On The Mixed *r* Th Modulus of Smoothness

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Abstract

We introduce the mixed 1st modulus of smoothness of functions in $\mathcal{L}p(X)$, for p<1, for Peano continuum X. Then we define a mixed rth modulus of smoothness of functions in $\mathcal{L}p(X)$. Some properties and direct theorems for these moduli of smoothness are proved.
Key words. Mixed modulus of smoothness. Degree of approximation. Direct theorem.

الخلاصة

قدمنا في هذا البحث مقياس النعومة المختلط من الرتبتين الأولى و الرتبة r للدوال في الفضاءات $\mathcal{L}p(X)$ عندما p<1, مع بعض خصائصهما. كذلك برهنا بعض المبرهنات المباشرة في التقريب باستخدام هذين المقياسين.
الكلمات المفتاحية. مقياس النعومة المختلط. درجة التقريب الأفضل. مبرهنة مباشرة.

1. The first mixed modulus of smoothness

In our work we use X as a compact space unde the metric d_X also we use $\mathcal{L}p(X)$, $p < 1$, the space of all functions *f*: X ⟶ R satisfying $\|f\|p = (\int_X |f|^p)^{1/p} < \infty$. We mean by the Peano continuum, any locally connected, compact metric space.

Let X and Y be two compact spaces under the matrices d_x and d_Y respectively, and if g a real function on $X \times Y$, it mean in $\mathcal{L}p(X \times Y)$. Then we define a version of mixed modulus of smoothness of first order as

$$\omega_{1,1}(g,\sigma_1,\ \sigma_2)_p = \sup_{\substack{d_X(z_1,z_2)\le\sigma_1 \\ d_Y(w_1,w_2)\le\sigma_2}} \left\| g(z_1, w_1) - g(z_1, w_2) - g(z_2, w_1) + g(z_2, w_2) \right\| p$$

Let us collect some properties of the first mixed modulus of smoothness by the following theorem, of easy direct proof.

Theorem 1.1. If $g \in \mathcal{L}p(X \times Y),\ p < 1$ then

1.1.1. $\omega_{1,1}(g, 0,\ 0)_p = 0$

1.1.2. $\omega_{1,1}(g,\sigma_1,\ \sigma_2)_p$ is monotone function of $(\sigma_1,\ \sigma_2)$

1.1.3. $\omega_{1,1}(f,\lambda_1\sigma_1,\ \lambda_2\sigma_1)_p \le c(p)\ \ \lambda,\lambda_2\ \ \omega_{1/1}(f,\sigma_1,\ \sigma_2)_p$

1.1.4.

2. rth order mixed modulus for measuring smoothness

In this section we will define the mixed rth modulus of smoothness and introduce some theorems as applications of it.

If f is a real function on $X \times Y$ belongs to $\mathcal{L}p(X \times Y)$ define the mixed rth modulus of smoothness, for r ≥ 2 as

$$\omega_{r,r}(f,\delta_1,\ \delta_2)_p = \sup_{\substack{0<h_1<\delta_1 \\ 0<h_2<\delta_2}}$$

$$\left\| \sum_{i=0}^{r} \binom{r}{i}\ (-1)^{r-i}\ \left(f\ \left(x, y - \frac{rh_1}{2} + ih_1\right) - f\ (x - \frac{rh_2}{2} ih_2, y)\right)\right\|_p$$

$\delta_1,\ \delta_2\ > 0$, when x $\pm \frac{rh_i}{2} \in\ X$, = 1,2. .
In the following theorem let us collect some properties of our mixed modulus of smoothness

Theorem 2.1. Let $f \in \mathcal{L}p(X\ )\ p < 1$, where X is a Peano cotinum metric space under the metric d, then

2.1.1 $\omega_{r,r}\ (f, 0,\ 0)_p = 0$

2.1.2 $\omega_{r,r}\ (f, \delta_1,\ \delta_2)_p \geq 0$

2.1.3 $\omega_{r,r}\ (f, \delta_1,\ \delta_2)_p \leq\ \omega_{r,r}\ (f, \bar{\delta_1},\ \bar{\delta_2})_p$,
when $\delta_1 \leq\ \bar{\delta_1}$, $\delta_2 \leq\ \bar{\delta_2}$.

2.1.4 $\omega_{r,r}\ (f, \lambda_1\delta_1, \lambda_2\delta_1)_p \leq c(p)\ \ \lambda_1\,\lambda_2\ \omega_{r,r}\ (f, \delta_1, \delta_2)_p$.

2.1.5 $\omega_{r,r}\ (f, \delta_1 + \delta_{2\ ,}\ \lambda_1 + \lambda_2\ )_p \leq c(p)(\omega_{r,r}\ (f, \delta_1, \lambda_1\ )_p + \omega_{r,r}\ (f, \delta_{2\ ,}\ \lambda_2\ )_p)$

proof: the proofs of 2.1.1 and 2.1.2, are direct. Now let us prove 2.1.3. Let $\delta > 0$, we have $h_2 \leq\ \lambda\ \delta_2$, by a result from functional analysis there exists $\acute{h}_2$ satisfy C_1 $h_2 \leq \acute{h}_2 \leq$ C_2 h_2, C_1 , C_2 are positive constants. Since X is a compact space, using a version of Hilbert theorem we obtain that there exists a shortest arc Γ connecting any two points from

$\{t_i\ \}i^r = 0 = \{\ X - \frac{rh_2}{2} +\ ih_2\}_{i=0}^{r}$ r,

and h_2 = d $(t_i,\ t_{i+1})$, i = 0, 1, ….., r and

$\bar{h_2}$ = d $(t_i,\ t_{i+1})$. Since X is convex metric space we obtain that length Γ = $h_2 \leq C_2$ h_2 $\leq C_2\ \lambda_2\delta_2 \leq C_1 \cap\ \delta_2$.

Proof of 2.1.4. If $\delta_1 = \delta_2 = 0$ the proof is trivial, so let us assume δ_1 , $\delta_2 > 0$, and let $(x_1\ , y_1)$ and $(x_2\ , y_2)$are two points in X × Y : d_x $(x_1\ , x_2) \leq\ \ \lambda_1\,\delta_1$ and d_y $(y_1\ , y_2) \leq \lambda_2\,\delta_2$.

From analysis we can find metrices f_x and f_Y on X and Y respectively equivalent to d_x and d_y respectively. Because of the compactness of X and Y, from analysis there is an arc Γ_1 connecting $x_1$and y_1 , also there is an arc Γ_2 connecting $x_2$and y_2, and

f_x $(x_1\ , x_2)$ is the length of the are Γ_1, and f_Y $(y_1\ , y_2)$ is the length of the are Γ_2. Then the length of Γ_1= f_x $(x_1\ ,x_2) \leq$ c d_x $(x_1\ , x_2) \leq c\ \lambda_1\,\delta_1$, also the length of Γ_2= f_y $(y_1\ ,$ $y_2) \leq$ c d_Y $(y_1\ , y_2) \leq c\ \lambda_2\,\delta_2$.

Let ℓ_i= $\frac{i}{n}$, i = 0, 1, 2, 3, ….. , n, we can find a parametrization
ψ_1 with $z_i = \psi_1\ (\ell_i)$, and ψ_2 with $w_i=\ \psi_2\ (\ell_i)$, and
$d_x(\ \psi_{1,}\ \ell_i), \psi_1\ (\ell_{i+1}) \leq$ c f_x $(\psi, \ell_i), \psi\ (\ell_{i+1}) \leq$ c. The length of Γ_1
connecting z_i and z_{i+1}
= c $(\ell_{i+1}\ -\ \ell_i)$
=The length of Γ_1 connecting x_1 and x_2. $d_Y((\psi_2, \ell_i)$, $\psi_2\ (\ell_{i+1})) \leq cf_Y((\psi, \ell_i), \psi\ (\ell_{i+1})) \leq$ c the length of Γ_2 connecting w_i and w_{i+1}
= c $(\ell_{i+1}\ -\ \ell_i)$ the length of Γ_2 connecting y_1 and y_2.

Then

d_x $((z_i\ , z_{i+1}) \leq\ \delta_1$ and $d_y(w_i\ , w_{i+1}) \leq \delta_2$, for I = 0, 1, 2,., n – 1.

If we assume λ_1, λ_2 = cn , this leads to

$$\|f(x_1, y_1) - f(x_2, y_2) - f(x_2, y_1) +\ f(x_2, y_2)\ \|_p \leq\ \ \sum_{i=0}^{n-1} \omega_{1,1}, (f) d_X\big(z_{i\ ,}z_{i+1}\ \big),$$

$d_y, (w_i\ , w_{i+1})_p \leq n^2\ \omega(f, \delta_1,\ \delta_2)_p$

$\omega_{1,1}\ (f, \lambda_1\,\delta_1, \lambda_2\,\delta_2)_p \leq n^2\ \omega_{1,1}\ (f, \delta_1,\ \delta_2)_p$ = c $\lambda_1\,\lambda_2\ \omega_{1,1}\ (f, \delta_1,\ \delta_2)_p$

where c is an absolute constant that may differ from each step to another.

$$F\,(\mathrm{x}\,,\mathrm{y}) = \;(\sum_{q=1}^{\ell}\;\sum_{i=1}^{r}\binom{r}{i}\;(-1)^{r-i-1}\;f\left(\mathrm{x}-\frac{rh_2}{2}+ih_2,\mathrm{y}_q\right)-$$
$$\sum_{j=1}^{k}\;\sum_{i=1}^{r}\binom{r}{i}(-1)^{r-i}f\left(\mathrm{x}_j,\mathrm{y}-\frac{rh_1}{2}+ih_1\right))\varphi_q\,(\mathrm{x}),\;\psi_j\,(\mathrm{y})$$

belongs to $\mathcal{L}$p (X) $\otimes\, \psi_k + \Phi_1 \otimes \mathcal{L}$p (Y)

Therefore

$$\|f-F\|\mathrm{p}\;\le\;\mathrm{c(p)}\;\;\sum_{q=1}^{\ell}\;\;\sum_{j=1}^{r}\|(-1)^r\;f\left(\mathrm{x}-\frac{rh_2}{2},y_q\right)+(-1)^r\,f$$
$$+\sum_{i=1}^{r}\binom{r}{i}(-1)^{r-i}f\left(\mathrm{x}-\frac{rh_2}{2}+ih_2,\mathrm{y}_q\right)$$
$$+\sum_{j=1}^{k}\;\;\sum_{i=1}^{r}\binom{r}{i}(-1)^{r-i}f\;(\mathrm{x}_j,\mathrm{y}-\frac{rh_1}{2}+ih_1)\Big\|_p\;\;\psi_q(\mathrm{x}),\;\psi_j\;(\mathrm{y})$$

$$\le\;\mathrm{c(p)}\;\;\sum_{q=1}^{\ell}\;\;\sum_{j=1}^{k}\omega_{r,r}\;(f,\delta_1,\delta_2)_p\;\;\;\psi_q(\mathrm{X})\,\psi_j\;(\mathrm{y})$$

$= \mathrm{c(p)}\;\omega_{r,r}\;(f,\delta_1,\delta_2)_p$.

This completes the proof of 2.1.4.

Proof of 2.1.5 Using definition of $\omega_{r,r}\;(f,\delta_1,\;\lambda_1)_p$, we get

$\omega_{r,r}\;(f,\delta_1+\delta_{1\,,}\,\lambda_1+\lambda_2\;)_p$ = sup

$$0<\mathrm{h}_1\le\delta_1+\;\delta_2$$
$$0<\mathrm{h}_2\le\lambda_1+\lambda_2$$

$$\left\|\sum_{i=0}^{r}\binom{r}{i}\;\;(-1)^{r-i}\;\left(f\;(\mathrm{X,y}\;\frac{rh_1}{2}+ih_1)+f\;(\mathrm{X}-\;\frac{rh_2}{2}ih_2,\mathrm{y})\right)\right\|_p$$

$\le\;$ c(p) $(\omega_{r,r}\;(f,C_1\;\delta_1,\;C_2\lambda_1)_p+\;\omega_{r,r}\;(f,C_1\;\delta_1,\;C_2\lambda_2)_p+\;\omega_{r,r}\;(f,C_1\;\delta_2,\;C_2\lambda_1)_p+$ $\omega_{r,r}\;(f,C_1\;\delta_2,\;C_2\lambda_2)_p$

Using (3) we obtain

$\omega_{r,r}\;(f,\delta_1+\delta_{2\,,}\,\lambda_1+\lambda_2\;)_p \le$ c(p) $(\omega_{r,r}\;(f,\delta_1,\;\lambda_1)_p$ + $\omega_{r,r}\;(f,\delta_2,\;\lambda_2)_p)$.

Theorem 2.2. For any two positive aumbers δ_1,δ_2 and any f $\in \mathcal{L}$p(X $\times$ Y)**,** $p<1$ and X and Y are two compact metric space we have

$$\inf\,\inf_{\substack{\delta_1>\acute{\delta}_1,\\ \delta_2>\acute{\delta}_2}}\;\;\;\omega_{r,r}\;(f,\delta_1,\;\delta_2)_p=\omega_{r,r}\;(f,\acute{\delta}_1,\;\acute{\delta}_2)_p$$

Proof: We must show that, if $\delta_{1,n}$, and $\delta_{2,n}$ are two decreasing sequences with limits δ_1 and δ_2 respectively we have

$$\omega_{r,r}\;(f,\delta_{1,n}\;,\;\delta_{2,n}\;)_p \text{ converges to } \omega_{r,r}\;(f,\delta_1,\;\delta_2) \text{ as } n\longrightarrow\infty$$

Suppose there exists an $\in\;>0$ such that

$$\omega_{r,r}\;(f,\delta_{1,n}\;,\;\delta_{2,n}\;)_p>\omega_{r,r}\;(f,\delta_1,\;\delta_2)+\in$$

This implies that there exist

$y-\frac{rh_1,n}{2}+ih_{1,n}$ in Y, with

$$\mathrm{d_Y}(y-\frac{rh_1,n}{2}+ih_{1,n}\;,\;\mathrm{y}\;-\;\frac{rh_1,n}{2}+jh_{1,n})<\;\;\delta_{1,n}$$

and

$x-\frac{rh_2,n}{2}+ih_{2,n}$ in X, with

$$d_x\,(X - \frac{rh_{2},n}{2} + ih_{2,}n,\ X - \frac{rh_{2},n}{2} + jh_{2,n}) < \delta_{2,n}$$

Therefore

$$\left\| \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left(f\,(x, y\,\frac{rh_{1,n}}{2} + ih_{1,n}) + f\,(x - \frac{rh_{2,n}}{2} + ih_{2,n}, y) \right) \right\|_p$$

$> \omega_{r,r}\,(f, \delta_1,\ \delta_2)_p + \in$ (1)

Since X and Y are compact spaces, we get the above two sequences in X and Y have converging subsequences. This leads to $h_{1,n_\kappa} \longrightarrow h_{1,o}$ and $h_{2,n_\kappa} \longrightarrow h_{2,o}$, $h_{1,o} \in X$ and $h_{2,o} \in$ Y so

$$\left\| \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left(f\,(X, y\,\frac{rh_{1,n_\kappa}}{2} + ih_{1,n_\kappa}) + f\,(X - \frac{rh_{2,n_\kappa}}{2} + ih_{2}, n_k, y) \right) \right\|_p$$

Converges to

$$\left\| \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left(f\,(x, y - \frac{rh_{1,0}}{2} + ih_{1,0}) + f\,(x - \frac{rh_{2,0}}{2} + ih_{2,0}, y) \right) \right\|_p$$

From (1) we have:

$\left\| \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left(f\,(x, y - \frac{rh_{1,0}}{2} + ih_{1,0}) + f\,(x - \frac{rh_{2,0}}{2} + ih_{2,0}, y) \right) \right\|_p \geq$ $\omega_{r,r}\,(f, \delta_1,\ \delta_2)_p + \in$.

But $h_{1,n_k} \longrightarrow h_1$ and $h_{1,n_k} < \delta_{1,n_k} \longrightarrow \delta_1$,

Also

$h_{2,n_k} \longrightarrow h_2$ and $h_{2,n_k} < \delta_{2,n_k} \longrightarrow \delta_2$, Therefore

$$\omega_{r,r}(f, \delta_1, \delta_2) \geq \left\| \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \begin{pmatrix} f\left(x, y - \frac{rh_{1,0}}{2} + ih_{1,0}\right) + \\ f\,(x - \frac{rh_{2,0}}{2} + ih_{2,0}, y) \end{pmatrix} \right\|_p$$

$\geq \omega_{r,r}\,(f, \delta_1,\ \delta_2)_p + \in$.

Which is a contradiction. Thus our result is satisfied.

We can strength the above result by the following example it mean the above result not true in general.

Example 2.3. Define $G\colon \{0,1\} \times \{0,1\} \longrightarrow R$, as

$$G\,(z, w) = \begin{cases} 1 & (z, w) = (1,1) \\ 0 & (z, w) \neq (1,1) \end{cases}$$

It is clear that G is a continuous function

$$\omega_{r,r}\,(f, \delta_1,\ \delta_2)_p = \begin{cases} 1 & \delta_1,\ \delta_2 \geq 1 \\ 0 & 0.\text{w} \end{cases}$$

If we choose $\acute{\delta}_1,\ \acute{\delta}_2 = 1$, we get that the above result is not true.

3. A version of Jackson Theorem

A classical type theorem due to Jackson, for the approximation of functions f ∈ $\mathcal{L}$p[a, b] by polynomials says

$$E_n\,(f)_p \leq c(p)\,\omega_r\,,(f, \tfrac{1}{n})_p,\ \infty \geq p > 0\ , \qquad (2)$$

where c(p) is a positive constant depends on p only, for p < 1,

$$E_n\,(f)_p = in\,f_{p \in P_n}\ \ \|f - p\|,$$

P_n is the space of polynomials of degree less than or equal to n, and

$$\omega_r(f,\delta) = \sup_{|h|\leq\delta} \|\Delta_h^r f\|_p$$

The inequality (2) was given in terms of the nth entropy number

$$\delta_n([a,b]) = \frac{b-a}{2n},$$

which generalized using the compact metric space X.

In [Stephani, 1992] I. Stephani was proved

$$E_n(f) \leq \omega_1(f, \in_n(X)),$$

where ω_1 is the modulus of continuity of one variable for a function f ∈ C (X), and E_n (f) is the error of the function f to some class Φ_n in

$$\emptyset_1 \subseteq \emptyset_2 \subseteq \ldots.. \subseteq \emptyset_n \subseteq \ldots..$$

with union dense in C (X).

Now let us introduce the blending Jackson version theorem

$$E_{m,n}(f)_p \leq c(p)\, \omega_{r,r}(f, \frac{1}{m}, \frac{1}{n})_p, \qquad (3)$$

where f defined on $X \times Y$, and,

$$E_{m,n}(f)_p = \inf \left\|f - P_{m,n}\right\|_{p(X\times Y)},$$

the infimun is taken on all pseudo polynomials that have the form

$$P_{m,n}(x,y) = \sum_{i=o}^{n} \propto_i(x) y^i + \sum_{j=0}^{m} \beta_j(y) x^j$$

And $\propto_i$ and , β_j are bounded function coefficients. Inequality (3) was proved by Yu . A. Brudnyi [Brudnyi,1992; Gonska, Jetter, 1985]. By [Hbing,1949] for $\omega_{1,1}$, and a continuous function f, X = [a, b], y = [c,d]. And (3) also proved in [Gonska, 1985; Jetter,1989; Cottin,1988; Cottin,1992] for blending Jackson theorem using trigonometric pseudo polynomials and continuous function in C(X).

Let us define the blending space C(X)⊕ M(Y) + M (X)⊕ C(Y) = B$\mathcal{L}$, with respect to a suitable norm M(Y) and M (X) space of bounded functions equipped with the uniform norm on the compact metric space X or Y. ⊗ is the tensor product defined by$f_1 \otimes f_2$ ∈ C(X) ⊗ M (Y), defined by

$$f_1 \otimes f_2 (x,y) = f(x)g(y)$$

Let X be a compact metric space under the metric d_x, with

$\psi_1 \subseteq \psi_2 \subseteq \ldots\ldots.. \subseteq \psi_{n,} \subseteq \ldots.$, its nested subspaces and partition .

$$\hat{E}_{m,n}(f) = \inf\{\left\|f - P_{m,n}\right\|;\ \varphi_{m,}\ \psi_m\},$$

in$f$ is on all pseudo polynomials:

$$P_{m,n}(x,y) = \sum_{i=o}^{m} A_i(y)x^i + \sum_{j=0}^{n} B_j(x)\, y^j,$$

where A_i, B_j are bounded functions, is the degree of the approximation of f using the blending space of pseudo polynomials as an approximation space.

$$B(M(X), M(Y), A_x, A_y) = A_x \otimes M(y) + M(X) \otimes A_y$$

If X is a compact space under the metric dx , a partition of unity

$\varphi_1, \varphi_2, \ldots\ldots.. \varphi_n$ on X , it mean φ_j ∈ C(X),

$$0 \leq \varphi_j(t) \leq t, \quad \sum_{j=1}^{n} \varphi_j(t) = 1, \qquad t \in X, \qquad j \text{ is natural}$$

with n greater than 2, to be controllable if the supports

$$\text{supp}(\varphi_j) = \overline{\{t \in X: \varphi_j(t) \neq 0\}}$$

Have the property

$$\in_1,(\text{supp}(\varphi_j)) < \in_{n-1}(X), j = 1, 2, \ldots.., n.\ [6]$$

Theorem 3.1: If $(X, \|.\|_p)$ and $(Y, \|.\|_p)$ are compact quasi normed spaces for $0 < p < 1$, and let $f \in \mathcal{L}p(X \times Y)$, then

$$\hat{E}_{m,n}(f)_p \leq \inf_{\substack{\delta_1 > \delta_{1,m}(X) \\ \delta_2 > \delta_{2,m}(y)}} \omega_{r,r}(f, \delta_1, \delta_2)_p \qquad (4)$$

Proof: Let m = 1, assume $\delta_1 > \delta_{1,1}(X)$, then
we can find $x_1 \in X$, satisfy $X \subseteq B(x_1, \delta_1)$. Also if n=1, assume $\delta_2 > \delta_{2,1}$ (Y) then we can fined
$y_1 \in y$, satisfy $Y \subseteq B(y_1, \delta_2)$. Then we map

$$F(x, y) = \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i-1} \left(f\left(x, y - \frac{rh_1}{2} + ih_1\right) + f\left(x - \frac{rh_2}{2} + ih_2, y\right) \right)$$

$\in \mathcal{L}p(X) \otimes \psi_1 + \Phi_1 \otimes \mathcal{L}p(Y)$. Therefore we have

$|f(x,y) - F(x,y)| \leq$

$$\mathbf{C}\left|(-1)^r f\left(x, y - \frac{rh_1}{2}\right) + f\left(x - \frac{rh_2}{2}, y\right) - \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i-1} \left(f\left(x, y - \frac{rh_1}{2} + ih_1\right) + f\left(x - \frac{rh_2}{2} ih_2, y\right) \right)\right|.$$

Therefore

$\|f - F\|_{\mathcal{L}p(X \times Y)} \leq c(p)\ \omega_{r,r}(f, \delta_1, \delta_2)_p$.

Thus the inequality of our theorem is satisfied for $m = n = 1$, for any $\delta_1 > \delta_{1,1}$ (X) And $\delta_2 > \delta_{2,1}$ (Y). If $m = 1$ and $n > 1$, we can find $k \leq n$, satisfy $\delta_{1,n}(Y) = \delta_{1,k}(Y)$. If $k = 1$, we can choose $\delta_2 > \delta_{1,1}(Y)$, and apply the same lines of the case above. Then $k > 1$, implies $\delta_{1,k}(Y) < \delta_{1,k_{-1}}(Y)$, so we have $\delta_{1,k}(Y) < \delta < \delta_{1,k_{-1}}(Y)$. Using the entropy definition we can fined the points $x_1, x_2, \dots, x_k \in Y$ such that

$$Y \subseteq \bigcup_{j=1}^{k} B_{\delta_2}(x_j)$$

Using the same lines used in [Cottin,1988] We can get a partition $\psi_1, \psi_2, \dots \dots \psi_k$, satisfying

$$\mathrm{supp}(\psi_j) \subseteq B_{\delta_2}(x_j),\ j = 1, 2, \dots\dots, k.$$

Then since $\delta_2 < \delta_{1,k_{-1}}$ (Y), we can obtain $(\psi_j)_{i=1}^{k}$ satisfy the condition of controllability.

The map

$$F(X, y) = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i-1} f\left(X - \frac{rh_2}{2} + ih_2, y\right) - \sum_{j=1}^{k} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i}$$
$$f\left(x - \frac{rh_2}{2} + ih_2, y_j\right) \psi_j(y),$$

belongs to $\mathcal{L}p(X) \otimes \psi_k + \Phi_1 \otimes \mathcal{L}p(Y)$, and $B\mathcal{L}(\mathcal{L}p(X), \mathcal{L}p(Y), \Phi_1, \psi_n)$. We have using the conditions of controllability that

$$\|f - F\|_p \leq c(p) \sum_{j=1}^{r} \|(-1)^r f\left(x - \frac{rh_2}{2}, y\right) + \sum_{i=1}^{r} \binom{r}{i} (-1)^{r-i}$$
$$f\left(x - \frac{rh_2}{2} + ih_2, y\right) + \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f\left(x - \frac{rh_2}{2}, ih_2, y_j\right)\|$$
$$\psi_j(y) \leq c(p) \sum_{j=1}^{k} \omega_{r,r}(f, \delta_1, \delta_2)_p\, \psi_j(y) = c(p)\omega_{r,r}(f, \delta_1, \delta_2)_p$$

Thus (4) satisfied when *m = 1*, and $n > 1$. Using the same lines above we can prove the case when $m > 1$ and *n = 1*. It remain the case when $m, n > 1$. Let ℓ and k are two naturals with $\ell \le m, k \le n$ and $\delta_{1,\ell}$ (X) = $\delta_{1,m}$ (X) and $\delta_{1,k}(Y) = \delta_{1,n}(Y)$. When ℓ = k = 1, we shall return to the case above . Let us assume k, $\ell > 1$, we shall prove (4) for δ_1 and δ_2, satisfying $\delta_{1,\ell}$ (X) < $\delta_1 < \delta_{1,\ell}$ -1(X) and $\delta_{2,k}(Y) < \delta_2 < \delta_{2,k_{-1}}$ (Y). By entropy numbers definition, we can find $x_1, x_2, \dots, x_\ell$ and $y_1, y_2, \dots, y_k \in$ Y , satisfying

$$X \subseteq \cup_{q=1}^{\ell} B_{x_q}(\delta_1),\ Y \subseteq \cup_{j=1}^{k} B_{y_j}(\delta_2) \qquad (5)$$

As in the case above, the partition of unity (φ_q), (ψ_j) subordinate to the open cover in (5) satisfying the condition of controllability because of $\delta_1 < \delta_{1,\ell_{-1}}$ (X), $\delta_2 < \delta_{2,k_{-1}}$ (Y). Then define the

Theorem 3.4. Let X have the Peano property, and let P be a positive linear operator from $\mathcal{L}$p(X) to $\mathcal{L}$p(X), satisfying $P(f(x)) = f(x)$, where $f(x)$is the identity function. Then for any $f \in \mathcal{L}$p(X), and $\delta > 0$ we have $\|P(f) - f\|_p \le$ c(p) $\omega_{r,r}(f, \delta_1, \delta_2)_p$

Proof: Since G satisfy Peano property, so for any two points with distance $\le \delta_1$ or $\le \delta_2$, we have

$$\left\| \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left(f(x, y - \frac{rh_1}{2}, ih_1 \right) + f\left(x - \frac{rh_2}{2}, ih_2, y \right) \right\|$$

$$\le \omega_{r,r}(f, \delta_1, \delta_2)_p \quad \sum_{i=0}^{r} 1 + \frac{d\left(\left(x, y - \frac{rh_1}{2} + ih\right), (x - \frac{rh_2}{2} + ih_2, y)\right)}{\min\{\delta_1, \delta_2\}}$$

Then

$\|P(f) - f\|_p \le$ c(p) $\omega_{r,r}(f, \delta_1, \delta_2)_p$.

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