Parallel Second Order Runge – Kutta new method of the Geometric Mean

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Abstract :

We present in this paper a new parallel Runge – Kutta method of the second order second stage for solving initial value problems (IVPs) in Ordinary Differential Equations (ODEs) .This new method depending on using the predictor − corrector method in the Geometric mean (GM) Runge – Kutta formula, and make it parallel by using the broadening of the computation front.

Key words: Runge − Kutta methods, predictor − corrector method, the stability, the computation front.

طريقة رونج كوتا مخمن مصحح متوازية جديدة ذات وسط هندسي

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ملخص البحث :

نستعرض في هذا البحث طريقة متوازية من الرتبة والمرحلة الثانية من طرائق رنج كتا المشهورة المستخدمة لحل مسائل القيم الابتدائية من المعادلات التفاضلية الاعتيادية وذلك بالمزج بين صيغة التخمين والتصحيح واستخدام المتوسط الهندسي لطريقة رنج كتا التقليدية كذلك استخدمنا اسلوب تعريض جبهة الحساب لإيجاد طريقة متوازية جديدة.

Introduction

Runge - Kutta methods are one of the best methods for numerically solving (ODEs), and the search for better methods is always up to date [1]. Our concern here is with present a new method for solving Initial Value Problems (IVPs) using a mixing between techniques and formulas and obtain a new formula suitable for parallel computers, (see [2]). As we know a first step toward developing a parallel algorithm for the numerical solution of Initial Value Problems (IVPs), how we might widen the front of computation .The predictor – corrector (PC) methods of numerical integration provide a means for doing this, (see [3,4]) ."Evans Introduce a new Runge - Kutta method using the Geometric mean (GM) formula [5]" (see [6]). Here we collected these ideas and using the Implicit Runge - Kutta methods (IRK) which can be derived directly from Explicit Runge - Kutta methods (ERK) , these implicit methods " that were derived " , it is "the (backward) form of the explicit (forward) form [7]", to present our new parallel method which we called (PPCGM2) formula.

2.1. Definition :

"The front of computation is the imaginary straight line that separate the values are next to be computed from all previously computing value problems [8,9,2]".

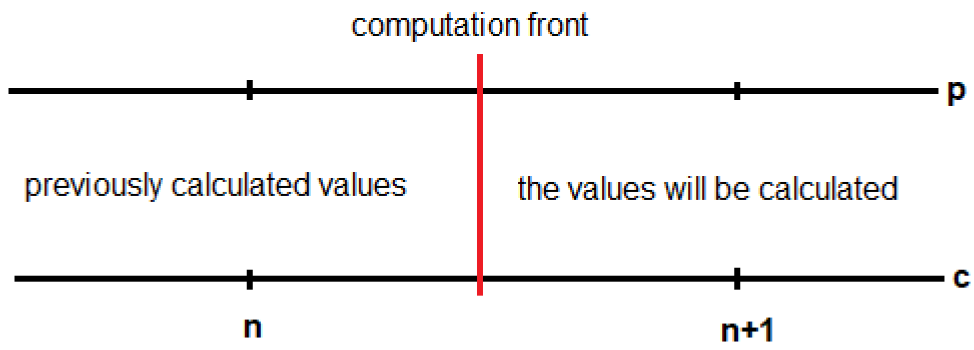

Figure 1 :explain of the definition above.

3. The (GM) formula.

Evans, and Sague [5] introduce new method which has the form

$$y_{n+1} - y_n = \frac{h}{n}\left(\sum_{i=1}^{n}\sqrt{k_i k_{i+1}}\right) \qquad (1)$$

Where,

$$k_i = f\left(x_n + e_i h_i\,, y_n + h\sum_{j=1}^{i-1} g_{ij} k_j\right) \quad i = 1,2,3,\dots,r \qquad (2)$$

Where $e_i\,, g_{ij}\ \epsilon\ [0,h]$ and h_i the step size.

We have considered f as a function of y only. "This will reduce the lengthy Taylor series expansions of $k_i\,,\ i = 1,2,,$

Thus (2) becomes,

$$k_i = f\left(y_n + h\sum_{j=1}^{i-1} a_{ij} k_j\right) \quad i = 1,2,3,\dots,r \ \ [10]". \qquad (3)$$

4. New parallel method namely PPCGM2 :

The new parallel method PPCGM2 calculates y_{n+1}^{p} and y_n^{c} depending on y_n^{p} and y_{n-1}^{c} , which has the form,

$$y_{n+1}^{p} - y_n^{p} = h\left(\sqrt{k_1 k_2}\right) \qquad (4)$$

Where,

$$k_1 = f(x_{n-1}\,,\, y^c_{n-1})\ ,\ k_2 = f(x_{n-1} + 2h\,,\, y^c_{n-1} + 2hk_1) \qquad (5)$$

And,

$$y^c_n - y^c_{n-1} = h\left(\sqrt{J_1 J_2}\right) \qquad (6)$$

Where,

$$J_1 = f(x_n\,,\, y^p_n)\ ,\ J_2 = f(x_n - h\,,\, y^p_n - hJ_1) \qquad (7)$$

Illustrating the computation process of the PPCGM2 mode in fig.2,

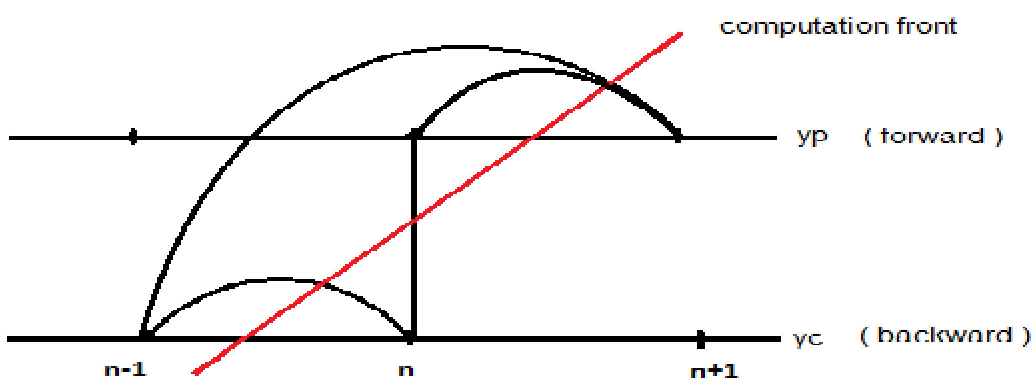

Figure 2 : the PPCGM2 mode.

4.1. Derivation of PPCGM2.

The parallel predictor Geometric mean method of two stage second order has the form,

$$y^p_{n+1} - y^p_n = h\left(\sqrt{ak_1 bk_2}\right) \qquad (8)$$

Where,

$$k_1 = f(y^c_{n-1})\ ,\ k_2 = f(y^c_{n-1} + \alpha h k_1) \qquad (9)$$

To derive our new predictor PPCGM2- method, expansion of k_1 and k_2 gives ,

$k_1 = f(y^c_{n-1}) = f$ **and ,**

$$k_2 = f(\, y_{n-1}^{c} + \alpha h k_1\,) = f + \alpha h k_1 f_{y_{n-1}^{c}} + o(h^2)$$

$$k_2 = \qquad\qquad = f + \alpha h f f_{y_{n-1}^{c}} + o(h^2)$$

Then,

$$ak_1 * bk_2 = af * b(\, f + \alpha h f f_{y_{n-1}^{c}}) + o(h^2)$$

$$abk_1k_2 = abf^2 + ab\alpha h f^2 f_{y_{n-1}^{c}} + o(h^2)$$

Now,

$$(abk_1k_2)^{1/2} = \left(abf^2 + ab\alpha h f^2 f_{y_{n-1}^{c}} + o(h^2)\right)^{1/2}$$

$$= (ab)^{1/2} f \left(1 + \alpha h f_{y_{n-1}^{c}} + o(h^2)\right)^{1/2}$$

$$(abk_1k_2)^{1/2} = (ab)^{1/2} f \left(1 + \frac{\alpha h}{2}\, f_{y_{n-1}^{c}} + o(h^2)\right) \qquad (10)$$

Substitute (10) in (8) we obtain,

$$y_{n+1}^{p} - y_{n}^{p} = (ab)^{1/2} f \left(h + \frac{\alpha h^2}{2}\, f_{y_{n-1}^{c}} + o(h^3)\right) \qquad (11)$$

Comparing equation (11) with Taylor series [11,2] which has the form,

$$y_{n+1} - y_n = hf + \frac{h^2}{2} f f_{y_{n-1}} + o(h^3) \qquad (12)$$

From eq. (11) and (12) we get,

$$(ab)^{1/2} = 1 \ , \alpha = 1$$

We get two equations with three parameters , so that is means one degree of freedom ,choosing $a = 1$ then $b = 1$.

So we get the following system :

$$y_{n+1}^{p} - y_{n}^{p} = h\left(\sqrt{k_1 k_2}\right) \qquad (13)$$

Where,

$$k_1 = f(x_{n-1}\,,\, y_{n-1}^{c})\ ,\ k_2 = f(x_{n-1} + 2h\,,\, y_{n-1}^{c} + 2hk_1) \tag{14}$$

Now to get the corrector method , from the backward formula,

$$y_{n-1}^{c} - y_{n}^{c} = -h(\sqrt{aJ_1 bJ_2}\,) \tag{15}$$

Where,

$$J_1 = f(\ y_n^p\ )\quad ,\ J_2 = f(\ y_n^p - \alpha h J_1\ ) \tag{16}$$

J_1 **and** J_2 **expansion in (16) gives,**

$$J_1 = f(\ y_n^p\ ) = f$$

$$J_2 = f(\ y_n^p - \alpha h J_1\ ) = f - \alpha h f f_{y_n^p} + o(h^2)$$

then,

$$aJ_1 = af$$

$$bJ_2 = b\left(f - \alpha h f f_{y_n^p} + o(h^2)\right)$$

$$bJ_2 = bf - b\alpha h f f_{y_n^p} + o(h^2)$$

$$abJ_1J_2 = abf^2 - ab\alpha h f^2 f_{y_n^p} + o(h^2)$$

$$(abJ_1J_2)^{1/2} = \left(abf^2 - ab\alpha h f^2 f_{y_n^p} + o(h^2)\right)^{1/2}$$

$$= (ab)^{1/2} f\left(1 - \alpha h\, f_{y_n^p} + o(h^2)\right)^{1/2}$$

$$= (ab)^{1/2} f\left(1 - \frac{\alpha h}{2}\ f_{y_n^p} + o(h^2)\right) \tag{17}$$

Equation (17) Substituting in equation (15) we get,

$$y_{n-1}^{c} - y_{n}^{c} = -h(ab)^{1/2} f + \frac{\alpha h^2}{2}\ f_{y_n^p} + o(h^3) \tag{18}$$

Comparing equation (18) with Taylor series [11] below which has the form,

$$y_{n-1} - y_n = -hf + \frac{h^2}{2} f f_{y_n} + o(h^3) \quad (19)$$

Getting ,

$$(ab)^{1/2} = 1 \text{ , } \alpha = 1$$

Which is two equations of three parameters, so we have one degree of freedom .

Choosing $a = 1$ then we have $b = 1$.

Thus the new corrector formula has the form,

$$y_n^c - y_{n-1}^c = h(\sqrt{J_1 J_2}\,) \quad (20)$$

Where,

$$J_1 = f\left(x_n \text{ , } y_n^p\right) \text{ and } J_2 = f\left(x_n - h \text{ , } y_n^p - hJ_1\right) \quad (21)$$

Equations (13) , (14) , (20) and (21) together represent our new PPCGM2-method , Where y_{n+1}^p is the predictor form and y_n^c is the corresponding corrector form.

5. Analysis the stability of PPCGM2 :

The important advantage of Runge - Kutta methods that they are stable, when we have a good quite of stability, "if we take a suitable small step size h".

"Testing the stability of Runge - Kutta methods by the test equation $\acute{y} = \lambda y$ where $\lambda = \partial f / \partial y$ constant [12]".

We examine the stability of the predictor of the PPCGM2 method which has the form,

$$y_{n+1}^p - y_n^p = h\left(\sqrt{k_1 k_2}\right)$$

Where,

$$k_1 = f(x_{n-1} \text{ , } y_{n-1}^c) \text{ , } k_2 = f(x_{n-1} + 2h \text{ , } y_{n-1}^c + 2hk_1)$$

Using "the test equation $\acute{y} = \lambda y$ [13,14]" to evaluate the interval of absolute stability,

$$k_1 = f(x_{n-1}\,,\, y^c_{n-1}) = \lambda\, y^c_{n-1} \quad (22)$$

$$k_2 = f(x_{n-1} + 2h\,,\, y^c_{n-1} + 2hk_1) = \lambda(\, y^c_{n-1} + 2h\lambda\, y^c_{n-1}) \quad (23)$$

Equations (22) and (23) substituting in equation (13) we get ,

$$y^p_{n+1} - y^p_n = h\left((\lambda\, y^c_{n-1})(\lambda\, y^c_{n-1} + 2h\lambda^2\, y^c_{n-1})\right)^{1/2}$$

$$y^p_{n+1} - y^p_n = h\lambda\, y^c_{n-1}(\,1 + 2h\lambda)^{1/2} \quad (24)$$

Dividing (24) by y^c_{n-1} and putting $z = h\lambda$ we get,

$$\Upsilon_1 = \frac{y^p_{n+1} - y^p_n}{y^c_{n-1}} = z(\,1 + 2z)^{1/2} \quad (25)$$

The equation (25) satisfies the absolute stability condition if $|\Upsilon_1| < 1$ where z identify the condition when z ∊ (-2, 0.6570) which is stability region of this method.

6. Numerical Example :

x	y	y^c	$\lvert y^c - y\rvert$	y^p	$\lvert y^p - y\rvert$
0	2	2	0	1.980198	0.019802
0.005	1.999950	2	4.993e-05	2	4.999e-05
0.010	1.999800	2.000141	0.000341	2	0.000199
0.015	1.999550	2.000386	0.000836	2.000173	0.000623
0.020	1.999200	2.000733	0.001532	2.000456	0.001255
0.025	1.998750	2.001180	0.002430	2.000843	0.002092
0.030	1.998201	2.001729	0.003527	2.001333	0.003131
0.035	1.997552	2.002378	0.004825	2.001925	0.004372
0.040	1.996805	2.003129	0.006324	2.002619	0.005813
0.045	1.995958	2.003981	0.008023	2.003414	0.007455
0.050	1.995012	2.004934	0.009922	2.004310	0.009298
0.055	1.993968	2.005989	0.012021	2.005308	0.011340
0.060	1.992825	2.007146	0.014320	2.006408	0.013582
0.065	1.991585	2.008405	0.016820	2.007610	0.016024
0.070	1.990247	2.009768	0.019520	2.008913	0.018666
0.075	1.988812	2.011233	0.022420	2.010320	0.021507
0.080	1.987281	2.012802	0.025520	2.011829	0.024548
0.085	1.985653	2.014474	0.028821	2.013441	0.027788
0.090	1.983930	2.016252	0.032322	2.015157	0.031227

0.095	1.982111	2.018135	0.036023	2.016978	0.034866
0.100	1.980198	2.020123	0.039925	2.018903	0.038704

Table 1: represents the results of PPCGM2 method applied to the example $\acute{y} = -xy^2$, y(0)=2 , when h=0.005.

7. Discussion of the numerical results :

Table 1 is results obtained from the PPCGM2 formula . Comparison of the numerical solution of the new method with the exact solution of the considered problem shows that the new method is stable in the z ∈ (-2, 0.6570) interval . This conclusion is clear from reviewing the above table.

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