Applications of Variational Iteration Method for Solving A Class of Volterra Integral Equations

Mohammed S. Mechee Adil M. Al Ramahi Raad M. Kadum
Department of Math., Faculty of Computer Science and Math., Kufa University
mohammeds.abed@uokufa.edu.iq

Abstract
In this paper, one type of Volterra integral equations (VIEs) is classified as nth-order VIE of fourth-kind. This class of nth-order, fourth-kind VIE usually occurs in many fields of physics and engineering. A new iteration technique is proposed to solve a class of Volterra integral equations. The nth-order VIE of fourth-kind is converted to nth-order ordinary differential equation (ODE). The ODE is solved by using variational iteration method (VIM). It shows that the variational iteration method (VIM) is efficient and powerful integrator for dealing with this class of integral equations. Some examples are solved to illustrate the effectiveness and simplicity of the proposed method.The comparison of the results using VIM with analytical solutions reveals that VIM is very effective, convenient and quite accurate to both linear and nonlinear problems.
Keywords: Integral equations, Volterra integral equation VIE, nth-order,VIM, ODEs.

الخلاصة

في هذا البحث, صنفت احدى معادلات فولتيرا التكاملية كمعادلة رتبة n ذات النوع الرابع. يظهر هذا الصنف من معادلات فولتيرا التكاملية في تطبيقات الحقول المختلفة للفيزياء والهندسة. فقد فرضت التقنية التكرارية الجديدة لحل صنف المعادلات التكاملية. حيث تحول معادلات فولتيرا التكاملية ذات النوع الرابع والرتبة n الى معادلة تفاضلية اعتيادية ذات الرتبة n , عندئذ تحل المعادلة التفاضلية باستخدام طريقة التغاير التكرارية. أظهرت طريقة التغاير التكرارية كاداة كفوءة وفعالة لمعالجة صنف المعادلات التكاملية. أختيرت عدة امثلة لأظهار فعالية وبساطة هذه الطريقة . أظهرت المقارنات للنتائج العددية لطريقة التغاير التكرارية مع الحلول التحليلية للمسائل المقارنة بان الطريقة ذات فعالية ملائمة ودقيقة للمسائل الخطية وغير الخطية.
الكلمات المفتاحية: معادلات تكاملية, معادلة فولتيرا التكاملية, رتبة n, ومعادلات تفاضلية اعتياديه.

1 Introduction

Some of mathematical formulation of physical phenomena contains integral equations. The nonlinear Volterra-Fredholm integral equation, which is special type of integral equations, arises in many physical applications and biological models. The analytical and approximated methods of solutions of the integral equations have an important role in the fields of engineering and applied science. Due to some of these integral equations cannot be solved explicitly, it is often necessary to resort to approximated or numerical techniques. Several of new methods, for solving Volterra integral equations have been developed in recent years

In recent years, Some of different basic functions such as orthonormal bases and wavelets have been used to estimate the solutions of nonlinear Volterra-Fredholm integral equations, (Mirzaee and Hoseini (2013)). In survey to the integrators for solving the Volterra integral equations such as a Taylor method, transform method, the method of variation, the collocation method, numerical technique, the direct quadrature method, HPM method, Adomian decomposition method and homotopy perturbation method. In this paper, one type of Volterra integral equations (VIEs) is classified as nth-order VIE of fourth-kind. A new iteration technique is proposed to solve a class of integral equations. VIE is converted to nth-order ordinary differential equations (ODE) and then is solved using variational iteration method (VIM).

2 Preliminary

In this section we have introduced the background and some definitions which are used in this paper.

2.1 Background and Classification of VIE

In mathematics, the integral equation is an equation in which an unknown function appears under one or more integral signs naturally, such an equation there can occur other terms as well. The Fredholm Volterra integral equation is a special type of integral equations. They are divided into three groups referred to as the first-, second- and third-kind.

A linear Fredholm integral equation of the first-kind is

$$y(x) = \lambda \int_a^b k(x,t)u(t)dt \;,\; t > 0 \tag{1}$$

a linear Fredholm integral equation of the second-kind is

$$u''(x) = y(x) + \lambda \int_a^b k(x,t)u(t)dt \;,\; t > 0 \tag{2}$$

and the linear Fredholm integral equation of third-kind is

$$u(x)g(x) = y(x) + \lambda \int_a^b k(x,t)u(t)dt \;,\; t > 0 \tag{3}$$

where *u; g* and the kernel *k* is a given function and *u* is an unknown function to be solved. Volterra integral equations of first-, second- and third-kind are defined precisely as above except that $b = x$ is the variable upper limit of integration. Here, we give some new definitions for a class of linear integral equations.

Definition 2.1. Volterra integral equation of fourth-kind has the following form*:*

$$u(x) = y(x) + \int_a^b (x-t)^n u(t)dt \;,\; t > 0 \tag{4}$$

Definition 2.2. Order of VIE

We define the order in Volterra integral equation (4) as VIE of order n.

Theorem 2.1. The Volterra integral equation of order n in (4) can be converted to the following ordinary differential equation

$$\mathrm{u}^{(\mathrm{n}+1)}(x) = \mathrm{f}^{(\mathrm{n}+1)}(x) + n!\,u(x) = F(x,u(x))$$

with initial conditions

$$u^i(x) = \alpha_i$$

for i = 0,1,… ,*n*-1.

Proof: using Lipshitze theorem. In recent years, the approximated methods have been applied to wide classes of VIEs problems in many fields of mathematics, physics and engineering. Such methods approximated the solutions for some types of VIEs. The purpose of this paper is to develop the theory of the direct method for the Voltera integral equations. The nth-order VIE of fourth-kind is converted to nth-order ODEs and then, solved using the the VIM method. The new results based on the VIM are compared with the exact solutions; the results show that the method is highly accurate.

2.2 A Quasi-Linear *nth*-Order Ordinary Differential Equation(ODE).

Generally, we define qusi-linear *nth*-order ODEs, It is frequently found in many physical problems such as electromagnetic waves, thin film flow and gravity can be written in the following form

$$y^n(x) = f\big(x, y(x), y'(x), y''(x), y'''(x), \dots, y^{n-1}(x)\big); x \geq x_0 \tag{5}$$

with initial conditions,

$$y^i(x_0) = \alpha^i$$

for $i = 0,1,\dots,n.$

where, $f: R * ⍰^n \rightarrow ⍰^n$

and

$$y(x) = [y_1(x), y_2(x), y_3(x), \dots y_n(x)]$$

$$f(x,y) = [f_1(x,y), f_2(x,y), f_3(x,y), \dots f_n(x,y)]$$

$$\alpha^i = [\alpha_1^i, \alpha_2^i, \alpha_3^i, \dots \alpha_n^i]$$

when the ODE (5) in n dimension space, then we can simplify to

$$\mathrm{z}^{\mathrm{n}}(x) = g\big(z(x), z'(x), z''(x), z'''(x), \dots, z^{n-1}(x)\big) \tag{6}$$

using the following assumption,

$$z(x) = \left\{\begin{matrix} y_1(x) \\ y_2(x) \\ . \\ . \\ . \\ y_n(x) \\ x \end{matrix}\right\} \quad , \quad g(x) = \left\{\begin{matrix} f_1(z_1, z_2, \dots, z_n, z_{n+1}) \\ f_2(z_1, z_2, \dots, z_n, z_{n+1}) \\ . \\ . \\ . \\ f_n(z_1, z_2, \dots, z_n, z_{n+1}) \\ 0 \end{matrix}\right\}$$

with the initial conditions

$$z^i(x_0) = \alpha^{-i}$$

for $i = 0,1,2,\dots, n$

where

$$\alpha^{-i} = \left[\alpha_1^i, \alpha_2^i, \alpha_3^i, \dots, \alpha_n^i, x_0\right]$$

Most researchers, scientists and engineers used to solve (5) or (6) numerically solve by converting the *n- th* order differential equation to a system of first order equations *nth* times the dimensions (Faires and Burden (2003)). However, Mechee et al. (2013) & You& Chen (2013) solved (5) or (6) directly when n=3. Sometimes some researchers can solve this equation analytically. However, it is more efficient if the equation can be solved using variational iteration methods (He (1997), Odibat & Momani (2006), Shatalov (2011)& Struwe (2008)).

3 The Analysis of the Method

In this section, we will introduce the analysis of variational iteration method(VIM).

3.1 Variational Iteration Method

The main idea of VIM is to approximate the solution of differential equation by using an iteration formula in the form of a correctional functional which involves Lagrange multiplier. By applying variational theory, Lagrange multiplier can be determined. The iteration is initiated by a simple function such as a linear function. To illustrate the main concept of this method consider the following system of differential equations:

$$\mathrm{Tu(t)} = \mathrm{g(t)} \tag{7}$$

where T is a differential operator that acts on a sufficiently smooth function u defined on such an interval $I \subseteq R$. The given function g is also defined in I. Initially, split T into its linear and nonlinear part, namely

$$\mathrm{Tu(t)} = \mathrm{Lu(t)} + \mathrm{Nu(t)} = \mathrm{g(t)} \tag{8}$$

where L and N are linear and nonlinear differential operator respectively. A correctional functional for equation (8) is then defined iteratively as

$$u_{n+1}(t) = u_n(t) + \int_0^1 \lambda(t,s)(\mathrm{Lu(s)} + \mathrm{Nu(s)} - \mathrm{g(s)})ds \tag{9}$$

Where, $\lambda(t, s)$ is Lagrange multiplier, u_n is the *nth* approximate solution, and u_n is the

restricted variations so that $u_n = 0$. By choosing such a simple initial function u_0, iterations performes until it converges to a fixed point, under a condition where

$$u_{n+1}(t) = u_n(t)$$

When this condition is reached, then we obtain

$$\int_0^1 \lambda(t,s)(\mathrm{Lu(s)} + \mathrm{Nu(s)} - \mathrm{g(s)})ds = 0 \tag{10}$$

which is equivalent to the condition

$$\mathrm{Tu(t)} = \mathrm{Lu(t)} + \mathrm{Nu(t)} = \mathrm{g(t)} \tag{11}$$

This means that $u_{n+1}(t)$ can be considered as an approximated solution for equation.

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\mathrm{Lu(t)} + \mathrm{Nu(t)} - \mathrm{g(t)})ds \tag{12}$$

We will study the general approximated method for solving nth-order VIE of fourth-kind by using VIM. Firstly, we consider the following nth-order VIE of fourth-kind:

$$u(x) = f(x) + \int_0^x (x-t)^n f(t,u(t))dt \ , \ \ t > 0 \tag{13}$$

with the initial conditions

$$\mathrm{u^i(0)} = ß^i \tag{14}$$

for $i = 0,1,2,\dots,n$

The steps of the approximated method of the problem (13) are :

1. Differentiate equation (13) (n+1)th times, to the following (n+1)th-order ODE:

$$\mathrm{u}^{(\mathrm{n}+1)}(x) = \mathrm{f}^{(\mathrm{n}+1)}(x) + n!\, f\big(x, u(x)\big), \quad x \square 0 \qquad (15)$$

with the initial conditions

$$\mathrm{u}^{\mathrm{i}}(0) = ß^{i} \qquad (16)$$

for $i = 0, 1, 2, \ldots, n$
where f is a given function.

2. We will solve the (n+1)th-order ODE (15) with initial condition in (16) using theVIM.
This algorithm is for solving any nth-order VIE of type IV (13) with initial conditions(14).

4 Implementation(Problems Tested)
Problem 1 (Volterra integral equation of fourth-kind) The Volterra integral equation

$$y(t) = 1 - \int_0^x y(t)dt, \;\; 0 < t \le 1 \qquad (17)$$

With initial conditions,

$$y(0) = 1$$

The exact solution:

$$y(t) = e^{-t}$$

The VIE (17) is equivalent to following ODE

$$y'(t) + y(t) = 0, \quad 0 < t \le 1$$

The general correct functional is as following:

$$y_{n+1}(t) = y_n(t) + \int_0^1 \lambda(t,s)(\mathrm{L}(y_n(\mathrm{s})) + \mathrm{N}(y_n(\mathrm{s})) - \mathrm{g}(\mathrm{s}))ds$$

Hence, the correct functional for this ODE is as following:

$$y_{n+1}(t) = y_n(t) + \int_0^1 \lambda(t,s)(\mathrm{y}'(\mathrm{s}) + \mathrm{y}(\mathrm{s}))\, ds$$

Yields the following stationary condition:

$$\lambda(t,s) = -1$$

So, the functional correct has the following form:

$$y_{n+1}(t) = y_n(t) - \int_0^1 (x - t)(\mathrm{y}'(\mathrm{t}) + \mathrm{y}(\mathrm{t}))\, ds \qquad (18)$$

Consider the initial condition has the form

$$y_0(x) = 1$$

Hence, using the formula (18) we get the approximation terms:

$$y_1(x) = 1 - \mathrm{x}$$

$$y_2(x) = 1 - \mathrm{x} + \frac{x^2}{2}$$

$$y_3(x) = 1 - \mathrm{x} + \frac{x^2}{2} - \frac{x^3}{3!}$$

$$y_4(x) = 1 - \mathrm{x} + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

Then, we get

$$y_n(x) = 1 - \mathrm{x} + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^2 \frac{x^n}{n!}$$

Problem 2 (Volterra integral equation of first-order, fourth-kind)
The Volterra integral equation

$$\mathrm{y(t)} = 1 - \mathrm{x} + \int_0^x (x-t)y(t)dt, \quad 0 < t \leq 1 \tag{19}$$

with initial conditions,

$$y(0) = y'(0) = 1$$

The exact solution:

$$y(t) = e^{-t}$$

The VIE (19) is equivalent to following ODE

$$y''(t) - y(t) = 0, \quad 0 < t \leq 1$$

The general correct functional is as following:

$$y_{n+1}(t) = y_n(t) + \int_0^1 \lambda(t,s)(\mathrm{L}(y_n(\mathrm{s})) + \mathrm{N}(y_n(\mathrm{s})) - \mathrm{g(s)})ds$$

Hence, the correct functional for this ODE is as following:

$$y_{n+1}(t) = y_n(t) + \int_0^1 \lambda(t,s)(\mathrm{y''(s)} - \mathrm{y(s)})\, ds$$

Yields the following stationary condition:

$$\lambda(t,s) = x - t$$

So, the correct functional has the following form:

$$y_{n+1}(t) = y_n(t) - \int_0^1 (x-t)(\mathrm{y''(t)} + \mathrm{y(t)})\, ds \tag{20}$$

Consider the initial condition has the form

$$y_0(x) = 1 - x$$

Hence, using the formula (20) we get the approximation terms:

$$y_1(x) = 1 - \mathrm{x} + \frac{x^2}{2}$$

$$y_2(x) = 1 - \mathrm{x} + \frac{x^2}{2} - \frac{x^3}{3!}$$

$$y_3(x) = 1 - \mathrm{x} + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$y_4(x) = 1 - \mathrm{x} + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

Then, we get

$$y_n(x) = 1 - \mathrm{x} + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^2 \frac{x^n}{n!}$$

Problem 3 (Volterra integral equation of fourth-kind, first-order)
The Volterra integral equation

$$\mathrm{y(t)} = 2\mathrm{x} - 4\int_0^x (x-t)y(t)dt, \quad 0 < t \le 1 \tag{21}$$

with initial conditions,

$$y(0) = y'(0) = 0$$

Problem 4 (Volterra integral equation of fourth-kind)
The Volterra integral equation

$$\mathrm{y(t)} = 2\mathrm{x} - 4\mathrm{x}^2 + 2x^3 - \frac{x^4}{4} + \int_0^x y(t)dt, \quad 0 < t \le 1 \tag{23}$$

with initial conditions,

$$y(0) = 0, y'(0) = 2, y''(0) = -6$$

Exact solution:

$$y(x) = x^3 - 3x^2 + 2x"$$

The VIE (23) is equivalent to following ODE

$$y'''(t) - y''(t) - 18t + 6 = 0, \qquad 0 < t \le 1$$

Hence, the correct functional for this ODE is as following:

$$y_{n+1}(t) = y_n(t) + \int_0^1 \lambda(t,s)(\mathrm{y'''(t)} - \mathrm{y''(t)} - 18\mathrm{t} + 6)\, dt$$

Yields the following stationary condition:

$$\lambda(t,s) = -1$$
$$1 + \lambda'(t,s) = 0$$
$$"1 - \lambda''(t,s) = 0$$

So, the correct functional has the following form:

$$y_{n+1}(t) = y_n(t) - \int_0^1 (t-s)^2(\mathrm{y'''(t)} - \mathrm{y''(t)} - 18\mathrm{t} + 6)\, ds \tag{24}$$

Consider the initial condition has the form

$$y_0(x) = 1 - x$$

Hence, using the formula (24) we get the approximation terms:

$$y_1(x) = 1 - \mathrm{x} + \frac{x^2}{2!}$$
$$y_2(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!}$$
$$y_3(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$
$$y_4(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

Then, we get

$$y_n(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^n}{n!}$$

Problem 5 (Volterra integral equation of fourth-order)
The Volterra integral equation

$$\mathrm{y(x)} = 4\mathrm{x}^3 + \int_0^x (t-x)y(t)dt, \quad 0 < t \leq 1 \qquad (25)$$

with initial conditions,

$$y(0) = y'(0) = y''(0) = 0, y'''(0) = 6$$

Exact solution:

$$y(t) = t^3$$

The VIE (25) is equivalent to following ODE

$$y^{(4)}(t) - y''(t) = 6t, \qquad 0 < t \leq 1$$

Hence, the correct functional for this ODE is as the following:

$$y_{n+1}(t) = y_n(t) + \int_0^1 \lambda(t,s)(y^{(4)}(s) - y''(s) - 6s)\, ds$$

Yields the following stationary condition:

$$\lambda(t,s) = -1$$
$$1 + \lambda'(t,s) = 0$$
$$1 - \lambda''(t,s) = 0$$
$$" \lambda'''(t,s) = -1$$

So, the correct functional has the following form:

$$y_{n+1}(t) = y_n(t) - \int_0^1 (t-s)^3 \big(y^{(4)}(s) - y''(s) - 6s\big)\, ds \qquad (26)$$

Consider the initial condition has the form

$$y_0(x) = 1 - x$$

Hence, using the formula (26) we get the approximation terms:

$$y_1(x) = 1 - \mathrm{x} + \frac{x^2}{2!}$$
$$y_2(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!}$$
$$y_3(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

Then, we get

$$y_n(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^n}{n!}$$

Problem 6 (Volterra integro equation of third-order)
The Volterra integral equation

$$\mathrm{y(t)} = \frac{1}{6!}\int_0^3 (t-x)^3 \big(y^4(t) - y''(t) + y'(t)\big)dt, \quad 0 < t \leq 1 \qquad (27)$$

with initial conditions,

$$y(0) = y'(0) = y''(0) = y'''(0) = y''''(0) = y'''''(0) = 0$$

Exact solution: $y(t) = x^6 - 10x^4$

The VIE (27) is equivalent to following ODE

$$y^{(6)}(t) - y^{(5)}(t) + y^{(4)}(t) - 360t^2 + 720t - 480 = 0, \qquad 0 < t \leq 1$$

Hence, the correct functional for this ODE is as following:

$y_{n+1}(t) = y_n(t) + \int_0^1 \lambda(t,s)(y^{(6)}(s) - y^{(5)}(s) + y^{(4)}(s) - 360s^2 + 720s - 480)\, ds$

Yields the following stationary condition:

$$\lambda(t,s) = -1$$
$$1 + \lambda'(t,s) = 0$$
$$1 - \lambda''(t,s) = 0$$
$$"\lambda'''(t,s) = -1$$

So, the correct functional has the following form:

$$y_{n+1}(t) = y_n(t) - \int_0^1 (t-s)^3 \left(y^{(4)}(t) - y''(t) + y'(t)\right) ds \tag{28}$$

Consider the initial condition has the form

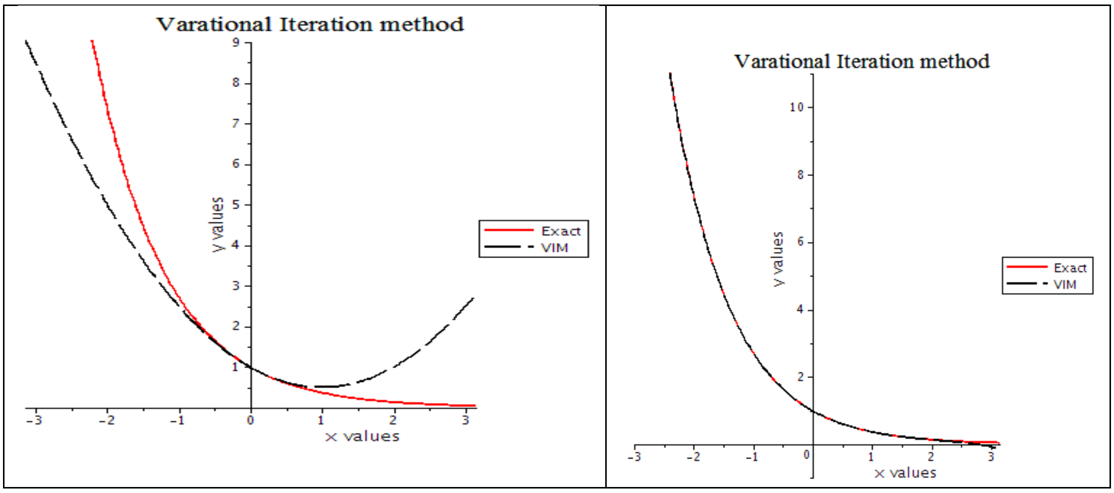
$$y_0(x) = 1 - x$$

Hence, using the formula (28) we get the approximation terms:

$$y_1(x) = 1 - \mathrm{x} + \frac{x^2}{2!}$$
$$y_2(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!}$$
$$y_3(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

Then, we get

$$y_n(x) = 1 - \mathrm{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^n}{n!}$$



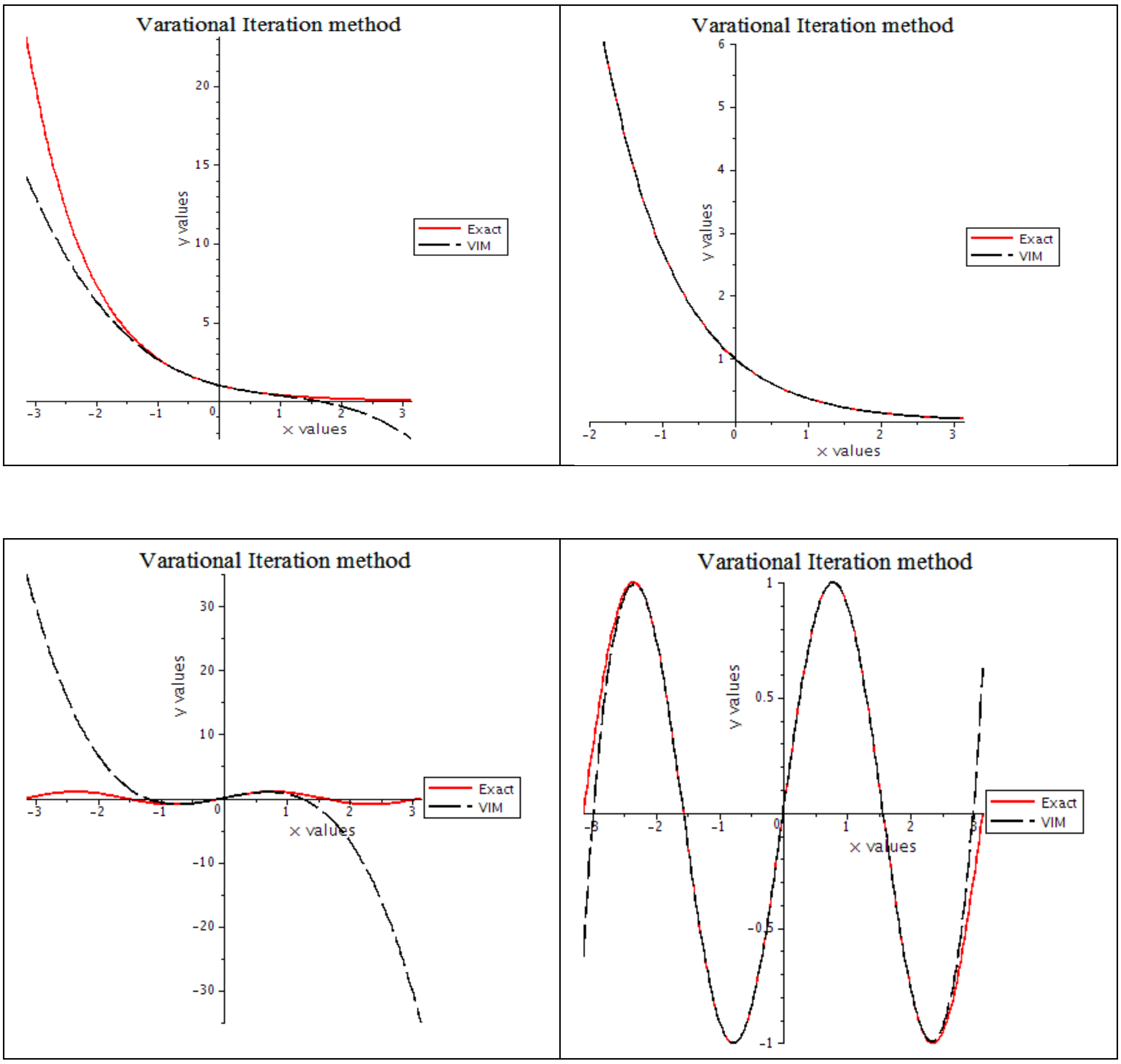


Figure 1: Comparisons on approximated solutions versus exact solutions for (1) first iteration and (2) sixth iteration using variational iteration method in Problems (a) 1 , (b) 2 and (c) 3

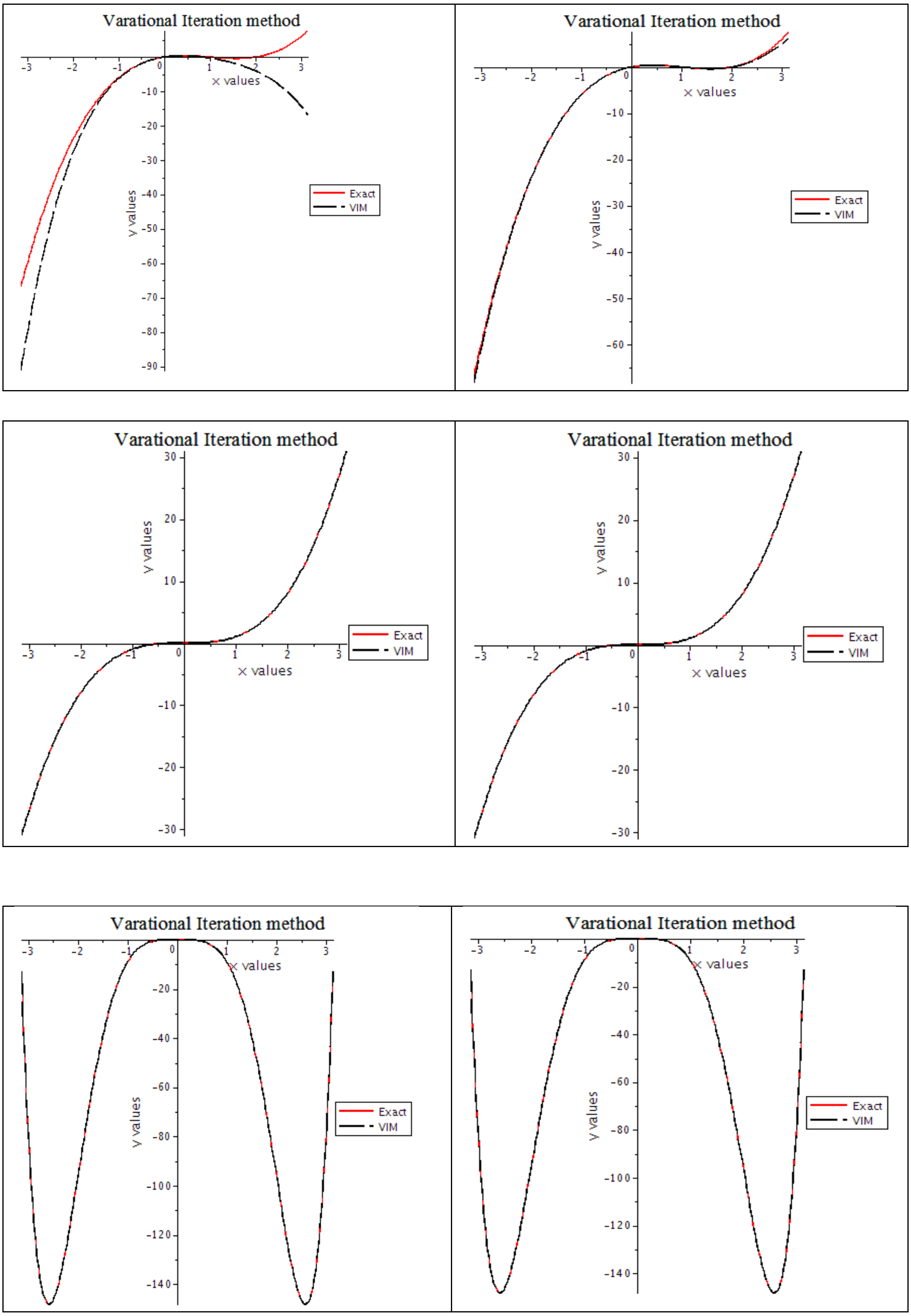


Figure 2: Comparisons on approximated solutions versus exact solutions for (1) first iteration and (2) sixth iteration using variational iteration method in Problems (a) 4 , (b) 5 and (c) 6

5 Discussion and Conclusion

Computing a solution for VIEs directly by using classical methods can be difficult.

The proposed direct method technique in this paper requires less computational work in addition to great features such as fast and effective computation. We compare the approximated solution for VIM with exact solutions. This comparison is intended to establish the validity of the method. From the approximated results of the method, we observe that the method is applicable for a class of VIEs and has good agreement with the exact solutions. The new method is efficient and provides encouraging results. In this paper, a new type of VIEs is classified as Type IV. This class of VIEs of type IV usually occurs in many fields of physics and engineering. VIE of type IV is converted to the nth-order ODE. The nth-order ODE is then solved by using VIM. Hence, we can conclude that VIM can be used analytically as an efficient method with less complicity time for solving the type IV of VIEs.

References

Abbasbandy, S. 2006, Numerical solutions of the integral equations: Homotopy perturbation method and Adomians decomposition method, Applied Mathematics and Computation, 173, 493

Cardone, A., Ixaru, L. G., Paternoster, B., & Santomauro, G. 2013, Ef-Gaussian direct quadrature methods for Volterra integral equations with periodic solution, Mathematics and Computers in Simulation

Chang, K.-C. 1981, Variational methods for non-differentiable functionals and their applications to partial differential equations, Journal of Mathematical Analysis and Applications, 80, 102

Chen, H., & Zhang, C. 2012, Block boundary value methods for solving Volterra integral and integro-differential equations, Journal of Computational and Applied Mathematics,236, 2822

Faires, J. D., & Burden, R. 2003, Numerical Methods, Thomson Learning, Inc., Pacific Grove

Ghasemi, M., Kajani, M. T., & Babolian, E. 2007, Numerical solutions of the nonlinear Volterra Fredholm integral equations by using homotopy perturbation method, Applied Mathematics and Computation, 188, 446

He, J. 1997, Variational iteration method for delay differential equations, Communications in Nonlinear Science and Numerical Simulation, 2, 235

Izzo, G., Russo, E., & Chiapparelli, C. 2012, Highly stable Runge Kutta methods for Volterra integral equations, Applied Numerical Mathematics, 62, 1002

Jiang, W., & Chen, Z. 2013, Solving a system of linear Volterra integral equations using the new reproducing kernel method, Applied Mathematics and Computation, 219, 10225

Kanwal, R., & Liu, K. 1989, A Taylor expansion approach for solving integral equations, International Journal of Mathematical Education in Science and Technology, 20, 411

Malolepszy, T. 2014, Osgood type condition for the Volterra integral equations with bounded and nonincreasing kernels, Journal of Mathematical Analysis and Applications, 410, 411

Mechee, M., Senu, N., Ismail, F., Nikouravan, B., & Siri, Z. 2013, A Three-Stage Fifth-Order Runge-Kutta Method for Directly Solving Special Third-Order Differential Equation with Application to Thin Film Flow Problem, Mathematical Problems in Engineering, 2013

Mirzaee, F., & Hoseini, A. A. 2013, Numerical solution of nonlinear Volterra Fredholm integral equations using hybrid of block-pulse functions and Taylor series, Alexandria Engineering Journal, 52, 551

Odibat, Z., & Momani, S. 2006, Application of variational iteration method to nonlinear differential equations of fractional order, International Journal of Nonlinear Sciences and Numerical Simulation, 7, 27

Ramos, J., et al. 2010, A Volterra integral formulation for determining the periodic solutions of some autonomous, nonlinear, third-order ordinary differential equations, Applied Mathematics and Computation, 216, 2635

Sezer, M. 1994, Taylor polynomial solutions of Volterra integral equations, International Journal of Mathematical Education in Science and Technology, 25, 625

Shatalov, M. 2011, Variational formulation of the method of lines and its application to the wave propagation problems

Sorkun, H. H., & Yalcinbas, S. 2010, Approximate solutions of linear Volterra integral equation systems with variable coefficients, Applied Mathematical Modelling, 34, 3451

Struwe, M. 2008, Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems, Vol. 34 (Springer)

Wang, K., & Wang, Q. 2014, Taylor collocation method and convergence analysis for the Volterra Fredholm integral equations, Journal of Computational and Applied Mathematics, 260, 294

You, X., & Chen, Z. 2013, Direct integrators of Runge Kutta type for special third-order ordinary differential equations, Applied Numerical Mathematics, 74, 128

Yusufoglu, E. 2008, A homotopy perturbation algorithm to solve a system of Fredholm Volterra type integral equations, Mathematical and Computer Modelling, 47, 1099

Zhang, K., Li, J., & Song, H. 2012, Collocation methods for nonlinear convolution Volterra integral equations with multiple proportional delays, Applied Mathematics and Computation, 218, 10848