Especial Case of Compactness in Bitopological Spaces

Asaad M. A. Alhosaini
University of Babylon, College of Education, Dept. of Mathematics

Abstract

By using the notion of δ-open sets, the author define δ-compactness in-bitopological spaces ,and studied the concepts and theorem related to compactness in view of this notion and obtained some similar results to that in topological spaces for example: "If (X,T,Ω) is a δ-compact bitopological space ,then every infinite subset of X has at least one δ-limit point",

and others which disagree with that in topological spaces ,for example ,we gave an example for a δ-compact subset of a δ-T_2 bitopological space which is not δ-closed.

الخلاصة

بالاستفاده من مفهوم(المجموعات δ – المفتوحه) عرف المؤلف (δ– التراص) في الفضاءات الثنائي التوبولوجيه وبحث المفاهيم والمبرهنات المتعلقه بالتراص من هذا المنظور وحصل علىبعض النتائج المشابهه لمثيلاتها في الفضاءات التوبولوجيه مثل (اذا كان (X,T,W) فضاء ثنائي التوبولوجي فأن كل مجموعه جزئيه غير منتهيه تحتوي على الاقل نقطه δ – غائيه).

واخرى تخالف مثيلاتها في الفضاءات التوبولوجيه , حيث اعطينا مثالا ل مجموعه جزئيه δ–متراصه في فضاء ($T\delta$ $_2$ -) ثنائي التوبولوجي وهي غير δ – مغلقه.

1. Introduction

In (Alswidi and Alhosaini, 2005) we worked on connectedness in bitopological spaces(the definition of bitopological spaces appeared firstly in (Kelly, 1963).The auther will continue, in this paper, with the notion of compactness .We first remember the definitions and results appeared in (Alswidi and Alhosaini, 2005) that we need in the present work and also refer to the part of (Jaleel, 2003) related to the axioms of separation in bitopological spaces and correct some wrong results appeared in that part.

(X,T,Ω) is said to be a bitopological space if (X,T) and (X,Ω) were two topological spaces.

A subset A of X (in a bitopological space (X,T,Ω)) is said to be δ-open set if A$\subset$T-int(Ω-Cl(T-intA))).In this paper we will use the symbol B.S to mean a bitopological space and the symbol δ -O(X) for the collection of all δ-open sets of (X,T,Ω). The complements of δ-open sets are called δ-closed sets.

δ-int(A) is defined to be the union of all δ -open sets contained in A .

δ-Cl(A) is defined to be the intersection of all δ -closed sets containing A.

A function f from (X,T,Ω) to (Y,T',Ω') is said to be δ -continuous if for every δ-open subset B of Y , f^{-1} (B) is δ-open in X.

Similarly δ-open, δ -closed and δ -homeomorphism between two bitopological spaces are defined.

A B.S (X,T,Ω) is said to be δ-T_0 if for each pair of distinct points x and y of X there exists a δ-open set G containing x but not y.

Similarly the notions of δ-T$_1$, δ-T$_2$, δ-regular, δ-normal and δ-limit point were defined in (Jaleel, 2003) by replacing open (closed) sets by δ-open (δ-closed) sets.The following results (and others) were given in [3]:In a B.S (X,T,Ω) if (X,T) is T$_0$ (T$_1$, T$_2$) then (X,T,Ω) is δ-T$_0$ (δ-T$_1$, δ-T$_2$).

1.1 Remark.

If (X,T) is a regular space then it is not necessarily that a B.S (X,T,Ω) be δ-regular.(The converse i.e, If (X,T) is a regular space then a B.S (X,T,Ω) is δ-regular,was wrongly proved in (Jaleel, 2003). See the following example:

Let X={a,b,c,d},T={X, ϕ,{a},{d},{a,d},{b,c},{a,b,c},{b,c,d}}and Ω={X, ϕ,{a} } then (X,T,Ω) is a B.S and

δ-O(X)={X, ϕ,{a},{d},{a,d},{b,c},{a,b,c},{b,c,d},{a,b},{a,c},{a,b,d},{a,c,d}}

note that (X,T) is a regular space ,but (X,T,Ω) is not δ-regular.(take the point b and the δ-closed set {c,d}).

1.2 Remark. If (X,T) is a normal space then it is not necessarily that a B.S (X,T,Ω) be δ-normal.(The converse i.e, If (X,T) is a normal space then a B.S (X,T,Ω) is δ-normal,was wrongly proved in (Jaleel, 2003) referring to the same example above and taking F={c,d} and H={b} which are both δ-closed sets and they are disjoint where there are no disjoint δ-open sets that containing F and H respectively.

2. δ-compactness.

2.1 Definition. Let (X,T,Ω) be a B.S ,a δ-open cover of X is a collection {U$_i$},i$\in\Lambda$,of δ-open subsets of X such that $\bigcup_{\Lambda} U_i$ =X. A collection {V$_j$},j$\in\Lambda'$,is said to be a δ-open subcover of {U$_i$},i$\in\Lambda$ if

$\{V_j \mid j\in\Lambda'\}\subseteq\{U_i \mid i\in\Lambda\}$ and {V$_j$},j$\in\Lambda'$,is itself a δ-open cover of X.

2.2 Remark. Let (X,T,Ω) be a B.S,if {U$_i$},i$\in\Lambda$ is an open cover of (X,T) then {U$_i$},i$\in\Lambda$ is a δ-open cover of (X,T,Ω).

2.3 Definition. A B.S (X,T,Ω) is said to be a δ-Lindelof B.S if every δ-opencover of X has a countable δ-open subcover.

It is obvious that If (X,T,Ω) is a δ-Lindelof B.S then (X,T) is a Lindelof space,but the converse is not true see the following example.

2.4 Example. Let R be the set of real numbers,T , the absolute value topology on R (the usual topology) , Ω ,the cofinite topology on R.Then (R,T, Ω) is a B.S. (R,T) is a Lindelof space (see (Gemignani, 1971)) .

Now we will show that (R,T, Ω)is not a δ-Lindelof B.S:

We first remind that δ-O(X) consists of all the elements of T and any set containing a nonempty element of T(see (Alswidi and Alhosaini, 2005)).

Let $U_x = (-x,0]\cup\{x\}$ then { U_x }, x$\in$R$^+$ is a δ-open cover of (R,T, Ω) which has no countable δ-open subcover.

We know that a regular Lindelof space is a normal space (see (Gemignani, 1971)).

2.5 Remark. A δ-regular, δ-Lindelof B.S is not necessarily δ-normal.

Let N be the set of positive integers ,A$_n$={2n-1,2n},n$\in$N,

T={ ϕ }$\cup${all possible unions of A$_n$,s},T is a topology on N (see Naoum, (1974)).

and let Ω be the cofinite topology on N,then (N,T,Ω) is a B.S and:

i) δ-O(N)=T$\cup\{B\subset N \mid$ B is infinite and cntains some A$_n$ }.

ii) (N,T,Ω) is a δ-Lindelof B.S (because N is a countable set).
iii) (N,T,Ω) is a δ-regular B.S

Assume F is a δ-closed subset of N and $x \notin F$

case1: F is a finite set ,choose j and k such that A_j, $A_k \subset$N-F ,$x \notin A_k$ and take M=max{2j,2k,the numbers in F}, U={x} $\cup A_j \cup$ {all odd numbers >M}.

V=F$\cup A_k \cup$ {all even numbers >M} ,then U and V are disjoint δ-open sets (see (i) above) where $x \in U$ and $F \subset V$.

case2: F is infinite and N-F is finite ,in this case X-F must be the union of some A_n,s(see (i) above),let $x \in A_j$ (for some j), take U= A_j and V=N- A_j .so U and V are disjoint δ-open sets where $x \in U$ and $F \subset V$.

case3: F and N-F are infinite ,so N-F must contain some A_n ,say A_k ,(see (i) above),if $x \in A_k$ take U= A_k and V=N- A_k .If $x \notin A_k$,assume $x \in A_j$ for some j, and choose $y \in$N-F such that $y \notin A_j$ but $y \in A_p$ {for some p) then take U= $\{x\} \cup A_k \cup$((N-F)- A_j) and V=F$\cup A_j$, any way in this case U and V are disjoint δ-open sets where $x \in U$ and $F \subset V$.So (N,T,Ω) is a δ-regular B.S.

iv) (N,T,Ω) is not a δ-normal B.S ,for if F={3,5,7,…}and H={4,6,8,…} then F and H are disjoint δ-closed subsets of N .Note that any δ-open set containing F must contain A_1 or some element of H ,similarly any δ-open set containing H must contain A_1 or some element of F(see (i) above) i.e there are no two disjoint δ-open sets U , V such that $F \subset U$ and $H \subset V$.

2.6 Definition. A B.S (X,T,Ω) is said to be δ-compact if for any δ-open cover $\{U_i\}, i \in \Lambda$ of X ,there is a finite subcover .Suppose $A \subset X$, A is said to be δ-compact if every δ-open cover of the B.S A (with the relative topologies) has a finite subcover.

2.7 Remark. If (X,T,Ω) is a δ-compact BS., then (X,T) is a compact space (because $T \subset \delta$-O(X)).The convers is not true,see the following example:
Let X=[0,1] ,T be the usual topology, and Ω be the cofinite topology on X.
Then (X,T,Ω) is a B.S. , (X,T) is a compact space ,where (X,T,Ω) is not δ-compact ,for if we take U_x =[0,½) $\cup$ {x},x $\in$[½,1] ,then { U_x } is a δ-open cover of X which has no finite δ- subcover.

2.8 Theorem. Let (X,T,Ω) be a B.S ,then X is δ-compact if and only if given any family $\{F_\alpha\}$, $\alpha \in \omega$ of δ-closed subsets of X such that the intersection of any finite number of the F_α is nonempty , $\bigcap_{\omega} F\alpha \neq \phi$.

The proof of this theorem is similar to the corresponding theorem in topological spaces.(see Gemignani, 1971))

2.9 Theorem. If (X,T,Ω) is δ-compact B.S ,then every infinite subset of X has at least one δ-limit point.

proof: Let A be an infinite subset of X which has no δ-limit point. A contains a denumerable subset B,let B={x_1,x_2,…},since A has no δ-limit point,then B has no δ-limit point too,which implies that δ-Cl(B)=B i.e B is δ-closed.

Let D=X-B, D is δ-open in X ,since B has no δ-limit point, then for each x_i,i=1,2,3,…there exists a δ-open set U_i such that $U_i \cap B = \{x_i\}$,clearly,

$D\cup(\bigcup_N U_i)$=X. The family $\{U_i\}$,$i\in N \cup\{D\}$ forms a δ-open cover of X and since X is δ-compact ,there exist $i_1, i_2, i_3, \ldots, i_n$ such that $X=D\cup U_{i1}\cup\ldots\cup U_{in}$

Now $B=B\cap X=B\cap(D\cup U_{i1}\cup\ldots\cup U_{in})$,therefore

$B=\phi\cup\{x_{i1}\}\cup\{x_{i_2}\}\cup\ldots\cup\{x_{in}\}$ is a finite set which is a contradiction, so A has a δ-limit point.(see [1]).

2.10 Theorem. Any δ-closed subset of a δ-compact B.S is δ-compact.(see Gemignani, 1971)).

2.11 Remark. In topological spaces (any compact subset of a T_2 space is closed),(see [5]).This theorem in bitopological spaces is not valid,see the following example:

Let X={a,b,c,d},T={X,ϕ,{a},{b},{a,b},{c,d},{a,c,d},{b,c,d}},and Ω={X,ϕ,{a}}.

Then δ-O(X)= {X,ϕ, {a}, {b}, {a,b}, {c,d}, {a,c,d}, {b,c,d}, {a,b,c}, {a,c}, {a,d}, {b,c}, {b,d},{a,b.d}}

(X,T,Ω) is a δ- T_2 B.S,see [3],also it is δ-compact (because X is finite), and so any subset of X. But A={a,b,d} is not δ-closed in X.

2.12 Remark. In topological spaces (any compact T_2 space is a regular and a normal space.) This is not true in bitopological spaces ,referring to the example of remark 1.1 we see that (X,T,Ω) is a δ-compact B.S (because X is finite), it is a δ- T_2 (can be easily checked) and we showed in 1.1and 1.2, that (X,T,W)is neither δ-regular nor δ-normal.

2.13 Theorem. Suppose f is δ-continuous function from a δ-compact B.S (X,T,Ω) onto a B.S (Y,T',Ω') .Then (Y,T',Ω') is δ-compact.

2.13.1 Corollary. If (X,T,Ω) is δ-compact B.S ,then any B.S δ-homeomorphic to X is δ-compact.

The proofs of the above theorems and corollary are omitted because similar to them are true in topological spaces and the same technique can be used to prove them.(see (Gemignani, 1971)).

2.14 Definition. A B.S (X,T,Ω) is said to be locally δ-compact if given any x εX and any δ-neighborhood U of x ,there is a δ-compact set A such that x ε δ-int(A)$\subset$A$\subset$U.

2.15 Remark. If a B.S (X,T,Ω) is locally δ-compact ,then (X,T) must be locally compact (because every T-neighborhood of x is a δ-neighborhood of x ,and if A is δ-compact in (X,T,Ω) then A is compact in (X,T)).

The converse is not true ,i.e, if (X,T) is locally compact ,it is not necessary that (X,T,Ω) be locally δ-compact. Refereing to example 2.4 we know that (R,T) is locally compact (see [5]),where the only δ-compact subsets of (R,T,Ω) are finite subsets and δ-int (finite subset of R) =ϕ, so (R,T,Ω) is not locally δ-compact.

2.16 Theorem. If f is a δ-continuous, δ-open function from a B.S (X,T,Ω) onto a B.S (Y,T',Ω'), then if X is locally δ-compact ,Y is also.

The proof of this theorem is similar to that in topological spaces, see (Gemignani, 1971).

References

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