Neural Network Trigonometric Approximation

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Abstract

In this paper, we define a weighted norm to construct a weighted $L_{p,\alpha}$-space for 2π-periodic functions. Then, we prove that any $f \in L_{p,\alpha}([-\pi,\pi]^m)$ is approximated by a feedforward neural network with sigmoidal hidden neuron in terms of $\omega_k(f,\delta)_{p,\alpha}$, That's what we called Neural Networks Trigonometric Approximation.

Keywords: Neural Network, Trigonometric Approximation, Modulus of Smoothness.

الخلاصة

قمنا في هذا البحث بتعريف معيار موزون لنشكل من خلاله فضاءً موزوناً $L_{p,\alpha}([-\pi,\pi]^m)$ للدوال الدورية التي تمتلك 2π من الدورات. ذلك واستخدمنا هذا المفهوم أعلاه لاثبات أنّ من الممكن تقريب كل دالة تنتمي إلى الفضاء المذكور آنفاً بواسطة شبكة عصبية تقدمية ذات عصبونات سينية وذلك بدلالة $\omega_k(f,\delta)_{p,\alpha}$. وهذا ما يسمى بالتقريب المثلثاتي للشبكات العصبية.

الكلمات المفتاحية: شبكات عصبية ، تقريب مثلثاتي ، مقياس النعومة.

1. Introduction

In recent years, trigonometric polynomials have a main rule in approximation functions in L_p-space as well as other more general spaces. They are widely used to find neural networks as approximators for those functions, see for examples [Coa and Zhang 2007,Wang and Xu 2011,Wang,Chen and Yang 2012], for the great importance for neural networks in different fields, and the essential need for approximated neural networks in different applications, such as [Dineva, *et al.*, 2015, Ferrari and Stengel 2005, Lin and Cao 2015].

Three-layer feedforward neural network(FNN) is an important class of neural networks that can approximate the desired function well, see [Hornik *et al.*, 1989, Suzuki 1998]. Most papers deal with the rate of approximation as a tool to understand the approximation capability, see [Lin and Cao 2015, Li and Xu 2007, Wang and Xu 2010].

In this work, we care to find a new space with a new norm to shit light on the relationship between the approximation error and the properties of the used neural network. For our space, a feedforward neural network (FNN) of three layer can be existed to approximate the present function well.

2. Preliminaries

First of all, we need to define a three layer FNNs with one hidden layer as follow [Wang Chen & Yang 2012]

$$"N(x) = \sum_{i=1}^{m} C_i \emptyset\left(\sum_{j=1}^{d} w_{ij}.x_j + \varphi_i\right), x \in R^d, d \geq 1" \qquad (2.1)$$

that generates one output of that d inputs, with thresholds $\varphi_i \in R$, connection weights $w_{ij} \in R^d$ and $\emptyset$ is the activation function. We use the sigmoid activation function that satisfies

$$\emptyset(t) = \begin{cases} 1, & as\ t \to +\infty \\ 0, & as\ t \to -\infty \end{cases} \qquad (2.2)$$

Given $m \in N$, $\boldsymbol{x} = (x_i)_{i=1}^m \in N_0^m$, where $|\boldsymbol{x}| = \sum_{i=1}^m |x_i|$, $\boldsymbol{rx} = \sum_{i=1}^m r_i x_i$. We define the weighted L_p-space $L_{p,\alpha}([-\pi,\pi]^m), p \geq 1, \alpha \geq \frac{1}{p}$ such that for any $f \in L_{p,\alpha}([-\pi,\pi]^m)$, we define the norm

$$\|f\|_{p,\alpha} = (2\pi)^{-m} \left(\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(\boldsymbol{x})|^p |\boldsymbol{x}|^{-\alpha p} dx_1 \dots dx_m \right)^{\frac{1}{p}} \qquad (2.3),$$

We estimate the trigonometric approximation in terms of modulus of smoothness

$$\omega_k(f,\delta)_p = \sup_{\|\mathbf{t}\| \leq \delta} \left\| \Delta_{\mathbf{t}}^k(\mathbf{t}) \right\|_p \qquad (2.4),$$

where,

$$\Delta_{\mathbf{t}}^k f(\mathbf{x}) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(\mathbf{x} + (\tfrac{\boldsymbol{k}}{\mathbf{2}} - \boldsymbol{j})\mathbf{t}), \qquad (2.5)$$

Our main result is as follow:

Theorem 1. For $\in L_{p,\alpha}([-\pi,\pi]^m), p \geq 1, \alpha \geq \frac{1}{p}$, there exist FNN of the form (2.1) with $x \in [-\pi,\pi]^m, C_i \in R$, such that

$$\|N(x) - f(x)\|_{p,\alpha} \leq \omega_m(f,\delta)_{p,\alpha} \qquad (1.6)$$

3. Auxiliary Lemma

This lemma is useful to construct a FNN that approximate trigonometric function of m-dimension. FNN (that has piecewise linear and activation function hidden layer units of sigmoid type) constructed by Suzuki [8] and then developed by Wang and Xu[10] can be more accurate and suitable for $L_{p,\alpha}([-\pi,\pi]^m)$ in the next lemma.

Lemma: For $\sigma \in N$, we can find networks of three layers $NS_\sigma(\boldsymbol{rx})$ and $NC_\sigma(\boldsymbol{rx})$ to approximate $\sin(\boldsymbol{rx})$ and $cos(\boldsymbol{rx})$, respectively, each one has q hidden layer units , where $q = 4|\boldsymbol{r}|\sigma$ related to $NL_\sigma(\boldsymbol{rx})$ s.t.

$$NS_\sigma(\boldsymbol{rx}) = 2(-1)^{|\boldsymbol{r}|} sin\frac{\pi}{4\sigma} \sum_{k=0}^{q-1} (-1)^{q-1-k} \binom{q-1}{k} cos\left(\boldsymbol{x} + \left(\frac{q-1}{2}\right)\boldsymbol{r}\right) NP_\sigma(\boldsymbol{rx})$$

and

$$NC_\sigma(\boldsymbol{rx}) = (-1)^{|\boldsymbol{r}|} - 2(-1)^{|\boldsymbol{r}|} sin\frac{\pi}{4\sigma} \sum_{k=0}^{q-1} (-1)^{q-1-k} \binom{q-1}{k} sin\left(\boldsymbol{x} + \left(\frac{q-1}{2}\right)\boldsymbol{r}\right) NP_\sigma(\boldsymbol{rx})$$

where

$$NP_{\sigma}(\boldsymbol{rx}) = \frac{1}{(1+e^{-\boldsymbol{rx}})}$$

then

$$\|\sin(\boldsymbol{rx}) - NS_{\sigma}(\boldsymbol{rx})\|_{p,\alpha} = \|\cos(\boldsymbol{rx}) - NC_{\sigma}(\boldsymbol{rx})\|_{p,\alpha} \le c(p)\frac{1}{|\boldsymbol{r}|^{\alpha}}$$

Proof:

It is enough to prove the first estimate,

$$\|\sin(\boldsymbol{rx}) - NS_{\sigma}(\boldsymbol{rx})\|_{p,\alpha}$$

$$= (2\pi)^{-m}\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}|\sin(\boldsymbol{rx}) - NS_{\sigma}(\boldsymbol{rx})|^{p}|\boldsymbol{rx}|^{-\alpha p}dx_1 \ldots dx_m\right)^{\frac{1}{p}}$$

$$\le (2\pi)^{-m}(2)^{-m}\left\{\int_{0}^{\pi}\cdots\int_{0}^{\pi}|\sin(\boldsymbol{rx})|^{p}|\boldsymbol{rx}|^{-\alpha p}dx_1 \ldots dx_m\right\}^{\frac{1}{p}}$$

$$+2(-1)^{|r|}(2\pi)^{-m}(2)^{-m}sin\frac{\pi}{4\sigma}\sum_{k=0}^{q-1}\left\{\int_{0}^{\pi}\cdots\int_{0}^{\pi}\left|(-1)^{q-1-k}\binom{q-1}{k}cos\left(\boldsymbol{x}\right.\right.\right.$$

$$\left.\left.\left.+\left(\frac{q-1}{2}\right)\boldsymbol{r}\right)\right|^{p}\frac{|\boldsymbol{rx}|^{-\alpha p}}{(1+e^{-\boldsymbol{rx}})}dx_1 \ldots dx_m\right\}^{\frac{1}{p}}$$

$$\le 2^{p-1}\left\{(\pi)^{-m}\pi^{m}(|\boldsymbol{r}|\pi)^{-\alpha} + (\pi)^{-m}\sum_{k=0}^{q-1}\binom{q-1}{k}\pi^{m}(|\boldsymbol{r}|\pi)^{-\alpha}\right\}$$

$$\le c(p)\frac{1}{|\boldsymbol{r}|^{\alpha}}$$

where

$$|\boldsymbol{rx}| = \left|\sum_{i=1}^{m} r_i x_i\right| \le \sum_{i=1}^{m}|r_i x_i| \le \sum_{i=1}^{m}|r_i|\pi = |\boldsymbol{r}|\pi\,.$$

since $\boldsymbol{x} \in [-\pi,\pi]^{m}$. ■

4. Proof of Theorem 1.

Let > 0, "$t(x) = \sum_{|r|<n}\left(a_r sin(rx) + b_r cos(rx)\right)$", and, by lemma 1, choose FNNs

$$N(x) = \sum_{|r|<n} 2(-1)^{|r|} + \sum_{|r|<n}(-1)^{|r|}\,sin\frac{\pi}{4\sigma}\sum_{k=0}^{q-1}(-1)^{q-1-k}\binom{q-1}{k}\left\{cos\left(\boldsymbol{x}+\left(\frac{q-1}{2}\right)\boldsymbol{r}\right)\right.$$

$$\left. - b_r sin\left(\boldsymbol{x}+\left(\frac{q-1}{2}\right)\boldsymbol{r}\right)\right\}NP_{\sigma}(\boldsymbol{rx}),$$

such that

$$\|N - t\|_{p,\alpha} \leq \varepsilon$$

Let $t_n^*(x)$ be the the nth best approximation using trigonometric transformation of N, s.t.

$$\|N - t_n^*(x)\|_{p,\alpha} \leq \varepsilon$$

Now, by triangle inequality

$$\|f - N\|_{p,\alpha} \leq \|N - t_n^*(x)\|_{p,\alpha} + \|f - t_n^*(x)\|_{p,\alpha} \leq \varepsilon + \omega_m(f,\delta)_{p,\alpha}$$

by taking $\omega_m(f,\delta)_{p,\alpha} \geq \varepsilon$, the proof is done.■

5. Conclusions and Future Work

Estimating the rate of function approximation in weighted space with a three layer neural network using trigonometric approximation and module of smoothness is studied. This paper shows the development of the upper bound estimating approximation. In the future, it is useful to think about estimating the lower bound approximation.

6. References

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