ there exists $d_i \in D$, $W_i \subset G(d_i)$. If $C = \{\alpha < \kappa : V \cap U_\alpha \ne \phi\}$, then $C \subset \bigcup_{i=1}^{n} d_i$, so C is finite. Hence $\mathcal{U}$ is locally finite.

Definition 1.9

A space X is called ω - s - expandable if and only if every locally finite collection $\{F_i : i < \omega\}$ of subsets of X has a locally finite collection $\{G_i : i < \omega\}$ of s - open subsets of X such that $F_i \subseteq G_i$ for each $i < \omega$.

Theorem 1.10

A space X is ω - s - expandable if and only if every countable open cover of X has a locally finite s – open refinement.

Proof:-

($\rightarrow$) Let $\mathcal{U} = \{U_i : i < \omega\}$ be a countable open cover of X . Let $F_1 = U_1$ and $F_i = U_i \setminus \bigcup_{k=1}^{i-1} U_k$ for each $i = 2,3,...,$ then the collection $\{F_i : i < \omega\}$ is a locally finite cover of X . Since X is ω - s - expandable , there is a locally finite s – open collection $\{G_i : i < \omega\}$ such that $F_i \subseteq G_i$ for each i . Let $V_i = U_i \cap G_i$ for each i , then the collection $\{V_i : i < \omega\}$ is a locally finite s – open refinement of $\mathcal{U}$.

($\leftarrow$) let $\mathcal{F} = \{F_i : i < \omega\}$ be a locally finite closed collection of X . Let $U_i = X \setminus \bigcup_{j=i+1}^{\infty} F_j$, $i < \omega$. Now U_i is open , U_i meets only finitely many elements of $\mathcal{F}$, and $\{U_i : i < \omega\}$ covers X . Then there is a locally finite s – open refinement $\mathcal{V} = \{V_i : i < \omega\}$ of $\{U_i : i < \omega\}$. set $G_i =$ st(F_i,$\mathcal{V}$)$= \cup\{ V \in \mathcal{V} : V \cap F_i \ne \phi\}$, $i < \omega$. Clearly $F_i \subseteq G_i$ and G_i is s – open for each $i < \omega$.

We claim that $\{G_i : i < \omega\}$ is locally finite. Each $x \in X$ belongs to an open set O which meets only finitely many members of $\mathcal{V}$. Thus $O \cap G_i \neq \phi$ if and only if $O \cap V \neq \phi$ such that $V \cap F_i \neq \phi$ for some $i < \omega$. Since V contained in some U_i, so it meets only finitely many F_i. Thus $\{G_i : i < \omega\}$ is locally finite.

Corollary 1.11

If a space X is s - expandable then every countable open cover of X has a locally finite s – open refinement.

The result of this Corollary follows from Theorem.1.10.

Definition 1.12

A space X is called countably s–paracompact if and only if every countable open cover of X has a locally finite s–open refinement.

It is clear that every s–paracompact space is countable s–paracompact and a space X is ω - s - expandable if and only if it is countably s – paracompact.

Theorem 1.13

If every open cover of a space X has $\sigma-$locally finite open refinement, then X is s– paracompact if and only if X is countably s–paracompact.

Proof

($\rightarrow$) Is clear

($\leftarrow$) Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ be an open cover of X. Then there exists an open refinement $\mathcal{W} = \bigcup_{n<\omega} \mathcal{W}_n$ of $\mathcal{U}$ such that for each $n < \omega$ and each $x \in X$ there is a neighborhood G_n of X such that $|(\mathcal{W}_n)_{G_n}| < \omega$. Since $\{\cup \mathcal{W}_n : n < \omega\}$ is a countable open cover of X, there exists $\{V_n : n < \omega\}$ an s – open refinement of $\{\cup \mathcal{W}_n : n < \omega\}$ such that $V_n \subset \cup \mathcal{W}_n$ and for each $x \in X$ there is a neighborhood O of x such that $|\{n < \omega : O \cap V_n \neq \phi\}| < \omega$. Let $\mathcal{G} = \{V_n \cap W : W \in \mathcal{W}_n, n < \omega\}$. Hence $\mathcal{G}$ is an s– open refinement of $\mathcal{U}$. For each $x \in X$ there exists a neighborhood O of x such that $\{n < \omega : O \cap V_n \neq \phi\} = \{n_1, ..., n_k\}$. For each $i = 1, ..., k$ there is a neighborhood G_i of x such that $|(\mathcal{W}_{n_i})_{G_i}| < \omega$. Let $H = O \cap \bigcap_{i=1}^{k} G_i$. Then H is a neighborhood of x. Let $C = \{(n, W) : H \cap (V_n \cap W) \neq \phi, n < \omega, W \in \mathcal{W}_n\}$. Then $C \subset \bigcup_{i=1}^{k} (\{n_i\} \times (\mathcal{W}_{n_i})_{G_i})$. Therefore $|C| \leq \sum_{i=1}^{k} |(\mathcal{W}_{n_i})_{G_i}| < \omega$. Thus X is s – paracompact.

Corollary 1.14

If every open cover of a space X has $\sigma-$ locally finite open refinement , then the following are equivalent:

(One) X is s – paracompact

(Two) X is countably s – paracompact.

(Three) X is s – expandable.

(Four) X is ω - s – expandable.

The result of this Corollary follows from Theorems 1.8 , 1.10 , 1.13 and Corollary 1.11.

2. Products of two spaces

By a space we mean a Hausdorff topological space.

Lemma 2.1 [Engelking, 1998) **Theorem 3.10.3]**

For every space X the following conditions are equivalent:-

(a) The space X is countably compact.
(b) Every locally finite family of non – empty subsets of X is finite.
(c) Every locally finite family of one – point subsets of X is finite.
(d) Every infinite subset of X has an accumulation point.
(e) Every countably infinite subset of X has an accumulation point.

Lemma 2.2 [Engelking, 1998) **, Lemma 3.1.15]**

If A is a compact subspace of a space X and y a point of a space Y, then for every open set $W \subset X \times Y$ containing $A \times \{y\}$ there exist open sets $U \subset X$ and $V \subset Y$ such that $A \times \{y\} \subset U \times V \subset W$.

Lemma 2.3

Assume that Y has two covers $\{C_\alpha : \alpha < \kappa\}$ and $\{U_\alpha : \alpha < \kappa\}$ such that $\{U_\alpha : \alpha < \kappa\}$ is locally finite and for all $\alpha < \kappa, C_\alpha$ is compact and U_α is open with $C_\alpha \subset U_\alpha$. If X is s – expandable , then $X \times Y$ is s – expandable.

Proof

Let $\mathcal{F} = \{F_s : s \in D\}$be a locally finite family of closed subsets of $X \times Y$. For each $\alpha < \kappa$ and $s \in D$ define $p(\alpha, s) = \pi[F_s \cap (X \times C_\alpha)]$ where $\pi : X \times Y \to X$ be the projection. For any fixed $\alpha < \kappa$, $p_\alpha = \{p(\alpha, s) : s \in D\}$ is locally finite in X. [let $x \in X$. Since $\{x\} \times C_\alpha$ is compact , by [Lemma2.1(ii)] $D_0 = \{s \in S : (\{x\} \times C_\alpha) \cap F_s \neq \phi\}$ is finite. Since $(\{x\} \times C_\alpha) \cap (\bigcup_{s \in D|D_0} F_s) = \phi$, by [Lemma 2.2] , there exist an open set G in X and an open set H in Y such that $x \in G$, $G_\alpha \subset H$ and $(G \times H) \cap (\bigcup_{s \in D|D_0} F_s) = \phi$. Let $B = \{s \in D : G \cap p(\alpha, s) \neq \phi\}$. Then $B \subset D_0$. Thus B is finite and p_α is locally finite]. Since X is s–expandable, there is a family $\{W(\alpha, s) : s \in D\}$ of locally finite s – open subsets of X such that $p(\alpha, s) \subset W(\alpha, s)$ for all $s \in D$. For each $s \in D$ define $G_s = \bigcup_{\alpha < \kappa} (W(\alpha, s) \times U_\alpha)$. Since $\{C_\alpha : \alpha < \kappa\}$ covers Y, $F_\alpha \subset G_s$. It is clear that G_s is s – open. Now let $(x, y) \in X \times Y$. Then for G_1 a neighborhood of y, $A_0 = \{\alpha < k : G_1 \cap U_\alpha \neq \phi\}$ is finite. For each $\alpha \in A_0$ there is G_α a neighborhood of x such that if $D_\alpha = \{s \in D : G_\alpha \cap W(\alpha, s) \neq \phi\}$, then $|D_\alpha| < \omega$. Let$C = \left\{s \in D : ((\bigcap_{\alpha \in A_0} G_\alpha) \times G_1) \cap G_s \neq \phi\right\}$. Then $C \subset \bigcup_{\alpha \in A_0} D_\alpha$. It follows that $|C| < \omega$. Therefore $X \times Y$ is s – expandable.

Theorem 2.4

If X is s – expandable and Y is locally compact and paracompact, then $X \times Y$ is s – expandable.

Proof

For each $y \in Y$ there exists U_y a neighborhood of y such that $\overline{U_y}$ is compact. Then $\mathcal{U} = \{U_y : y \in Y\}$ is an open cover of Y. Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$, then $\mathcal{U}$

has a locally finite open refinement $\mathcal{V} = \{V_{\alpha} : \alpha < \kappa\}$. Since Y is T_2 paracompact space, then it is normal. Hence there exists an open cover $\mathcal{G} = \{G_{\alpha} : \alpha < \kappa\}$ of Y such that $\overline{G_{\alpha}} \subset V_{\alpha}$. Therefore $\{\overline{G_{\alpha}} : \alpha < \kappa\}$ and $\{V_{\alpha} : \alpha < \kappa\}$ are two covers of Y satisfy the conditions of Lemma 2.3. Therefore $X \times Y$ is s – expandable.

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