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DERIVATIVES OF MARGINAL ADDITION FUNCTIONS INDUCED IN EULERIAN PRODUCTS

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Abstract :

Study of gamma processes and functions as well as marginality shown that genre functions can be classified by the subtopic of boundaries Installable. There are these joint study and fringe game integrations For the mind function fσ, we mention and record several criteria for functional similarity functional sequence g_σ =(z) and Fσn = (z). Some necessary or essential requirements details of the existence of relevant Euler constants have been confirmed. Finally, we are we explain some of the various applications of this topic through several examples about gamma, as well as Euler's constants with other specific.

مشتقات دوال الإضافة الهامشية المحدثة في المنتجات الأويلرية

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مستخلص:

أظهرت دراسة عمليات ووظائف جاما بالإضافة إلى الهامشية أن وظائف النوع يمكن تصنيفها حسب الموضوع الفرعي للحدود القابلة للتثبيت. توجد هذه الدراسة المشتركة وتكاملات الألعاب الهامشية بالنسبة لوظيفة العقل fσ، نذكر ونسجل عدة معايير للتشابه الوظيفي للتسلسل الوظيفي (z) =g_σ و Fσn = (z). تم التأكد من بعض تفاصيل المتطلبات الضرورية أو الأساسية لوجود ثوابت أويلر ذات الصلة. وأخيرا، سنشرح بعض التطبيقات المتنوعة لهذا الموضوع من خلال عدة أمثلة عن جاما، وكذلك ثوابت أويلر مع غيرها من الخصوصيات.

1. Introduction and preliminaries

The 18th-century introduction of the Euler-Mascheroni constants, γ and s, makes them among the most well-known and practical mathematical constants. An extended class of -Euler constants is examined, Nevertheless, in 1997. Webster studied functions, of the form Γ that satisfy, the functional equation g = (z+1) g(z)f(z) (z > 0), the Boher - Mollerup theorem. is generalized to paper. However, in 2001, M. Hoooushmand proposed a new concept called Marginal addition function, addition function for every function It is define on the sub set of R that contains all natural number. and shows that Γ-Clerical work can be considered its sub-subject. Whether in newspapers[1]. He produced some related theories, such as the main theorem of Bohr-Möllrup and we hope to clarify some singularity conditions for marginal addition functions and Its relationship to functional equations This has been studied. We recall that Müller and Schleicher employed a similar strategy in 2010 to reduce assembly challenges by using a sequence of functions of rational groups without realizing it. In more recent times, analytical and functional groups.[2]

2. The derivative of limit summation functions

In this section' we will study. the derivative of marginal addition functions, as we will see These derivatives lead to generalizations of Euler's constant [3].

Let g be differentiable over Σ_g . sequence $g_{\sigma'_n}$ with the base

$$(g_{\sigma'_n})(z) = (g_{\sigma n}(z))' = g(n) - \sum_{q=1}^{n} g'(z+q)$$

Let's define that. If the end of this sequence is located at point z *, mark it with a symboll* $g_{\sigma'}(z)$ *signboard let's give*

Example1.1 $\lim_{n\to\infty}(\log\ )\sigma'_n(0) = -\gamma$

Where γ *is Euler's constant. Suppse* $g(z) = \log z$ *in this picture*

$$\begin{aligned} g_{\sigma'_n}(z) &= (g\sigma_n(z))' \\ &= g(z) - \sum_{q=1}^{n} g'(z+q) \\ &= \log n - \sum_{q=1}^{n} \frac{1}{z+q}. \end{aligned}$$

as a result of

$$g_{\sigma_n'}(0) = (1g)_{\sigma_n'}\, log\, = \log\, n - \sum_{q=1}^{n} \frac{1}{q}$$

so

$$\begin{aligned} g_{\sigma'}(0) &= (\log\,)_{\sigma'}(0) \\ &= \lim_{n\to\infty} g_{\sigma_n'}(0) \\ &= \lim_{n\to\infty}\left(1gn - \sum_{q=1}^{n}\frac{1}{q}\right) \\ &= -\gamma. \end{aligned}$$

Inspired by the previous example, let's define [4]*:*

$$\gamma_n(g,z) = -g_{\sigma_n'}(z)$$

And

$$\gamma_n\,(\text{g}) := -\gamma_n(g,0) = -g_{\sigma_n'}(0).$$

Also

$$\gamma(g,z) := \lim_{n\to\infty} \gamma_n\,(g,z).$$

Example1.2 suppose $g(z) = \log z$ *display*

$$\gamma\ \ (+log,z) = \frac{1}{z} + \Psi(z), \qquad (z > 0) \qquad (1.2)$$

where Ψ is the gamma function.

Sol

$$g_{\sigma_n'}(\text{z}) = \text{g(n)} - \sum_{q=1}^{n} g'(z+q) = \log\, n - \sum_{q=1}^{n}\frac{1}{z+q}$$

As a result of

$$\gamma(+log,z) = log_{\sigma'}(z)$$

$$= \lim_{n\to\infty}\left(logn - \sum_{q=1}^{n}\frac{1}{q} + \sum_{q=1}^{n}\frac{z}{q(z+q)}\right)$$

$$= \lim_{n\to\infty}\left(\log\, n - \sum_{q=1}^{n}\frac{1}{q}\right) + \sum_{q=1}^{\infty}\frac{z}{q(z+q)}$$

$$= -\gamma + \sum_{q=1}^{\infty}\left(\frac{1}{q} - \frac{1}{z+q}\right)$$

$$= -\gamma + \sum_{q=1}^{\infty}\left(\frac{1}{q+1} - \frac{1}{q+z+1}\right)$$

$$= \frac{1}{x} + \Psi(z), \qquad (z > 0).$$

The point we should pay attention to in the above calculations is that for every, $q \in N$

$$\frac{1}{q} - \frac{1}{z+q} = \frac{z}{q(z+q)}.$$

Also

$$\begin{aligned} \Psi(z+1) &= \frac{\Gamma'\,(z+1)}{\Gamma\,(z+1)} \\ &= \frac{(z\Gamma(z)\,)'}{z\Gamma\,(z)} \\ &= \frac{\Gamma(z) + z\Gamma'(z)}{z\Gamma(z)} \\ &= \frac{1}{z} + \Psi(z). \end{aligned}$$

We've already seen that if g is summable over $\mathbb{N}$ and $R(1) = 0$ ther for each every $s \in \mathbb{N}$

$$g_\sigma(s) = g(1) + (2) + \cdots + g(s)$$

And as a result

$$\frac{g_\sigma(s)}{s} = \frac{g(1) + g(2) + \cdots + g(s)}{s}$$

According to this issue, let us provide the following definition.

Definition1.1: If g is a function, then the average sum of the limit of g is as follows

$$g_{\tilde{\sigma}}(z) = \begin{cases} \frac{1}{z} g_\sigma(z) & z \neq 0 \\ \lim_{z \to 0} \frac{g_\sigma(z)}{z} & z = 0 \end{cases} \qquad (1.3)$$

Let's define that. Note, that the range of $g_{\sigma^\sim}$ *is* equal, to the range of fσ or equal to *Dgσ/*{0}

Theorem 1 *If* $g : [1 + \infty) \to \mathbb{R}$ *has a derivative and,* $R(g, 1) = 0$ *then the sequence* $g_{\sigma'_n}(z)$ *also true for* $(0 + \infty,)$ *and if* g' ascending, then $g_{\tilde{\sigma}}(z)$ *and* $g(z)$ in inequality

$$-\gamma(g, 1) \leq g_{\sigma'}(z) \leq g_{\sigma^\sim}(z) \leq -\gamma(g); \quad (0 < z \leq 1) \qquad (1.4)$$

To be true. Also, if g' In decreasing, the trend of the above inequality is reversed. To In addition, $g_{\sigma'(z)}$ is the solution of the function equation [5].

$$z(z) = g'(z)\ z\,(z - 1) \qquad (z > 1.) \qquad (1.5)$$

He is

Theorem 2

Suppose g' be ascending (if so g' rations the argument will be similarly presented) first, we show It $\lim_{z \to \infty} g'(z) = 0$ and on $[1, +\infty)$ he is getting off

We know that

$$R(g, 1) = \lim_{n \to \infty} R_n(g, 1) = \lim_{n \to \infty} \big(g(n) - g(n + 1)\big) = 0$$

As a result of

$$R(g, 1) = \lim_{n \to \infty} R_{n-1}(g, 1) = \lim_{n \to \infty} \big(g(n - 1) - g(n)\big)$$

So

$$\lim_{n \to \infty} \big(g(n) - g(n - 1)\big) = 0, \qquad \lim_{n \to \infty} \big(g(n + 1)_{g(n)}\big) = 0.$$

Now, according to the mean value theorem, the numbers $n < t_n < n + 1$ *and* $n - 1 < s_n < n$ available for this reason.

$$g(n+1) - g(n) = g'(t_n), \qquad g(n) - g(n-1) = g'(s_n)$$

From the previous relationships we conclude that

$$\lim_{n\to\infty} g'(s_n) = \lim_{n\to\infty} g'(t_n) = 0$$

But g' because it's up, so

$$g'(n-1) \le g'(s_n) \le g'(n) \le g'(t_n) \le g'(n+1)$$

Now it follows from the pressure theory

$$\lim_{n\to\infty} g'(n) = 0.$$

Now, if $1 \le z$ is a random number, then there are n such $n \le z \le n+1$ and as a result $g'(n) \le g'(z) \le g'(n+1)$. using pressure theory again, the result is obtained [6].

$$\lim_{z\to\infty} g'(z) = 0.$$

To prove that is decreasing, we first show $g'(z) \le 0$

Let $1 \le z$ be constant and $z < t$ random, we've got

$$g'(z) = \lim_{t\to\infty} g'(z) \le \lim_{t\to\infty} g'(t).$$

Suppose that $1 \le z_1 < z_2$ according to the theorem, the average value of the point $z_1 < z < z_2$ exists for this reason

$$\frac{g(z_2) - g(z_1)}{z_2 - z_1} = g'(z)$$

Because $g'(z) \le 0$ so $g(z_2) \le g(z_1)$ therefore, g decreases on $[1, +\infty)$ it is a lineage and hence to all $g(z) \le g(1), 1 \le z$ suppose that $0 < z < 1$ and $k \in \mathbb{N}$ in this case $q < q + z < q + 1$ applying the mean value theorem to the interval $[q, q + z]$,we obtain the point $z_q \in (q, q + z)$ there is that

$$-R_q(z) = g(q+z) - g(q) = g'(z_q).z.$$

As a result of

$$g_{\sigma_n}(z) = zg(n) + \sum_{q=1}^{n} R_q(z)$$

$$= zg(n) - \sum_{q=1}^{n} g'(z_q).z$$

$$= z\left(g(n)\sum_{q=1}^{n} g'(z_q)\right)$$

According to the assumption g' on $[1, +\infty)$ ascending hence for every , $q \in \mathbb{N}$

$$g'(q) \leq g'(z_q) \leq g(z+q) \leq g'(q+1)$$

As a result of

$$\sum_{q=1}^{n} g'(q) \leq \sum_{q=1}^{n} g'(z_q) \leq \sum_{q=1}^{n} g'(z+q) \leq \sum_{q=1}^{n} g'(q+1).$$

So

$$-\gamma_n(g,1) = g_{\sigma_n'}(1)$$

$$= g(n) - \sum_{q=1}^{n} g'(q+1)$$

$$\leq g(n) - \sum_{q=1}^{n} g'(z+q)$$

$$\leq g(n) - \sum_{q=1}^{n} g'(q)$$

As a result of $n = 1,2,\dots$

$$-\gamma_n(g,1) \leq g_{\sigma_n'}(z) \leq \frac{g_{\sigma_n'}(z)}{z} = g_{\sigma_n'}(z) \leq -\gamma_n(g,0) = -\gamma_n(g) \tag{1.6}$$

On the other hand, because g' on $[1, +\infty)$ ascending, so it is convex to g' on $[1, +\infty)$ and as a result

$$\frac{g(n+h) - g(n)}{h} \leq g(n+1) - g(n) \leq \frac{g\big((n+1)+h\big) - g(n+1)}{h} \tag{1.7}$$

Where $0 < h < 1$ is now as assumed , $h \to 0^+$ we will get[4].

$$g'(n) \leq -R_n(g,1) \leq g'(n+1).$$

So

$$R_n(g',1) = g'(n) - g'(n+1) \leq -g'(n+1) - R_n(g,1) \leq 0. \qquad (1.8)$$

Now let's show

$$g_{\sigma'_n}(z) - g_{\sigma'_{n+1}}(z) = R_n(1) + g'(n+1+z) \qquad (1.9)$$

We've got

$$g_{\sigma'_n}(z) - g_{\sigma'_{n+1}}(z)$$

$$= g(n)\sum_{q=1}^{n} \quad g'(q+z) - g(n+1) + \sum_{q=1}^{n} g'(q+z) = g(n) - g(n+1) + g'(n+1+z)$$

So

$$g_{\sigma'_{n+1}}(z) - g_{\sigma'_n}(z) = -R_n(1) - g'(n+1+z)$$

$$\leq -R_n(1) - g'(n+1) \leq 0, \qquad (z \geq 0)$$

That is, for every, $0 \leq z$ the function sequence $g_{\sigma'_n}(z)$ he is getting off. Now put $z = 0$ in We get relation (1.9) and using relation (1.8) [3].

$$R_n(g',1) = g'(n) - g'(n+1)$$

$$\leq -\left(R_n(g) + g'(n+1)\right)$$

$$= g_{\sigma'_{n+1}}(0) - g_{\sigma'_n}(0)$$

$$= -\gamma_{n+1}(g) + \gamma_n(g) \leq 0$$

That it

$$R_n(g',1) \leq -\gamma_{n+1}(g) + \gamma_n(g) \leq 0 \qquad (1.10)$$

Specially

$$-\gamma_{n+1}(g) \leq -\gamma_n(g) \tag{1.11}$$

This can be easily proven

$$-\gamma_n(g,1) = g'(1) - g'(n+1) - \gamma_n(g) \qquad n = 1,2,\ldots \tag{1.12}$$

From relation (1.11) we get the result

$$g'(1) - g'(n+1) - \gamma_{n+1}(g) \leq g'(1) - g'(n+1) - \gamma_n(g)$$
$$= -\gamma_n(g,1)$$

Also

$$g'(1) + g(n+1) - g(n) - \gamma_{n+1}(g) \leq g'(1) - g'(n+1) - \gamma_n(g).$$

As a result of

$$g'(1) + R_n(1) - \gamma_{n+1}(g) \leq g_{\sigma_n'}(z) \leq g_{\sigma_n'}(z) - \gamma_n(g), \qquad 0 < z \leq 1 \tag{1.13}$$

Now because $g_{\sigma_n'}(z) = \frac{1}{n}\, g_{\sigma_n}(z)$ it's the same thing, that's it $g_{\sigma_n'} \to g_\sigma(z)$ therefore for each,$0 < z \leq 1 \quad g_{\sigma_n'}(z)\ to\ g_{\sigma'}(z)$ be the same. As a result of

$$-\gamma(g,1) = g'(1) - \gamma(g) \leq g_{\sigma'}(z) \leq g_{\sigma_n'}(z) \leq -\gamma(f), \qquad 0 < z \leq 1 \tag{1.14}$$

Now consider that $g_{\sigma_n'}(z)$ on $(0,1]$ it's the same as using a relationship

$$g_{\sigma_n'}(z) - g_{\sigma_n'}(z-1) = g'(z) - g'(z+n)$$

This is the property that $\lim_{z\to\infty} g'(z) = 0$ we conclude that

$$g_{\sigma'}(z) = g'(z) + g_{\sigma'}(z-1), \qquad (1 < z \leq 2)$$

Continuing the above method, we conclude that

$$g_{\sigma'}(z) = g'(z) + g'(z-1) + \cdots + g'(z-s+1) + g_{\sigma'}(z-s) \qquad (s < z \leq s+1)$$

As a result of

$$g_{\sigma'}(z) = -\gamma(g,\{z\}) + \sum_{j=0}^{[z]} g'(z-j)$$

So $g_{\sigma_n'}(z)$ on $(0,+\infty)$ it is the same and the relation (1.6) holds for every $1 < z$ it was created

We can generalize the relationship (1.4) in the previous theorem as follows for every $0 < z$ Under the conditions of theorem1.1 then be bullish, then[6].

$$g'(1) - \gamma(g) + \sum_{j=1}^{\lfloor z \rfloor -1} g'(z-j) \le g_{\sigma'}(\{z\}) \sum_{j=0}^{\lfloor z \rfloor -1} g'(z-j)$$

$$\le g_{\sigma^{\sim}}(z) + \sum_{j=0}^{[z]-1} g'(z-j)$$

$$\le -\gamma(g) + \sum_{j=0}^{[z]-1z} g'(z-j), \quad (0 < z). \qquad (1.15)$$

The "Q" proof for unequal faces (1.2), expression $\sum_{j=0}^{[z]-1} g'(z-j)$ *add*

Now, in Theorem 1.1, replace the assumption $R_n(g.1) \to 0$ *with the limit assumption* $R_n(d.1)$ *And circulate it* [2].

A- everyone $g_{\sigma_n'}(z), 0 \le z$ convergent and divergent g' be bullish, then

$$g'(1) + R(1) - \gamma(g) \le g_{\sigma'}(z) \le g_{\sigma^{\sim}}(z) \le -\gamma(g). \qquad (0 < z \le 1) \qquad (1.16)$$

B- if g' so the lineage is

$$-\gamma(g) \le g_{\sigma^{\sim}}(z) \le g_{\sigma'}(z) \le g'(1) + R(1) - \gamma(g). \qquad (0 < z \le 1) \qquad (1.17)$$

C- $g_{\sigma'}(z)$ in the function equation

$$Z(z) = R(1) + g'(z) + Z(z-1), \qquad (z > 1) \qquad (1.18)$$

This as not true.

Proof: A- because g' Ascending, so g is convex. as a result of [1].

$$\frac{g(11) - g(10)}{11 - 10} \le \frac{g(12) - g(11)}{12 - 11} \le \frac{g(13) - g(12)}{13 - 12} \le \cdots$$

And so

$$\frac{g(10) - g(11)}{10 - 11} \le \frac{g(12) - g(13)}{12 - 13} \le \frac{g(14) - g(15)}{14 - 15} \le \cdots$$

So

$$g(10) - g(12) \geq g(13) - g(14) \geq g(15) - g(16) \ldots$$

This means sequence

$$R_n(g,1) = g(n) \ - g(n+1)$$

It is a descent (and therefore a new descent). Because it is assumed $R_n(g,1)$ so it is adjacent .$R_n(g,1)$ This is also true.

Now the function f with the rule

$$f(z) = g(z) + R(1)z, \qquad (1 \leq z)$$

Let's define that. In this case

$$f'(z) = g'(z) + R(1)$$

And

$$f_{\sigma_n}(z) \ = zf(n) + \sum_{q=1}^{n}(f(q) - g(z+q)$$

$$= z(g(n) + R(1)n)$$

$$+\sum_{q=1}^{n}[(g(q) + R(1)q) - (g(z+q) + R(1)(z+q))]$$

$$= zg(n) + \sum_{q=1}^{n}(g(q) - g(z+q))$$

$$= g_{\sigma_n}(z),$$

So (instead of g) in all assumptions of the theory theorem1.1 this is not true. as a result of

$$-\gamma(g,1) \leq g_\sigma(z) \leq g_{\sigma^-}z \leq -\gamma(g) \qquad (0 < z \leq 1)$$

And This (according to the definition rule of g) that it

$$g'(1) + R(1) - \gamma(g) \leq g_{\sigma'}(z) \leq g_{\sigma^-}(z) \leq -\gamma(g) \qquad (0 < z \leq 1)$$

Proof: B- if g' so the lineage is - g' We cannot fully complete the proof of part (A) to - g' let's repeat.

Proof C- Because $g_{\sigma'}(z) = g_{\sigma'}(z)$ Therefore, in both cases (A) and (B)

$$g_{\sigma'}(z) = g'(z)g_{\sigma'}(z-1) + R(1) \qquad (z > 1).$$

Example 3 : The function $g: \mathbb{R} \to \mathbb{R}$: f with the rule $g(z) = \tan^{-1} z$ it is considered using.

Theorem1: show

$$\begin{aligned}\frac{1}{2}(\pi \coth(\pi) - 2) &\leq \sum_{q=1}^{\infty} \frac{1}{1+(z+q)^2} \\ &\leq \frac{1}{z}\sum_{q=1}^{\infty}\left(\tan^{-1}(z+q) - \tan^{-1}(q)\right) \\ &\leq \frac{1}{2}(\pi \coth(\pi) - 1) \qquad (0 < z \leq 1). \qquad (1.19)\end{aligned}$$

Solve Because we want to use Theorem 1, so we must use all values , $-\gamma(g, 1)$ Let us calculate $-\gamma(g)$ *and* $g_{\sigma^{\sim}}(z), g_{\sigma'}(z)$

$$\begin{aligned} g_{\sigma'}(z) &= g(n) - \sum_{q=1}^{n} g'(z+q) \\ &= \tan^{-1} n - \sum_{q=1}^{n} \frac{1}{1+(z+q)^2}. \end{aligned}$$

as a result of

$$\begin{aligned} g_{\sigma'}(z) &= \lim_{n\to\infty} g_{\sigma_n'}(z) \\ &= \lim_{n\to\infty}\left(\tan^{-1}(n) - \sum_{q=1}^{n}\frac{1}{1+(z+q)^2}\right) \\ &= \frac{\pi}{2} - \sum_{q=1}^{\infty}\frac{1}{1+(z+q)^2} \end{aligned}$$

And

$$
\begin{aligned}
-\gamma(g) &= \lim_{n\to\infty}(-\gamma_n(g,0)) \\
&= \lim_{n\to\infty} g_{\sigma_n'}(0) \\
&= \lim_{n\to\infty}\left(\tan^{-1} n - \sum_{q=1}^{n}\frac{1}{1+q^2}\right) \\
&= \frac{\pi}{2} - \sum_{q=1}^{\infty}\frac{1}{1+q^2} \\
&= \frac{\pi}{2} - \frac{1}{2}(\pi\coth(\pi) - 1).
\end{aligned}
$$

Notice that in the last line of the relationship above, from the relationship

$$\cot h(\pi z) = \frac{1}{\pi x} + \frac{2z}{\pi}\sum_{q=1}^{\infty}\frac{1}{z^2+q^2}$$

We have used it

To calculate $-\gamma(g,1)$ we first show

$$\gamma_n(g) - \gamma_n(g,1) = g'(1) - g'(n+1)$$

We're got

$$
\begin{aligned}
\gamma_n(g) - \gamma_n(g,1) &= g_{\sigma_n'}(1) - g_{\sigma_n'}(0) \\
&= g(n) - \sum_{q=1}^{n} g'(q+1) - g(n) + \sum_{q=1}^{n} g'(q) \\
&= \sum_{q=1}^{n}(g'(q) - g'(q+1). \\
&= g'(1) - g'(n+1).
\end{aligned}
$$

Now consider that $g'(z) = \frac{1}{1+z^2}, g(z) = \tan^{-1}(z)$ As a result, if both parties to the relationship (1.13) When $n \to \infty$, we take the limit, we will have [4].

$$\gamma(g) - \gamma(g,1) = g'(1) = \frac{1}{2}$$

as a result of

$$-\gamma(g,1) = g'(1) - \gamma(g)$$

$$= \frac{1}{2} + \frac{\pi}{2} - \frac{1}{2}(\pi \ \cot h(\pi) - 1)\,. \qquad (1.23)$$

Finally,

$$g_{\sigma_n}(z) = zg(n) + \sum_{q=1}^{n}(g(q) - g(z+q))$$

$$= z\tan^{-1} n + \sum_{q=1}^{n}(\tan^{-1}(q) - \tan^{-1}(z+1)).$$

So

$$g_\sigma(z) = \lim_{n\to\infty} g_{\sigma_n}(z)$$

$$= \frac{\pi}{2}x + \sum_{q=1}^{\infty}(\tan^{-1}(q) - \tan^{-1}(z+q)) \qquad (1.24)$$

as a result of

$$g_{\sigma^\sim}(z) = \frac{g_\sigma(z)}{z}$$

$$= \frac{\pi}{2} + \frac{1}{z}\sum_{q=1}^{\infty}(\tan^{-1}(q) - \tan^{-1}(z+q)).$$

Now put these values in the relationship (1.4) and take that into account $g'(z) = \frac{1}{1+z^2}$ Descending, we will have

$$\frac{1}{2} + \frac{\pi}{2} - \frac{1}{2}(\pi \ \cot h(\pi) - 1 \geq \frac{\pi}{2} - \sum_{q=1}^{\infty}\frac{1}{1+(z+q)^2}$$

$$\geq \frac{\pi}{2} + \frac{1}{2}\sum_{q=1}^{\infty}\left(\tan^{-1}(q) - \tan^{-1}(z+q)\right)$$

$$\geq \frac{\pi}{2} - \frac{1}{2}(\pi \cot h(\pi) - 1)) \qquad (1.25)$$

If it is one of the aspects of the relationship (1.16), $\frac{\pi}{2}$ Removing and multiplying the sides by negative The relationship (1.10) is obtained,

3. Relationship with Euler type constant.

shows in reference that there are some Euler-type constants for the function $g:[1+\infty) \to \mathbb{R}$ It was proven that they have certain properties. In this section we show The theorem proven by Sander is a special case of Theorem 1.1 to $(z=0)$ He is. Before mentioning the theories involved, let us mention some relevant issues [1].

Assume Definition $g:[1+\infty) \to \mathbb{R}$ Let the function be complete. Sequel A_n and B_n Let us define as follows [6].

$$A_n = A_n(G) := \sum_{i=1}^{n} G(i) - \int_{1}^{n+1} G(z). \qquad (1.26)$$

$$B_n = B_n(G) = \sum_{i=1}^{n+1} G(i) - \int_{1}^{n+1} G(z)\,dz \qquad (n \geq 1). \qquad (1.27)$$

Sequel $(B_n)_1^{\infty}$ and $(A_n)_1^{\infty}$ dependent sequences are called F

If $G(z) = \frac{1}{z}$ then $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \gamma$ where Euler's constant is Macheron's constant.

Suppose Proposition 1.1 $g:[1,+\infty) \to \mathbb{R}$ function and let G be its primary function. In this he face

$$B(g) = \gamma(G) + G(1). \qquad (1.29)$$

S0 $\gamma_n(G)$ This is also true if and only if $B_n(g)$ be the same and in this case we have

$$B_{n-1}(g) = \gamma(G) + G(1) = \sum_{q=1}^{n} G'(q) - G(n) + G(1) \qquad (1.28)$$

Proof we've got

$$\begin{aligned} B_{n-1}(g) &= \sum_{q=1}^{n} g(q) - \int_{1}^{n} g(z)dz \\ &= \sum_{q=1}^{n} G'(q) - \int_{1}^{n} G'(z)dz \\ &= \sum_{q=1}^{n} G'(q) - (G(n) - G(1)) \\ &= \sum_{q=1}^{n} G'(q) - G(n) + G(1). \qquad (1.30) \end{aligned}$$

On the other side

$$G_{\sigma'_n}(z) = G(n) - \sum_{q=1}^{n} G'(z+q)$$

as a result of

$$\gamma_n(G) + (G) = -G_{\sigma'_n}(0) + G(1)$$

$$= \sum_{q=1}^{n} G'(q) - G(n) + G(1)$$

From the two relationships (1.30) and (1.31), it can be concluded that relationship (1.28) is correct. Now if who Both sides of equation (1.30) When $n \to \infty$ we take the limit, we obtain equation (1.20) .We now show that Sander's theorem follows from Theorem 1

Theorem 3. $G:[1,+1) \to \mathbb{R}$ It is a strictly positive and decreasing function In this case, the sequence is continuous $(A_n)_1^\infty$ Up Up and down precisely $(B_n)_1^\infty$ accurate proportions and Both sequences are identical. If $\lim_{x\to\infty\partial} G(z) = 0$ Both sequences are identical.

Proof we show that this theorem is a consequence of Theorem 1 for Assumption $[1,+\infty) \to \mathbb{R}$ In the assumptions of the previous theorem, it is true, that is,is continuous and strictly decreasing [5] .

And be positive. Since is continuous, its prime function is F in use $[1,+\infty)$ He is. Signboard Let G be true in the assumptions of Theorem 1 which is sufficient for this purpose Let us show that $R(G,1) = 0$ because $f = G'$ According to the average value theorem (actions done on) $c_n \in (n, n+1)$ for each number n $c_n \in (n, n+1)$ Available for that [6].

$$G(n) - G(n+1) = G'(c_n) = g(c_n) \qquad (1.31)$$

Because $\lim_{t\to\infty} g(t)$ *so*

$$\lim_{n\to\infty}\big(G(n) - G(n+1)\big) = \lim_{n\to\infty} f(c_n) = 0$$

as a result of

$$
\begin{aligned}
R(G,1) &= \lim_{n\to\infty} R_n(G,1) \\
&= \lim_{n\to\infty}\left(G(n) + \sum_{q=1}^{n}\left(G(q) - G(q+1)\right)\right) \\
&= \lim_{n\to\infty}\left(G(n) - G(n+1)\right) + G(1) \\
&= G(1) = 0. \qquad (1.32)
\end{aligned}
$$

Note that F is the basis function of g $[1,+\infty)$ *so*

$$G(z) = \int_1^x g(t)dt$$

As a result $G(1) = 0$. In short, we have proven it $R(G,1) = 0$ Hence G in the assumptions Theorem 1.1 is correct. Now it follows from Theorem 1. that $G_{\sigma_n}(z)$ For every $0 \le z$, this is also true $\gamma_n(G,z)$ for every this $0 \le z$ is especially true $\gamma_n(G) = \gamma_n(G,0)$ this is also true now Relationship (1.28) with the truth that $\lim_{t\to\infty} g(t) = 0$ It shows that the relationship (1.29) is correct [3]..

4. Conclusion

included in this research report. Examine marginal additive functions and their relationships. Additionally, Euler›s constants were examined. By studying the theory and examples contained in the article. An explanation of the connection and effective applications between derivatives and Euler›s constants is given.

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