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n-Refined Neutrosophic Fine Module and Some Applications

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ABSTRACT

We study and present an important concept in module theory, namely, neutrosophic fine module. Fine property for modules and rings. We denote the F module as a fine module. We investigate several facts about the relationships between a fine module and $P.P.F$ ring. Projective and injective modules have a direct relationship with the F-module. We prove that every continuous module is an F module, and the discrete module is an F module. When $R_n(I)$ has a perfect property, then every guasi-projective over $R_n(I)$ has a fine property. We show that every divisible module $M_n(I)$ over the field $F_n(I)$ is a fine module. We study a famous property of a submodule $N_n(I)$ of $M_n(I)$, and we present some concepts that have a direct relationship with a fine module, namely, finitely generated submodule and flat modules. We also prove that every cyclic module $M_n(I)$ in every submodule $N_n(I)$ of $M_n(I)$ is isomorphic to the summand of $M_n(I)$. Then, $M_n(I)$ is a fine module if it has a projective cover. We demonstrate that a fine finitely generated faithful multiplication module, $M_n(I)$, over ring $R_n(I)$ is a C-F-fine module if and only if $R_n(I)$ is a P-F-fine ring. We prove that fine ring $R_n(I)$ is called a P-F-fine ring if and only if every faithful multiplication fine module is a P-F-fine ring.

1. Introduction

In this work, we denote a neutrosophic fine ring as $(FR(I))$. All rings $R_n(I)$ are associative with unity. $U(R_n(I))$ is a symbol of all unities, and $id(R_n(I))$ is a symbol of all idempotent elements. Fine elements are defined in [1]. An element $(rI) \in$ $R_n(I)$ is said to be a fine element if $rI = uI + iI$, $(uI) \in U(R_n(I))$, and $(iI) \in id(R_n(I))$. Nicholson [1] showed that any ring $R_n(I)$ is said to be fine if every element $(rI) \in R_n(I))$ is a fine element. Thus, $R_n(I)$ is called a neutrosophic fine ring if $\forall (rI) \in R_n(I)$, (rI) is a fine element.

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In 1936, Von Neumann reported that any element $(aI),(bI) \in R_n(I)$, such that $aI = (aI)(bI)(aI)$ is a regular element, so $R_n(I)$ is a regular ring if $\forall (aI) \in R_n(I)$ and (aI) is a regular element. Hence, neutrosophic regular ring $R_n(I)$ means that every element in $R_n(I)$ is a neutrosophic regular element. A definition of a fine ring that has some properties similar to those of every regular ring was provided by [3]. The relation between a fine ring with projective and flat modules was examined by [6]. Some information on Nil–fine rings was given by [9]. If an element xI in $R_n(I)$ exhibits the relation $xI = uI + iI$, $(uI) \in U(R_n(I))$ and if $(iI) \in id(R_n(I))$ and

$(uI)(iI)=(iI)(uI)$, then xI is called a strongly neutrosophic fine element [7]. To obtain additional information on the dual nil-q–fine ring, we must study quasi-idempotent elements [10]. Additional details on quasi-fine rings and strongly quasi-fine rings are available in [11]. An element $(qI)\in R_n(I)$ is said to be quasi-idempotent if $qI^2=(uI)(qI),$ and unit $(uI)\in R_n(I)$. $QI(R_n(I))$ denotes the set of quasi-idempotent elements. Nil–fine rings were studied by [8]. The multiplicative dual of nil–fine rings was investigated in [10]. An element (aI) of ring $R_n(I)$ is dual nil–fine if $(aI)=(bI)(eI)$, where (bI) is a nilpotent element and (eI) is an idempotent element. A ring is called dual nil–fine if every non-unit element is dual a nil–fine element. In this study, we present new results on fine modules and provide a generalization of some fine module results in [14,15]. We study a fine module and related it to other concepts, such as C-F-fine and C-P-fine modules. Some applications are also presented. Further information about modules are available in [20, 21, 22, 23], and some properties of neutrosophic concepts are given in [24, 25, 26].

2. Preliminaries

Definition 2.1: Let R be any ring. Neutrosophic ring $R_n(I)$ is a ring generated by R and I under the operations of R [17].

Definition 2.2: A proper subset $K_n(I)$ of neutrosophic ring $R_n(I)$ is said to be a neutrosophic subring if $K_n(I)$ is a neutrosophic ring under the same operations of $R_n(I)$. If $K_n(I)=S_n(I)$ and n is positive, then $S_n(I)$ is a subring of $R_n(I)$ [17].

Definition 2.3 [18]**:** Let $R_n(I)$ be any neutrosophic ring. A nonempty subset $P_n(I)\subseteq R_n(I)$ is an ideal of $R_n(I)$ if

1) $P_n(I)$ is a neutrosophic subring of $R_n(I)$,
2) $\forall pI\in P_n(I)$ and $rI\in R_n(I),(rI)(pI)\in P_n(I)\wedge (pI)(rI)\in P_n(I)$.

Definition 2.4 [19]**:** Let $(M+,\cdot)$ be any R module over a commutative ring R, and let $M_n(I)$ be a neutrosophic set generated by M and I. The triple $(M_n(I),+,\cdot)$ is called a weak neutrosophic R module over ring R. If $M_n(I)$ is a neutrosophic R module over neutrosophic ring $R_n(I)$, then $M_n(I)$ is called a strong neutrosophic R module. The elements of $M_n(I)$ are called neutrosophic elements, and the elements of $R_n(I)$ are called neutrosophic scalars. If $m_1=x+yI, m_2=z+wI\in M_n(I)$, where x,y,z,w are elements in M and $\alpha=u+vI\in R_n(I)$. u,v are scalars in R and defined as follows:

$m_1+m_2=(x+yI)+(z+wI)=(x+z)+(y+w)I$,

$\alpha m=(u+vI)\cdot(x+yI)=ux+(uy+vx+vy)I$.

Definition 2.5 [20]: Let $M_n(I)$ be a strong neutrosophic R module over neutrosophic ring $R_n(I)$, and let $K_n(I)$ be a nonempty subset of $M_n(I)$. $K_n(I)$ is called a strong neutrosophic submodule of $M_n(I)$ if $K_n(I)$ is a strong neutrosophic R module over $R_n(I)$. $K_n(I)$ must contain a proper subset that is an R module.

Definition 2.6 [20]: Let $M_n(I)$ be a weak neutrosophic R module over ring R, and let $K_n(I)$ be a nonempty subset of $M_n(I)$. $K_n(I)$ is called a weak neutrosophic submodule of $M_n(I)$ if $K_n(I)$ is a weak neutrosophic R module over R. $K_n(I)$ must contain a proper subset that is an R module.

3. n-Refined Neutrosophic Fine Modules

In this part, we present neutrosophic fine module $M_n(I)$ with some results to explain the relationships between $M_n(I)$ and other concepts, such as C1 and C2, and to simplify the exposition.

Definition 3.1: Let $M_n(I)$ be a neutrosophic $R_n(I)$ module. $M_n(I)$ is called a neutrosophic continuous module if it satisfies C1 and C2, and it is called neutrosophic quasi-continuous if it satisfies C2 and C3. C1, C2, and C3 can presented by the following:

(C1) Every neutrosophic submodule $N_n(I)$ of $M_n(I)$ is essential to the direct summand of $M_n(I)$.
(C2) Every neutrosophic submodule $N_n(I)$ of $M_n(I)$ is isomorphic to the summand of $M_n(I)$.
(C3) If $N_n(I)$ and $K_n(I)$ are the summand of $M_n(I)$, then $N_n(I) \cap K_n(I) = 0I$, $N_n(I) \oplus K_n(I)$ is a summand of $M_n(I)$.

Lemma 3.2: [24] Let $M_n(I)$ be an extending neutrosophic $R_n(I)$ module. If $M_n(I)$ satisfies C2, then $M_n(I)$ is a continuous module.

Lemma 3.3 : [12] Every continuous module is a fine module.

Example 3.4:
(1) Neutrosophic discrete modules (neutrosophic semi-perfect modules) are fine modules.
(2) Neutrosophic projective $M_n(I)$ over perfect ring $R_n(I)$ is a fine module.

Remark 3.5: The following implications of the modules are true.

1- Injective module $\Rightarrow$ quasi injective module $\Rightarrow$ continuous module $\Rightarrow$ quasi continuous module $\Rightarrow$ extending module (C1 module).
2- Projective module $\Rightarrow$ injective module $\Rightarrow$ continuous module $\Rightarrow$ extending module.
3- $M_n(I)$ projective + factor module of $M_n(I)$ that has a projective cover $\Rightarrow$ $M_n(I)$ lifting module $\Rightarrow$ extending module (C1).

Proposition 3.6: Let $M_n(I)$ be a divisible module over the field $F_n(I)$. Then, $M_n(I)$ is a fine module.

Proof: Let $J_n(I)$ be an ideal in $F_n(I)$ and $gI \in J_n(I)$. For any $fI \in F_n(I)$, we obtain $fI = (fIgI^{-1})gI \in J_n(I)$. Therefore, $J_n(I) = F_n(I)$. Only $(0I)$ and $(1I)$ ideals exist in $F_n(I)$. In this case, $F_n(I)$ is a principle ideal domain, and $M_n(I)$ is a divisible module. If $0I \neq mI$ is a divisor in the ring such that $M_n(I) = mIM_n(I)$, then $M_n(I)$ is an injective module and continuous. Thus, $M_n(I)$ is a fine module.

Corollary 3.7: Suppose that $M_n(I)$ is a C2 projective module. If every factor module has a projective cover, then $M_n(I)$ is a fine module.

Proof: Suppose that $N_n(I) \le M_n(I)$ and that $g: M_n(I) \to \frac{M_n(I)}{N_n(I)}$ is an epimorphism. Then, $M_n(I) = M'_n(I) \oplus M''_n(I)$, such that $M'_n(I) \subset N_n(I)$ and $g: M''_n(I) \to \frac{M_n(I)}{N_n(I)}$ is a projective cover. However, $N_n(I) \cap M''_n(I) Ker(g)/M''_n(I)$, and it is small in $M''_n(I)$. Hence, $M_n(I)$ is a lifting module (C1 module). With the C2 property, we determine that $M_n(I)$ is a fine module.

Corollary 3.8: Every C2–cyclic module with a projective cover is a fine module.

Proof : The proof is trivial because cyclic with a protective cover provides lifting (C1 module).

4. Finitely Generated Flat – Fine Modules

In this section, we study a famous property of a submodule $N_n(I)$ of $M_n(I)$. We present some concepts that have a direct relationship with a fine module, namely, finitely (f.) generated submodule and flat modules.

Definition 4.1: If $ann(mI) = \{(rI)(mI) = 0I$, then $(rI) \in R_n(I), (mI) \in M_n(I)\}$ is a fine ideal of $R_n(I)$. $M_n(I)$ is called an f. generated fine module.

Definition 4.2: Let $M_n(I)$ be a module. $M_n(I)$ is an f. generated fine module if every f. generated fine submodule is a flat fine submodule.

Definition 4.3: A ring $R_n(I)$ is called a P-F-fine ring if every flat fine module is an f. generated fine module. In the next theorem, we present the equivalent meaning of a P-F-fine ring.

Theorem 4.4: Fine ring $R_n(I)$ is a P-F-fine ring if and only if every faithful multiplication fine module $R_n(I)$ is a P-F-fine ring.

Proof: Let $M_n(I)$ be a faithful multiplication $R_n(I)$ fine module. $M_n(I)$ is a flat fine module, so it is an f. generated fine module. Meanwhile, because $R_n(I)$ is a faithful multiplication fine module, then a fine ring is an f. generated fine module. Thus, $R_n(I)$ is a P-F-fine ring.

Definition 4.5: Let $M_n(I)$ be a fine module. Then, $M_n(I)$ is called torsion-free fine if $(rI)(mI)=0I,(rI)\in R_n(I),(mI)\in M_n(I),$ Then, $mI\neq 0I(rI=0I)$.

Remark 4.6: If $ann(mI)=0I, 0I\neq mI,$ then a torsion-free fine ring is an integral domain.

Proposition 4.7: Let $M_n(I)$ be any f. generated fine module over $R_n(I)$. For each prime fine ideal of $R_n(I)$, $M_n(I)$ is a torsion-free fine module over $R_n(I)$.

Proof: Take $0I\neq m_1I\in M_n(I)$. Then, $ann(m_1I)=ann(m_2I), 0I\neq m_2I\in M_n(I)$. Hence, $ann(m_1I)$ is a pure fine ideal in $R_n(I)$. Given that $R_n(I)$ is a ring that has a local property, $ann(m_1I)=R_n(I)$. However, $ann(m_1I)=0I$, so $M_n(I)$ is a torsion-free fine module.

Definition 4.8: Let $M_n(I)$ be a fine module. $M_n(I)$ is called an f. generated projective fine module if every f. generated submodule of $M_n(I)$ is projective.

Remark 4.9: $Ann(mI)=eI, eI=eI^2$ with $(mI)\in M_n(I)$. We know that every projective module is a fine flat module. Therefore, every f. generated projective fine module is an f. generated flat fine module.

Definition 4.10: Let $R_n(I)$ be a fine ring. If every principal ideal of $R_n(I)$ is projective, then $R_n(I)$ is called a P.P. fine ring. If every projective fine module is an f. generated projective fine module, then $R_n(I)$ is called a P.P. fine ring.

Definition 4.11: A non-unit element αI in $R_n(I)$ is called dual nil-q-fine if $\alpha I=(bI)(qI)$, where $(bI)\in N(R_n(I))$ and $(qI)\in QE(R_n(I))$. Ring $R_n(I)$ is called dual nil-q-fine if every non-unit αI in $R_n(I)$ is dual nil-q-fine.

Remark 4.12: Given that every idempotent is quasi-idempotent, dual nil-fine rings are dual nil q-F-fine.

Lemma 4.13: If $R_n(I)$ is either a local ring with $J(R_n(I))$ nil or a 2×2 matrix ring over a division ring, then $R_n(I)$ is dual nil-q-fine .

Proof: By [13, Theorem 2.3] and Remark 4.12.

Theorem 4.14: Let $R_n(I)$ be a ring. Every quasi-idempotent that is central in $R_n(I)$ is a dual nil-q-fine ring if and only if $R_n(I)$ is a local ring with nil $J(R_n(I))$.

Proof: Every quasi-idempotent is central in $R_n(I)$. Suppose that αI is a non-unit. If $(xI)\in \alpha IR_n(I)$ with every quasi-idempotent being central, then xI is a non-unit. Take $xI=(bI)(qI),(bI)\in N(R_n(I))$ and $(qI)\in QE(R_n(I))$, such that $qI^2=(uI)(qI)$ and $(uI)\in UC(R_n(I))$. Hence, $xI^n=bI^n uI^{n-1}qI, n\geq 1$. If bI is nilpotent, then so is xI. $\alpha IR_n(I)$ is nil, and $(\alpha I)\in J(R_n(I))$. $R_n(I)$ is local, and the Jacobson radical is nil. Conversely, it is fine in accordance with Lemma (4.13).

Proposition 4.15: Let $R_n(I)$ be a Noetherian $R_n(I)$, which is an abelian minimal prime with a unit number. If $R_n(I)$) is a unit direct product of a local ring, then $R_n(I)$ is a fine ring.

Proof: Every local is a fine ring, and the fineness is close to that of the unit direct product. Thus, $R_n(I)$ is fine.

Corollary 4.16: Let $R_n(I)$ be a ring. If $R_n(I)$ is semi-perfect, then it is fine and I-finite.

Proof: Suppose that $R_n(I)$ is a semi-perfect ring. $R_n(I)$ is I-finite with $1 = eI_1 + eI_2 + ... + eI_n$, where eI_n is a local idempotent for each n. However, local rings $eI_n R_n(I) eI_n$ are fine when ($R_n(I)$ is fine.

5. Some Applications

5.1. C-F-Neutrosophic Fine ModuleDefinition 5.1.1: Let $M_n(I)$ be an $R_n(I)$ module. We say that $M_n(I)$ is a C-F-fine module if every cyclic fine submodule is a flat fine submodule.

*) $M_n(I)$ is a C-F-fine module if $ann(xI)$ is a fine pure ideal, such that $xI \in M_n(I)$.

*) Ring $R_n(I)$ is a P-F-fine ring if every fine flat $R_n(I)$ module is a C-F-fine module.

Theorem 5.1.2: Fine ring $R_n(I)$ is a P-F-fine ring if and only if every fine faithful multiplication $R_n(I)-$ module is a C-F-fine module.

Proof: Let $M_n(I)$ be a fine faithful multiplication $R_n(I)$ module. $M_n(I)$ is a fine flat module, so $M_n(I)$ is a C-F-fine module. Given that $R_n(I)$ is a fine faithful multiplication $R_n(I)$ module, then $R_n(I)$ is a C-F-fine module. Thus, $R_n(I)$ is a P-F-fine ring.

Theorem 5.1.3: Let $M_n(I)$ be a fine f. generated faithful multiplication $R_n(I))$ module. $M_n(I)$ is a C-F-fine module if and only if $R_n(I)$ is a P-F-fine ring.

Proof: Let $R_n(I)$ be a P-F-fine module and $P_n(I)$ be any fine prime ideal in $R_n(I)$. Through [16], $R_n(I)$ is a C-F module and C-P module, and $Ann(M_p(I)) = (annM_n(I))_p = 0$. Then, $M_n(I)_p$ is fine faithful, so $M_n(I)$ is fine cyclic. Moreover, for each $xI \in M_n(I), \exists (yI) \in M_n(I)$, such that $(yI)_p = (xI)$ and $ann(yI)_p = (ann(yI))_p = ann(yI)$. However, $ann(yI)$ is a fine pure ideal in $R_n(I)$. Hence, $ann(xI)$ is a fine pure ideal in $R_p(I)$. $M_p(I)$ is a C-F-R_p-fine module. Thus, $R_p(I)$ is a P-F-fine ring. However, $R_n(I)$ is a local fine ring . Thus, $R_p(I)$ is an integral domain. From [14], we find that $R_n(I)$ is a P-F-fine ring.

Definition 5.1.4: Let $M_n(I)$ be a fine module. Then, $M_n(I)$ is fine torsion free if $(rI)(mI) = 0I$, $(rI) \in R_n(I)$, $(mI) \in M_n(I) \rightarrow mI \neq 0I \rightarrow rI = 0I$ $(ann(mI) = 0I$ for an integral domain).

Theorem 5.1.5: Let $M_n(I)$ be any C-F-fine module over $R_n(I)$. For each fine prime ideal of $R_n(I), M_p(I)$ is a fine torsion-free module over $R_p(I)$.

Proof: Take $0I \neq xI \in M_p(I)$. Then, $ann(xI) = ann(yI) \forall 0I \neq (yI) \in M_p(I) \rightarrow ann(xI)$ is a fine pure ideal in $R_n(I)$, but $R_n(I)$ is a fine local ring. Then, $ann(xI) = R_p(I)$. $0I \neq xI \rightarrow ann(xI) = 0I$, and $M_p(I)$ is fine torsion-free.

Note: The opposite of the theorem above is true if $M_n(I)$ is a fine faithful multiplication module.

Lemma 5.1.6: Let $P_n(I)$ be any fine flat $R_n(I)$ module, and let $A_n(I)$ be a fine ideal in $ann_R(p_n(I))$. Then, $P_n(I)$ is a fine flat $\frac{R_n(I)}{A_n(I)}$ module.

Theorem 5.1.7: Let $M_n(I)$ be a C-F-fine $R_n(I)$ module. Then, $\frac{R_n(I)}{ann(xI)}$ is a P-F-fine ring, that is, $xI \neq 0I$ in $M_n(I)$.

Proof: Let $\overline{R_n(I)} = \frac{R_n(I)}{ann(xI)}$ and $0I \neq (xI) \in M_n(I)$. We can easily map Φ: $\overline{R_n(I)} \rightarrow (xI)$: $\Phi\ (\overline{rI}) = (rI)(xI)$ as $R_n(I)$ isomorphism. However, $M_n(I)$ is a C-F-fine $\overline{R_n(I)}-$ module $\rightarrow$ $\overline{R_n(I)}$ is a C-F-fine $R_n(I)$ module. Let $\overline{\alpha I}$ element in $\overline{R_n(I)}$ be easily obtainable from $ann_R(xI) \subseteq ann_R(\overline{\alpha I})$. However, ($\overline{\alpha I}$) is a fine flat $R_n(I)$ module because $\overline{R_n(I)}$ is a C-

F-fine $R_n(I)$ module $\rightarrow$ by the above lemma, ($\overline{\alpha I}$) is a flat fine $\overline{R_n(I)}$ – module $\rightarrow$ and $\overline{R_n(I)}$ is a P-F-fine ring.

5.2. C-P-Fine Module

Definition 5.2.1: Any module $M_n(I)$ is called a cyclic projective fine module if every cyclic fine submodule of $M_n(I)$ is fine projective or if $ann(xI)$ is generated by an ideal for each $xI \in M_n(I)$ [7].

Remark 5.2.2: Every fine projective is fine flat. Thus, every cyclic projective fine module is a cyclic flat fine module. Any ring $R_n(I)$ is a P.P. fine ring if $R_n(I)$ is a cyclic projective fine module, or $R_n(I)$ is a P-P-fine ring if and only if every fine projective $R_n(I)$ module is a cyclic projective fine module.

Theorem 5.2.3: Ring $R_n(I)$ is a P-P-fine ring if and only if every faithful multiplication fine $R_n(I)$ module is a cyclic projective fine module.

Theorem 5.2.4: Let $M_n(I)$ be a faithful multiplication fine $R_n(I)$ module. If $R_n(I)$ is a P-P-fine ring, then $M_n(I)$ is a cyclic projective fine $R_n(I)$ module, and $M_S(I)$ is fine f. generated. Then, $R_n(I)$ is a P-P-fine ring.

Proof: If $R_n(I)$ is a P-P-fine ring $\Rightarrow$ Theorem (5.2.3), $M_n(I)$ is a cyclic projective fine $R_n(I)$ module by [5]. $R_S(I)$ is a F-fine regular ring. Hence, $M_S(I)$ is an F-fine regular $R_S(I)$ module. Conversely, assume that $M_n(I)$ is a cyclic projective fine $R_S(I)$ module. Therefore, $M_n(I)$ is a cyclic flat fine $R_n(I)$ module. In accordance with Theorem (5.1.3), $R_n(I)$ is a P-F-fine ring, and $R_n(I)$ is an integral domain. $M_S(I)$ is an f. generated fine $R_S(I)$ module then fine faithful. Therefore, if $M_S(I)$ is F-fine regular, then by [17], we determine that $R_S(I)$ a fine regular ring, and by [5], we find that $R_n(I)$ is a P-P-fine ring.

Corollary 5.2.5: Let $M_n(I)$ be a fine f. generated faithful multi $R_n(I)$ module. The following statements are equivalent.

1- $R_n(I)$ is a P-P-fine ring.

2- $M_n(I)$ is a cyclic projective fine $R_n(I)$ module.

3- $M_p(I)$ is a torsion fine $R_n(I)$ module $\ni P(I)$ ideal of $R_n(I)$, and $M_S(I)$ is F-regular fine.

4- $M_p(I)$ is a torsion-free fine $R_p(I)$ module, and $P_n(I)$ ideal of $R_n(I)$ and $M_S(I)$ is a Z-fine module. It is Z-fine regular if each fine cyclic (fine f. generated) submodule is fine projective and a direct summand of $M_n(I)$.

Conclusion

In this paper, we studied the concept of fine modules through a continuous module in neutrosophic theory. We proved that if $M_n(I)$ is a divisible module over field F, then $M_n(I)$ is a fine module. Moreover, every C_2-cyclic module that has a projective cover is a fine module. We showed that fine f. generated faithful multiplication module $M_n(I)$ over ring $R_n(I)$ is a C-F-fine module if and only if $R_n(I)$ is a P-F-fine ring. Fine ring $R_n(I)$ is called a P-F-fine ring if and only if every faithful multiplication fine module is a P-F-fine ring.

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الموديول النتروسوفي النظيف المشتق مع بعض التطبيقات

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الخلاصة:

في هذا البحث، ندرس ونقدم مفهوما مهما في نظرية الموديول والذي يسمى بالموديول النتروسوفي النظيف، مع الخصائص الجيدة للموديولات والحلقات. سوف نستخدم الرمز(F-module) للتعبير عن الموديول النظيف. كذلك قمنا بتحقيق بعض الخصائص حول العلاقة بين الموديول النظيف والحلقة من النوع ,(*P.P.F*) والموديولات من النوع(injective) و (projective) التي لها علاقة مباشرة مع الموديول النظيف. لقد اثبتنا أن كل موديول مستمر (Continuous)هو موديول نظيف وكذلك كل موديول متقطع(discrete) هو موديول نظيف، كذلك اذا كانت $R_n(I)$حلقة نتروسوفية لها خصائص(perfect) فان كل(guasi- projective) على الحلقة $R_n(I)$ له(fine property) وقد اثبتنا ان كل موديول $M_n(I)$ قابل للقسمة (divisible)على الحقل النتروسوفي ($F_n(I)$فانه موديول نظيف. وقد درسنا الخاصية الشهيرة جدا للموديول الجزئي ($N_n(I)$من الموديول ($M_n(I)$ وقد احتجنا الى تقديم بعض المفاهيم التي لها علاقة مباشرة مع الموديول النظيف والتي تسمى الموديول الجزئي المنتهي التولد كذلك الموديول المسطح. أخيرا اثبتنا أن كل موديول دائري ($M_n(I)$ حيث أن كل موديول جزئي منه ($N_n(I)$ هو (isomorphic to summand of $M_n(I)$) فأن $M_n(I)$ هو موديول نظيف اذا كان له (projective cover).

الكلمات المفتاحية :الحلقة النظيفة، المجموعة النتروسوفية، الموديول النتروسوفي، الموديول المسطح، الموديول المتباين.