Exact Stiffness Matrix for A Non-Prismatic Plane Frame Element with Parabolic Varying Depth

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Abstract

In this paper, the derivation of an exact stiffness matrix for a non-prismatic plane frame element with parabolic varying depth is presented .The derivation considered the coupling between the axial force, shear force and bending moment .A numerical example is carried out on a beam element using the derived stiffness matrix showing the difference between the exact solution and that obtained by finite element method by dividing the element into a number of prismatic elements is presented. The obtained results show the validity and efficiency of the derived stiffness matrix. It is found that the deflection obtained by using the finite element method is greater than the exact solution by (**1.14%**).

Keywords: **S**tiffness matrix; non-prismatic element; parabolic varying depth; plane frame element

الخلاصة

في هذا البحث تم اشتقاق مصفوفة الصلابة المضبوطة لعضو إنشائي ثنائي الأبعاد غير موشوري متغير السمك لاخطيا (منحني مكافئ) . في هذا الاشتقاق تم الأخذ بنظر الاعتبار تداخل التأثير بين كل من :القوة المحورية,قوة القص وعزم الانحناء .تم حل مثال واحد يتضمن تحليل عضو إنشائي باستخدام المصفوفة المشتقة ومقارنة النتائج التي تم الحصول عليها مع تلك التي تم الحصول عليها باستخدام طريقة العناصر المحددة بتقسيم العضو إلى عدد من العناصر الموشورية.من خلال النتائج التي تم الحصول عليها تبين صحة وفاعلية المصفوفة المشتقة .ومن خلال النتائج أيضا, تبين أن الهطول الناتج باستخدام طريقة العناصر المحددة أعلى من الحل المضبوط بـ (1,14%).

1.Introduction

Members having varying depth are frequently used in many engineering structures, such as haunched beams for bridges or portal frames, as precast roof girders, or as cantilever slabs. In addition, members with varying depth are commonly utilized in mechanical and aerospace engineering structures.In civil engineering construction, non-prismatic members offer several advantages such as;(1)larger stiffness at the ends of the span reduces the positive moment due to gravity loads and increases the overall stability and stiffness;(2) economical design that translated into larger or taller structures and (3) larger beam- column area (joints) to resist greater shear and moment. Up to now, there is no exact stiffness matrix for members with parabolic varying depth .**Head and Aristizabal-Ochoa (1987)** presented a computer algorithm that attempts to predict the behavior of slender ,linearly tapered,reinforced concrete columns.The conjugate beam method was used to calculate the second order effects (i.e.,the additional moment caused by the applied axial load)**. Feris and Kneene (1990)** proved that the elastic and inelastic analysis of members with continuously varying moment of inertia (I_x) and modulus of elasticity (E),along the length of the member, could be carried out by using linear equivalent systems of constant stiffness (EI). Avery good approximate solution was obtained that reduces the mathematical complexity of the problem. **Hashim (1999)** derived a numerical form stiffness matrix for element having linearly varying depth .The element is divided into three zones each zone have a specified flexural rigidity (EI).The derivation considered the coupling between the axial force,shear force and bending moment. The derived stiffness matrix is used in the inelastic analysis of plane frames.

The purpose of the present paper is to derive an exact stiffness matrix for a plane frame element with parabolic varying depth.

2. Derivation Of Stiffness Matrix For An Element With Parabolic Varying Depth:

Consider the element of length (L) shown in **Fig.1(a)**. The element is rectangular in cross-section with parabolic varying depth and constant width. Three degrees of freedom are assumed at each node Only the deformations in the plane of the element and the bending about centroidal main axis are considered. All displacements and forces are positive if they are in the directions shown in **Fig.1(b)**.

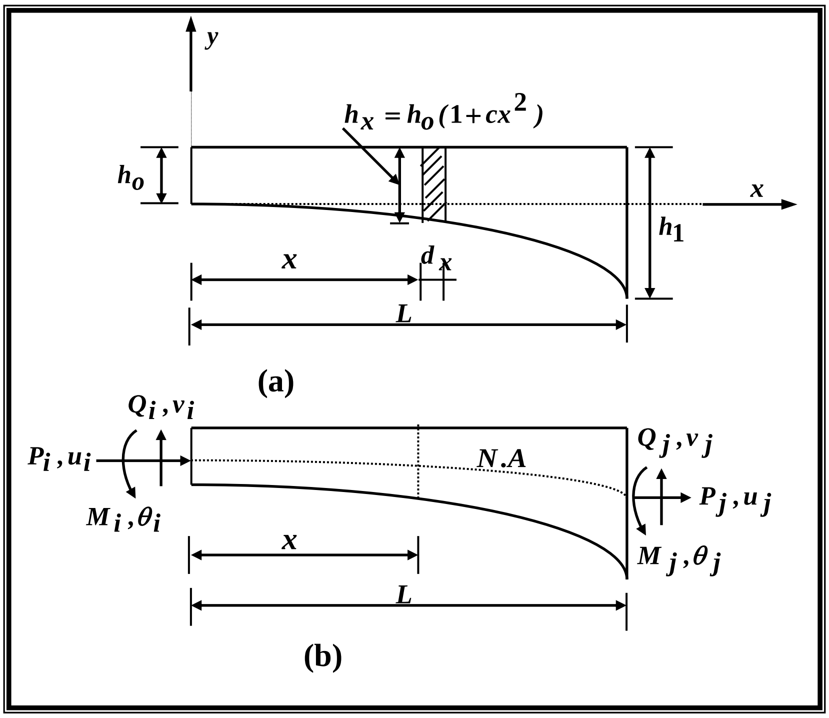

Fig.1:Plane Frame Element with Parabolic varying Depth (a) Typical Element (b) Degrees of Freedom

The stiffness coefficients corresponding to the degrees of freedom shown in **Fig.1(b)** can be obtained by using **Castigliano's** second theorm **Boresi, (2003),** which states that the deflection caused by an external force is equal to the partial derivative of the strain energy (U) with respect to that force .
The strain energy (U) is

$$U = \int \frac{P^2}{2AE} dx + \int \frac{M^2}{2EI} dx \tag{1}$$

Where P *and* M are the axial force and bending moment at section (x) from the left edge which can be found from equilibrium as follows

$$P = P_i \tag{2}$$

$$M_{(x)} = Q_i x + \frac{1}{2} P_i h_o (cx^2) - M_i \tag{3}$$

A and I are the area and the moment of inertia at the same section which can be written as

$$A = A_{(x)} = bh_o(1+cx^2) = A_o(1+cx^2) \tag{4}$$

$$I = I_{(x)} = \frac{bh_o^3}{12}(1+cx^2)^3 = I_o(1+cx^2)^3 \tag{5}$$

Where (c) is a constant depends on the greatest depth of the element as follows :
$h_{(x)} = h_o(1+cx^2)$, if $h_{(x)} = h_1$ at $x = L$ then,

$$c = \frac{h_1 - h_o}{h_o L^2} \tag{6}$$

The strain energy can be written as

$$U = \frac{1}{2E}\left[\int_0^L \frac{P_i^2}{A_{(x)}} dx + \int_0^L \frac{(M_{(x)})^2}{I_{(x)}} dx\right] \tag{7}$$

By substitution of eqs.(2),(3),(4),(5) in eq.(7) and after integrations , the final expression for the strain energy is

$$U = \frac{1}{2E}\left[P_i^2(a_o) + \frac{P_i^2 h_o^2}{4I_o}(a_1) - \frac{P_i M_i h_o}{I_o}(a_2) + \frac{M_i^2}{I_o}(a_3) + \frac{P_i Q_i h_o}{I_o}(a_4) - \frac{M_i Q_i}{I_o}(a_5) + \frac{Q_i^2}{I_o}(a_6)\right] \tag{8}$$

Where

$$a_o = \frac{\theta}{\sqrt{c} A_o} \tag{9}$$

$$a_1 = \frac{1}{\sqrt{c}}\left[\frac{3}{4}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right] - \frac{1}{4}(\sin\theta)^3 \cos\theta\right] \tag{10}$$

$$a_2 = \frac{1}{4\sqrt{c}}\left[\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right] \tag{11}$$

$$a_3 = \frac{1}{\sqrt{c}}\left[\frac{3}{4}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right] + \frac{1}{4}\left[(\cos\theta)^3 \sin\theta\right]\right] \tag{12}$$

$$a_4 = \frac{1}{2c}\left[\frac{1}{2(1+cL^2)^2} - \frac{1}{(1+cL^2)} + \frac{1}{2}\right] \tag{13}$$

$$a_5 = \frac{1}{2c}\left[1 - \frac{1}{(1+cL^2)^2}\right] \tag{14}$$

$$a_6 = \frac{1}{c}[a_2] \tag{15}$$

Where $\theta = tan^{-1}(\sqrt{c}L)$
The partial derivative of the strain energy (U) with respect to (P_i) is

$$\frac{\partial U}{\partial P_i} = u_i = \frac{1}{2E}\left[P_i(2a_o + \frac{h_o{}^2 a_1}{2I_o}) - \frac{M_i h_o a_2}{I_o} + \frac{Q_i h_o a_4}{I_o}\right] \tag{16}$$

Similarly, the partial derivative of the strain energy (U) with respect to Q_i and M_i are

$$\frac{\partial U}{\partial Q_i} = v_i = \frac{1}{2E}\left[\frac{P_i h_o a_4}{I_o} - \frac{M_i a_5}{I_o} + \frac{2Q_i a_6}{I_o}\right] \tag{17}$$

$$\frac{\partial U}{\partial M_i} = \theta_i = \frac{1}{2E}\left[\frac{-P_i h_o a_2}{I_o} + \frac{2M_i a_3}{I_o} - \frac{Q_i a_5}{I_o}\right] \tag{18}$$

The stiffness coefficient (k_{ij}) is the force of type (i) required to induce a unit displacement of type (j) and all other displacements equal to zero. Therefor eqs.(16),(17)and (18) will be used to find the stiffness matrix of the element.

2.1 Axial Stiffness

Consider the element shown in **Fig.(2)** which is subjected to a unit axial displacement. The stiffness coefficients corresponding to that displacement can be found by setting eqs.(16),(17)and(18) equal to 1 , 0and 0 respectively, hence

$$P_i(2a_o + \frac{h_o{}^2 a_1}{2I_o}) - \frac{M_i h_o a_2}{I_o} + \frac{Q_i h_o a_4}{I_o} = 2E \tag{19}$$

$$\frac{P_i h_o a_4}{I_o} - \frac{M_i a_5}{I_o} + \frac{2Q_i a_6}{I_o} = 0 \tag{20}$$

$$\frac{-P_i h_o a2}{I_o} + \frac{2M_i a3}{I_o} - \frac{Q_i a5}{I_o} = 0 \tag{21}$$

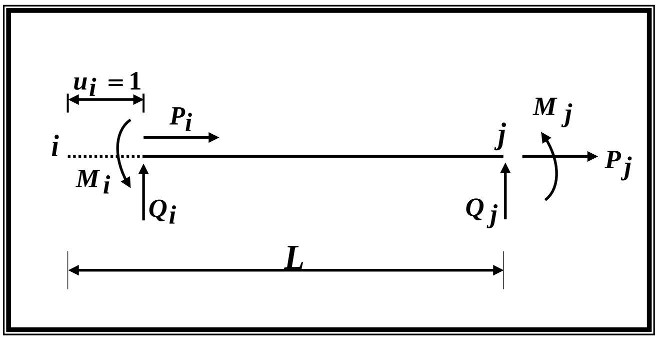

Fig.(2):Element with Parabolic Varying Depth Subjected to a Unit Axial Displacement

Solving eqs.(19),(20) and (21) simultaneously yields :

$$P_i = \frac{2EI_o}{h_o{}^2 Z_1} \tag{22}$$

$$Q_i = \frac{2EI_o}{a_5 h_o Z_1}(2a_3 Z_2 - a_2) \tag{23}$$

$$M_i = \frac{2EI_o}{h_o}\frac{Z_2}{Z_1} \tag{24}$$

Where $Z_1 = \left[\frac{2I_o}{h_o{}^2}a_o + \frac{1}{2}a_1 - a_2 Z_2 + \frac{a_4}{a_5}(2a_3 Z_2 - a_2)\right], Z_2 = \frac{(\frac{2a_6 a_2}{a_5} - a_4)}{(\frac{4a_6 a_3}{a_5} - a_5)}$

From equilibrium requirements

$$P_j = -P_i = \frac{-2EI_o}{h_o{}^2 Z_1} \tag{25}$$

$$Q_j = -Q_i = \frac{2EI_o}{a_5 h_o Z_1}(a_2 - 2a_3 Z_2) \tag{26}$$

$$\begin{aligned} M_j &= P_i\left(\frac{h_1 - h_o}{2}\right) + Q_i L - M_i \\ &= \frac{2EI_o}{Z_1}\left[\frac{(h_1 - h_o)}{2h_o{}^2} + \frac{(2a_3 Z_2 - a_2)L}{a_5 h_o} - \frac{Z_2}{h_o}\right] \end{aligned} \tag{27}$$

2.2 Transverse Stiffness

Proceeding as in the previous section,stiffness coefficients due to a unit lateral displacement at end *i* (**Fig.(3)**) can be found by making expressions (16),(17) and (18) equal to 0 , 1 and 0 respectively, hence

$$P_i(2a_o+\frac{h_o^2 a_1}{2I_o})-\frac{M_i h_o a_2}{I_o}+\frac{Q_i h_o a_4}{I_o}=0 \quad (28)$$

$$P_i(\frac{h_o a_4}{I_o})-\frac{M_i a_5}{I_o}+\frac{2Q_i a_6}{I_o}=2E \quad (29)$$

$$-P_i(\frac{h_o a_2}{I_o})+\frac{2M_i a_3}{I_o}-\frac{Q_i a5}{I_o}=0 \quad (30)$$

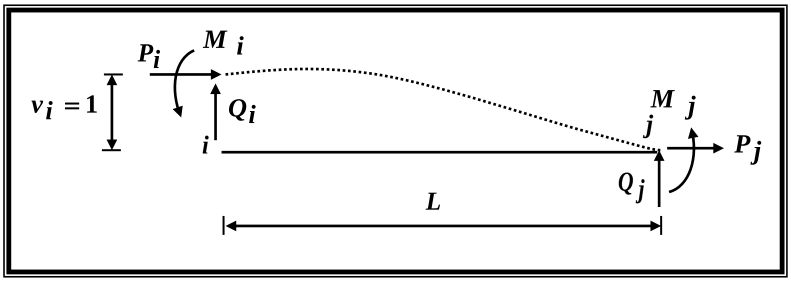

Fig.(3): Element with Parabolic Varying Depth Subjected to a Unit Lateral Displacement

Again, by solving eqs.(28),(29) and (30) simultaneously , the following expressions can be obtained

$$P_i=\frac{2EI_o}{a_2 h_o Z_3}\left[2a_3 Z_4-a_5\right] \quad (31)$$

$$Q_i=\frac{2EI_o}{Z_3} \quad (32)$$

$$M_i=\frac{2EI_o Z_4}{Z_3} \quad (33)$$

Where , $Z_3=\frac{a_4}{a_2}\left[2a_3 Z_{4-} a_5\right]-a_5 Z_4+2a_6$, $Z_4=\frac{\left[(a_1+\frac{4I_o a_o}{h_o^2})\frac{a_5}{2a_2}\right]-a_4}{\left[(a_1+\frac{4I_o a_o}{h_o^2})\frac{a_3}{a_2}\right]-a_2}$

Also, due to equilibrium requirements

$$P_j=\frac{2EI_o}{a_2 h_o Z_3}\left[a_5-2a_3 Z_4\right] \quad (34)$$

$$Q_j=\frac{-2EI_o}{Z_3} \quad (35)$$

$$M_j = P_i\left(\frac{h_1-h_o}{2}\right)+Q_i L-M_i$$
$$=\frac{2EI_o}{Z_3}\left[\frac{(2a_3Z_4-a_5)(h_1-h_o)}{2a_2h_o}+L-Z_4\right] \qquad (36)$$

2.3 Rotational Stiffness

Stiffness coefficients corresponding to a unit rotational displacement at node (i),as shown in **Fig.(4)** can be found by setting expressions (16),(17),and (18) equal to 0 , 0 and 1 respectively, so

$$P_i\left(2a_o+\frac{h_o^2a_1}{2I_o}\right)-\frac{M_ih_oa_2}{I_o}+\frac{Q_ih_oa_4}{I_o}=0 \qquad (37)$$

$$P_i\left(\frac{h_oa_4}{I_o}\right)-\frac{M_ia_5}{I_o}+\frac{2Q_ia_6}{I_o}=0 \qquad (38)$$

$$\frac{-P_ih_oa2}{I_o}+\frac{2M_ia3}{I_o}-\frac{Q_ia5}{I_o}=2E \qquad (39)$$

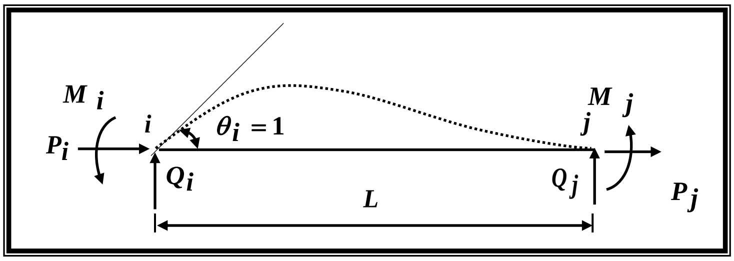

Fig.(4): Element with Parabolic Varying Depth Subjected to a Unit Rotational Displacement

Solving eqs.(37),(38) and (39) simultaneously , yields

$$P_i=\frac{4EI_o}{h_oZ_5}\frac{(a_2-a_4Z_6)}{\left(a_1+\frac{4I_oa_o}{h_o^2}\right)} \qquad (40)$$

$$Q_i=\frac{2EI_oZ_6}{Z_5} \qquad (41)$$

$$M_i=\frac{2EI_o}{Z_5} \qquad (42)$$

Where, $Z_5 = 2a_2 \dfrac{(a_4 Z_6 - a_2)}{(a_1 + \dfrac{4I_o a_o}{h_o^2})} - a_5 Z_6 + 2a_3$, $Z_6 = \dfrac{(a_5 - \dfrac{2a_4 a_2}{(a_1 + \dfrac{4I_o a_o}{h_o^2})})}{(2a_6 - \dfrac{2a_4^2}{(a_1 + \dfrac{4I_o a_o}{h_o^2})})}$

From equilibrium, the following expressions can be obtained

$$P_j = \frac{4EI_o}{h_o Z_5} \frac{(a_4 Z_6 - a_2)}{(a_1 + \dfrac{4I_o a_o}{h_o^2})} \tag{43}$$

$$Q_j = \frac{-2EI_o Z_6}{Z_5} \tag{44}$$

$$M_j = P_i\left(\frac{h_1 - h_o}{2}\right) + Q_i L - M_i$$
$$= \frac{2EI_o}{Z_5}\left[\frac{(a_2 - a_4 Z_6)(h_1 - h_o)}{(a_1 + \dfrac{4I_o a_o}{h_o^2})h_o} + Z_6 L - 1\right] \tag{45}$$

All other stiffness coefficients can be found from symmetry and equilibrium requirements. The coefficients of the $6*6$ stiffness matrix in the local coordinates system according to the degrees of freedom shown in **Fig.1(b)** is as follows :

$$K_{11} = \frac{2EI_o}{h_o^2 Z_1} \tag{46}$$

$$K_{21} = \frac{2EI_o}{a_5 h_o Z_1}(2a_3 Z_2 - a_2) \tag{47}$$

$$K_{31} = \frac{2EI_o}{h_o} \frac{Z_2}{Z_1} \tag{48}$$

$$K_{41} = \frac{-2EI_o}{h_o^2 Z_1} \tag{49}$$

$$K_{51} = \frac{2EI_o}{a_5 h_o Z_1}(a_2 - 2a_3 Z_2) \tag{50}$$

$$K_{61} = \frac{2EI_o}{Z_1}\left[\frac{(h_1 - h_o)}{2h_o^2} + \frac{(2a_3Z_2 - a_2)L}{a_5h_o} - \frac{Z_2}{h_o}\right] \quad (51)$$

$$K_{22} = \frac{2EI_o}{Z_3} \quad (52)$$

$$K_{32} = \frac{2EI_oZ_4}{Z_3} \quad (53)$$

$$K_{42} = \frac{2EI_o}{a_2h_oZ_3}\left[a_5 - 2a_3Z_4\right] \quad (54)$$

$$K_{52} = \frac{-2EI_o}{Z_3} \quad (55)$$

$$K_{62} = \frac{2EI_o}{Z_3}\left[\frac{(2a_3Z_4 - a_5)(h_1 - h_o)}{2a_2h_o} + L - Z_4\right] \quad (56)$$

$$K_{33} = \frac{2EI_o}{Z_5} \quad (57)$$

$$K_{43} = \frac{4EI_o}{h_oZ_5}\frac{(a_4Z_6 - a_2)}{(a_1 + \frac{4I_oa_o}{h_o^2})} \quad (58)$$

$$K_{53} = \frac{-2EI_oZ_6}{Z_5} \quad (59)$$

$$K_{63} = \frac{2EI_o}{Z_5}\left[\frac{(a_2 - a_4Z_6)(h_1 - h_o)}{(a_1 + \frac{4I_oa_o}{h_o^2})h_o} + Z_6L - 1\right] \quad (60)$$

$$K_{44} = \frac{2EI_o}{h_o^2Z_1} \quad (61)$$

$$K_{54} = \frac{2EI_o}{a_2h_oZ_3}\left[2a_3Z_4 - a_5\right] \quad (62)$$

$$K_{64} = \frac{2EI_o}{h_o}\left[\frac{(a_5 - 2a_3Z_4)L}{a_2Z_3} + \frac{2(a_2 - a_4Z_6)}{(a_1 + \frac{4I_oa_o}{h_o^2})Z_5} - \frac{(h_1 - h_o)}{2h_oZ_1}\right] \quad (63)$$

$$K_{55}=\frac{2EI_o}{Z_3} \tag{64}$$

$$K_{65}=\frac{2EI_o}{h_o}\left[\frac{(a_2-2a_3Z_2)(h_1-h_o)}{2a_5Z_1}+\frac{h_oZ_6}{Z_5}-\frac{h_{oL}}{Z_3}\right] \tag{65}$$

$$K_{65}=\left[(K_{61})\frac{(h_1-h_o)}{2}+(K_{62})L-(K_{63})\right] \tag{66}$$

The above coefficients can be written in a matrix form as follows

$$[K]=\begin{bmatrix} K_{11} & & & & & \\ K_{21} & K_{22} & & & sym. & \\ K_{31} & K_{32} & K_{33} & & & \\ K_{41} & K_{42} & K_{43} & K_{44} & & \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix}$$

3. Numerical Example

To verify the validity and efficiency of the derived stiffness matrix, a cantilever beam shown in **Fig.5(a)** is analyzed first by using the derived stiffness matrix then the beam is idealized to a number of equal lengths prismatic finite elements(**Fig.5(b)**) and analyzed using stiffness method. A computer program **“NASPAC”** is used for the analysis by using the finite element method **.**A comparison study is made between the exact solution using the derived stiffness matrix and the approximate solution.The effect of shear deformations is neglected in both analyses.The comparison is shown in **Fig.6**which represents the relationship between vertical deflection at free end with number of elements using both exact and approximate solution. The deflection obtained from the approximate method by dividing the beam into six elements is greater than the exact solution by (**1.14%**).It is clear from **Fig.6** that the exact stiffness matrix derived in this paper offer an efficient and less calculations solution than that by using finite elements.

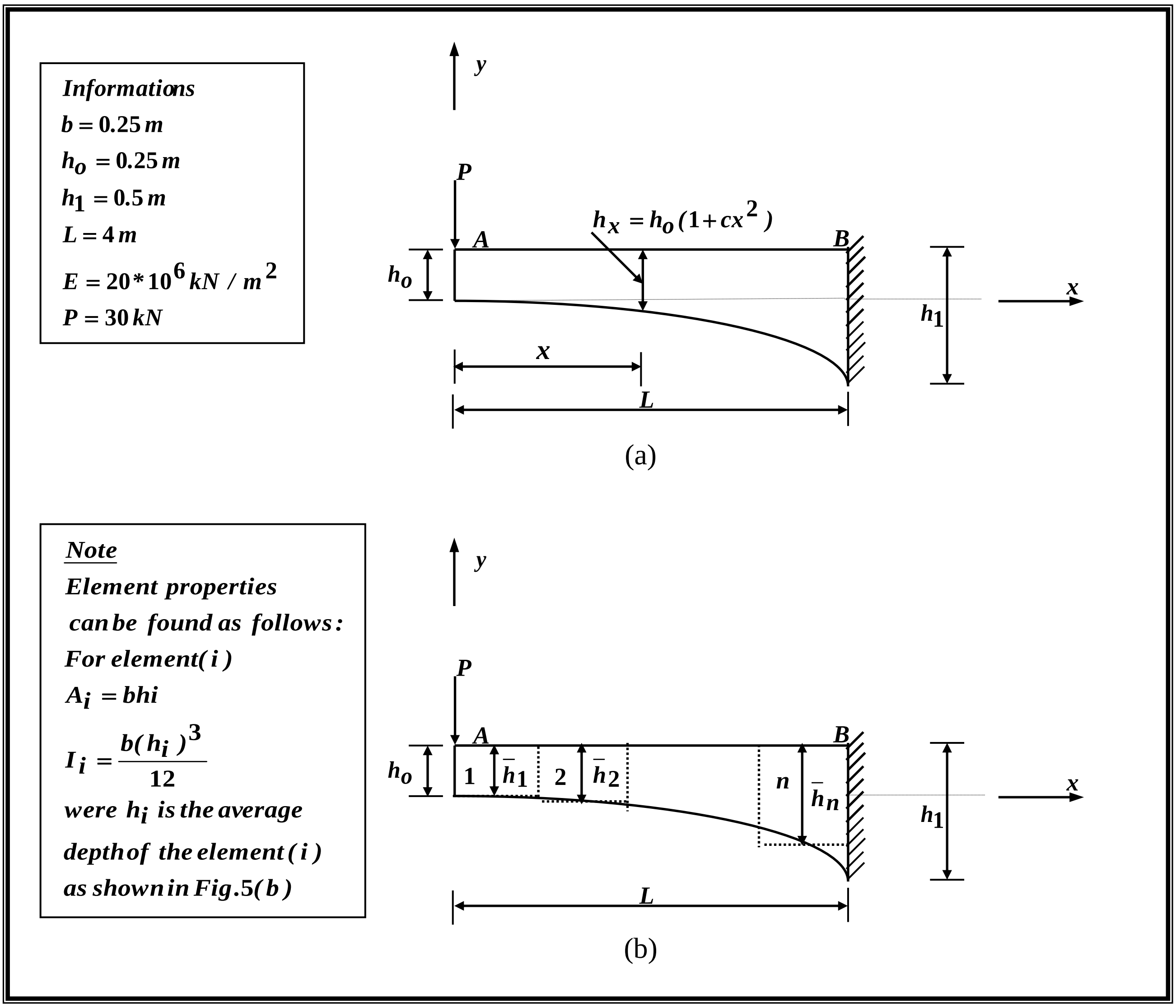

Fig.5: The Cantilever Beam of the Numerical Example
(a) Load and Dimensions (b) Approximating the Beam to a Number of Prismatic Elements

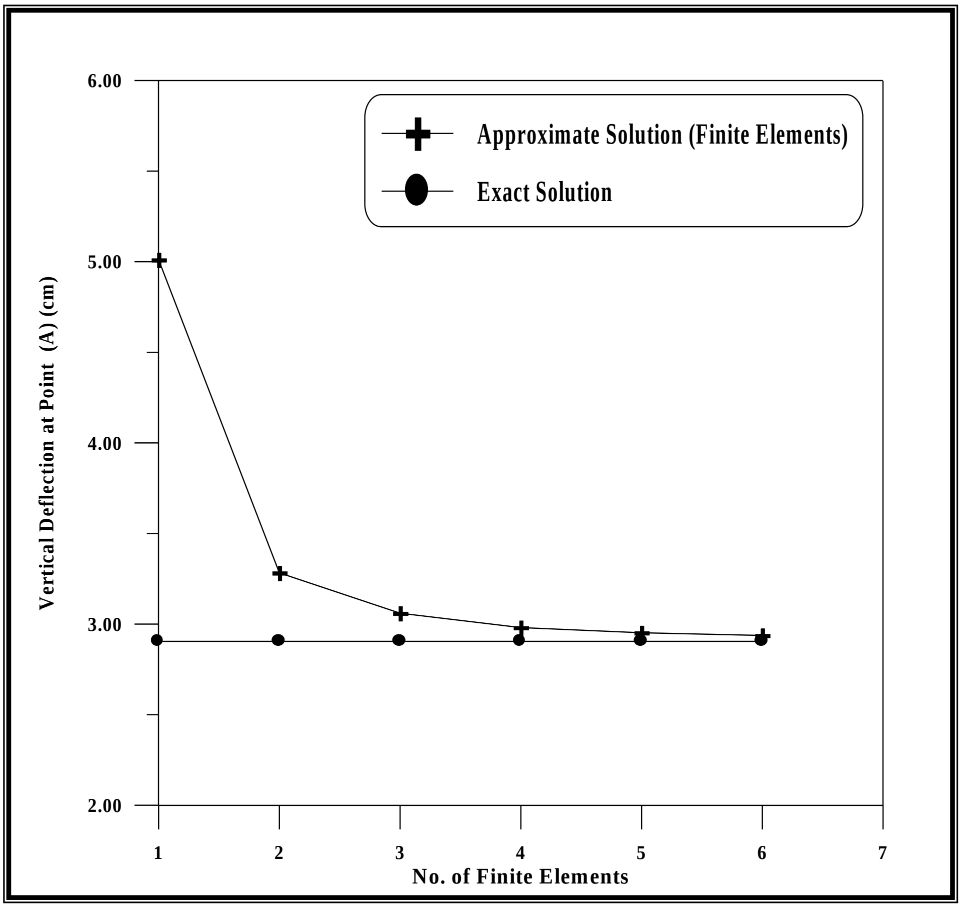

Fig.6: Vertical Deflection at Free End (Point A)

4.Conclusions

The exact stiffness matrix derived in this paper can be used directly and successfully in the analysis of girders and frames consisting of members having parabolic varying depth. The derived stiffness matrix offer an efficient and less numerical calculations solution than that by using finite elements .In addition,it is clear from the above figure that the non-prismatic member with parabolic varying depth is stiffer and more economic than that consisting of a number of prismatic elements. It is found that the deflection obtained by using the finite element method is greater than the exact solution by (**1.14%**).

5. References

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6.Notations

The following symbols are used in this paper

A_o **:** cross-sectional area of the smallest cross-section of the element

b **:** width of the element cross-section

E **:** modulus of elasticity

h_o **:** the smallest depth of the element

h_1 **:** the greatest depth of the element

I_o **:** the moment of inertia of the smallest cross-sectional area about the major axis

$[K]$ **:** the stiffness matrix

k_{ij} **:** the stiffness coefficient

L **:** length of the element

M **:** moment at a specified section

P **:** axial force at a specified section

Q **:** shear force at a specified section

U **:** the strain energy

u **:** the horizontal displacement

v **:** the vertical displacement