The Stability and Error Analysis for Linearized Crank-Nicolson-Galerkin Method for Euler Equations

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Abstract:

In this paper, we studied the stationary and the non-stationary incompressible Euler equations in two-dimensional domain by using mixed finite element method. By using the Linearized Crank-Nicolson-Galerkin Method we found the weak form to the above problem which is then improved to the approximate solution. we consider three cases, the first case for the steady-state and non steady-state Euler equations and proved some lemmas and theorems for the stability of the semi-discrete and fully-discrete mixed finite element method, the second case for the steady-state Euler equations and proved some lemmas for the ellipticity and continuity of this method and the third case for the steady-state and non steady-state Euler equations and these estimates are then applied to obtain quasi-optimal error analysis in the energy norm for velocity, pressure and velocity with pressure.

Keyword: Incompressible Euler Equations, Mixed finite element method, Linearized Crank-Nicolson-Galerkin Method, Error analysis.

1. Introduction

The classical numerical method for partial differential equations is the difference method where the discrete problem is obtained by replacing derivatives with difference quotients involving the values of the unknown at certain points.

The finite element method is a numerical analysis technique for obtaining approximate solutions to a wide variety of problems in mechanics and physics[6]. Although originally developed to study stresses in complex airframe structures, it has since been extended and applied to the broad field of continuum mechanics. Because of its diversity and flexibility as an analysis tool, it is receiving much attention in engineering schools and in industry. In this method, the discretization procedures reduce the problem to one of a finite number of unknowns by dividing the solution region into elements and by expressing the unknown field variable in terms of assumed approximating functions within each element. The approximating functions (sometimes called interpolation functions) are defined in terms of the values of the field variables at specified points called nodes or nodal points[١٠].

Mixed finite element methods are one of the important approaches for solving system of partial differential equations, for example, the stationary Navier-Stokes equations. However, fully discrete system of mixed finite element solutions for the stationary Navier-Stokes equations is of many degrees of freedom[9].

1.1 Notation

Let Ω be an open and bounded domain in R^2 with Lipschitz continuous boundary Γ .Throughout this paper we will use the standard notation for Sobolev spaces. Specially $H^r(\Omega)$, where r is an integer greater than zero, will denotes the Sobolev space of real-valued functions with square integrable derivatives of order up to r equipped with the usual norm which we denote $\|\cdot\|_r$. We will denote $H^0(\Omega)$ by $L^2(\Omega)$ and the standard L^2 inner product by $(\cdot,\cdot)$. Also $H^r(\Omega)$ will denote the space of vector-valued functions each of whose n components belong to $H^r(\Omega)$ and the dual space of $H^r(\Omega)$ will be denoted by $H^{-r}(\Omega)$.Of particular interest to us will be the constrained space see [10]

$$V = \left[H_0^1(\Omega)\right]^2 = \left\{v = (v_1, v_2) : v_i \in H_0^1, \quad i = 1,2\right\}$$

and

$$Q = \left\{ q \in L_2(\Omega) : \int_\Omega q \; dx = 0 \right\}$$

1.2 The weak formulations

We are interested in approximating the solution of the incompressible Euler equations written in the primitive variable formulation of the velocity $u = (u_1, u_2)$ and the pressure p. In particular, we consider the steady incompressible Euler equations, see [3], [5].

$$\frac{\partial u}{\partial t} + u.\nabla u + \nabla p = f \qquad in\,\Omega \tag{1.1a}$$

$$\nabla \cdot u = 0 \qquad in\,\Omega \tag{1.1b}$$

$$u.n = 0 \qquad on\;\; \partial\Omega = \Gamma \tag{1.1c}$$

Multiplying (1.1a) and (1.1b) by $v \in V$ and $q \in Q$, respectively, as a test functions and take integral over Ω

$$\int_\Omega \frac{\partial u}{\partial t} v\,dx + \int_\Omega (u \cdot \nabla) u\; v\,dx + \int_\Omega \nabla p\; v\,dx = \int_\Omega f\;\; v\,dx \quad ; \;\; v \in V$$

$$\int_\Omega div u\; q\,dx = 0\,; \quad q \in Q,$$

by using Green's formulation

$$\int_\Omega \frac{\partial u}{\partial t} v\,dx - \frac{1}{2}\int_\Omega u^2\;\nabla v\,dx - \int_\Omega p\; div\, v\,dx = \int_\Omega f\, v\,dx \quad ,$$

$$\int_\Omega div u\; q\,dx = 0\,.$$

We consider the following standard weak formulation of non- steady: seek $(u, p) \in V \times Q$ such that

$$(u_t, v) - n(u; u, v) - b(v, p) = (f, v) \; ; \quad v \in V , \tag{1.2a}$$

$$b(u, q) = 0 ; \quad q \in Q , \tag{1.2b}$$

where

$$n(u; u, v) = \frac{1}{2} \int_{\Omega} u^2 \, \nabla v \, dx ,$$

$$b(u, q) = \int_{\Omega} div u \, q \, dx .$$

Also, the weak formulation of the steady Euler equations is as follows:

Seek $(u, p) \in V \times Q$ such that:

$$-n(u; u, v) - b(v, p) = (f, v) \; ; \quad v \in V ,$$

$$b(u, q) = 0 ; \quad q \in Q ,$$

Continuity of the forms $n(\cdot, \cdot, \cdot)$ and $b(\cdot, \cdot)$ can be demonstrated. These conditions guarantee the existence and uniqueness of a solution (u, p) [3].

1.3 The discrete problems

Given finite dimensional spaces $V_h \subset V$ and $Q_h \subset Q$ where $0 < h < 1$ then the approximate solution (u_h, p_h) to (u, p) is the solution of the following equations:

$$(u_{t,h}, v) - n(u_h; u_h, v) - b(v, p_h) = (f, v) \; ; \quad v \in V_h , \tag{1.3a}$$

$$b(u_h, q) = 0 ; \quad q \in Q_h , \tag{1.3b}$$

Also, the discrete problem of the steady equations is as follows seek $(u_h, p_h) \in V_h \times Q_h$ such that:

$$-n(u_h; u_h, v) - b(v, p_h) = (f, v) \ ; \quad v \in V_h \ , \tag{1.4a}$$

$$b(u_h, q) = 0 \ ; \quad q \in Q_h \ , \tag{1.4b}$$

1.3 The Fully-Discrete Approximation

Now we turn our attention to some simple schemes for discretization with respect to the time variable.

1.3.1 Forward Euler Method

Letting n_k be the time step and u_h^k the approximation in V_h of $u(., t_k)$, $k = 0, \cdots, N$, at $t = t_k = k\, n_k$. This method is defined by replacing the time derivative $u_{t,h}$ in equation (1.3) by forward differences quotient $\frac{u_h^{k+1} - u_h^k}{n_k}$ with discretization error $O(n_k)$,

$$\left(\frac{u_h^{k+1} - u_h^k}{n_k}, v \right) - n(u_h^k; u_h^k, v) - b(v, p_h^k) = (f, v) \ ; \quad v \in V_h \ , k = 0, \cdots, N \tag{1.5a}$$

$$b(u_h^k, q) = 0 \ ; \quad q \in Q_h \ . \tag{1.5b}$$

1.3.2 Crank-Nicolson method

This method is defined by replacing the time derivative $u_{t,h}$ in equation (1.3) by forward differences quotient $\frac{u_h^{k+1}-u_h^k}{n_k}$ and the u_h and p_h by differences quotient $\frac{u_h^{k+1}+u_h^k}{2}$ and $\frac{p_h^{k+1}+p_h^k}{2}$ with the corresponding discretization error is $O(n_k^2)$,

$$\left(\frac{u_h^{k+1}-u_h^k}{n_k}, v\right) - n\left(\frac{u_h^{k+1}+u_h^k}{2}; \frac{u_h^{k+1}+u_h^k}{2}, v\right) - b\left(v, \frac{p_h^{k+1}+p_h^k}{2}\right) = \left(\frac{f(t_{k+1})+f(t_k)}{2}, v\right) \; ; \quad v \in V_h, k = 0, \cdots, N \tag{1.6a}$$

$$b\left(\frac{u_h^{k+1}+u_h^k}{2}, q\right) = 0 ; \quad q \in Q_h . \tag{1.6b}$$

1.3.3 Semi-implicit Oseen Method for Weak Formulations

In order to reduce the computational effort at each time step, it seems reasonable to replace the nonlinear stationary problems by linear ones in a similar way as in the Oseen iteration, which yields a modified equation at each step. Letting τ be the time step and u^k the solution in V of $u(\cdot, t_k), k = 1, 2, \cdots, N$, at $t = t_k = k\tau$. This method is defined by replacing the time derivative u_t in problem (1.2) by backward differences quotient $\frac{(u^k - u^{k-1})}{\tau}$ with the corresponding discretization error is $O(\tau^2)$,

$$\left(\frac{u^k - u^{k-1}}{\tau}, v\right) - n(u^{k-1}; u^k, v) - b(v, p^k) = (f^k, v) \quad ; \forall v \in V, \; k = 1, \cdots, N$$

$$b(u^k, q) = 0 \quad ; \forall q \in Q$$

1.3.4 Semi-implicit Oseen Crank-Nicolson-Galerkin Method

Letting τ be the time step and u^k the solution in V of $u(\cdot, t_k), k = 1, 2, \cdots, N$, at $t = t_k = k\tau$. This method is defined by replacing the time derivative u_t in problem (1.2) by backward differences quotient $\frac{(u^k - u^{k-1})}{\tau}$ and the u and p by differences quotient $\left(\frac{u^k + u^{k-1}}{2}\right)$ and $\left(\frac{p^k + p^{k-1}}{2}\right)$ with the corresponding discretization error is $O(\tau^2)$,

$$\left(\frac{u^k - u^{k-1}}{\tau}, v\right) - n\left(u^{k-1}; \frac{u^k + u^{k-1}}{2}, v\right) - b\left(v, \frac{p^k + p^{k-1}}{2}\right) = \left(\frac{f^k + f^{k-1}}{2}, v\right) \quad ; \forall v \in V,\ k = 1, 2, \cdots N \tag{1.7a}$$

$$b(u^k, q) = 0 \quad ; \forall q \in Q \tag{1.7b}$$

1.3.5 Linearized Oseen Crank-Nicolson-Galerkin Method for Weak Formulations

Problem (1.7) shares, however with backward Euler method discussed first above, the disadvantage of producing, at each time level, a nonlinear system of problem. For this reason we shall consider also a linearized modification in which the argument of $n(\cdot, \cdot, \cdot)$ is obtained by extrapolation from u^{k-1} and u^{k-2}, [2], with

$$\widehat{u}^k = \frac{3}{2}u^{k-1} - \frac{1}{2}u^{k-2}$$

$$\frac{1}{\tau}(u^k - u^{k-1}, v) - n(\widehat{u}^k, \widehat{u}^k, v) - b(v, \overline{p}^k) = (\overline{f}^k, v) \quad ; \forall v \in V, \quad k = 1, \cdots, N \tag{1.8a}$$

$$b\left(\bar{u}^{k}, q\right)=0 \quad ;\forall q \in Q \tag{1.8b}$$

where

$$\bar{u}^{k}=\frac{u^{k}+u^{k-1}}{2}, \bar{p}^{k}=\frac{p^{k}+p^{k-1}}{2}, \bar{f}^{k}=\frac{f^{k}+f^{k-1}}{2}.$$

1.3.6 Linearized Oseen Crank-Nicolson-Galerkin Method for Discrete Problem

Given finite dimensional spaces $V_h \subset V$ and $Q_h \subset Q$ where $0<h<1$ then the approximate solution (u_h, p_h) to (u, p) is the solution of the following problem:

$$\frac{1}{\tau}\left(u_h^k-u_h^{k-1}, v\right)-n\left(\hat{u}_h^k, \hat{u}_h^k, v\right)-b\left(v, \bar{p}_h^k\right)=\left(\bar{f}^k, v\right) \quad ;\forall v \in V_h, k=1,\cdots, N \tag{1.9a}$$

$$b\left(\bar{u}_h^k, q\right)=0 \quad ;\forall q \in Q_h \tag{1.9b}$$

The nonlinear equation (1.9a) will be solvable for u^k when u^{k-1} and u^{k-2} are given. Choosing $n(\cdot,\cdot,\cdot)$ at u^{k-1} as we did for the back ward Euler scheme will not be satisfactory here since this would be less accurate than necessary, whereas since

$$\hat{u}^k=\frac{3}{2}u^{k-1}-\frac{1}{2}u^{k-2}=u^{k-\frac{1}{2}}+O\left(\tau^2\right) \quad as \quad \tau \to 0$$

the choice just proposed will have the desired accuracy.

2. Abstract Results

Let V and Q be two real Banach spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_Q$ respectively. Let $a(\cdot,\cdot)$ and $b(\cdot,\cdot)\in L^{\infty}$ be continuous bilinear forms on $V\times V$ and $V\times Q$ respectively [4], $n(\cdot,\cdot,\cdot)\in L^{\infty}$ be continuous trilinear form on $V\times V\times V$ [9]:

$$|a(u,v)|\le \|a\|_{L^{\infty}}\cdot\|u\|_V\cdot\|v\|_V \qquad \forall\, u,v\in V, \tag{2.1}$$

$$|n(u,u,v)|\le \|n\|_{L^{\infty}}\cdot\|u\|_V^2\cdot\|v\|_V \qquad \forall\, u,v\in V, \tag{2.2}$$

$$|b(u,p)|\le \|b\|_{L^{\infty}}\cdot\|u\|_V\cdot\|p\|_Q \qquad \forall\, u\in V;\ \forall p\in Q. \tag{2.3}$$

we now state several further assumptions which we will require in the proofs of our main results [4].

(H1) There is a constant α >0 (α independent of h) such that

$$a(v,v)\ge \alpha\,\|v\|_W^2 \qquad \forall v\in Z_h\,,$$

where $$Z_h=\{v\in V_h : b(v,\varphi)=0\,,\ \forall\varphi\in Q_h\}$$

(H2) $S(h)$ is a number satisfying $\|v\|_V\le S(h)\|v\|_W\,;\quad \forall v\in V_h$.

(H3) There is a linear operator $\Pi_h : Y\to V_h$ satisfying

$b(y-\Pi_h y,\varphi)=0\ ;\quad \forall y\in Y \quad \mathit{and} \quad \varphi\in Q_h$.

Definition 2.1 [7] Cauchy-Schwarz inequalities:

$$\left|(v,w)_{L^2(\Omega)}\right|\le \|v\|_{L^2(\Omega)}\,\|w\|_{L^2(\Omega)}\ ;\quad v,w\in L^2(\Omega), \tag{2.4}$$

and

$$\left|(v,w)_{H^1(\Omega)}\right|\le \|v\|_{H^1(\Omega)}\,\|w\|_{H^1(\Omega)}\ ;\quad v,w\in H^1(\Omega)\ . \tag{2.5}$$

Lemma 2.1 There exists a linear operator $\Pi_h : H \to H_h$ such that, [6]

$$(div\,\Pi_h U, v_h) = (divU, v_h); \quad \forall v_h \in V_h, \forall U \in H,$$

$$\| \Pi_h U - U \| \le Ch^s \| U \|_s ; \quad for\, s = 1, 2.$$

3. The Stability

The arguments for stability and bound on the approximation error are useful for the analysis of the discrete formulations.

Lemma3.1 Suppose that u_h is the discrete solution of equations (1.3). Then there exists a constant $\delta > 0$ independent of h such that:

$$\|u_h(T)\| \le \delta . \tag{3.1}$$

Proof: By choosing $v = u_h \in V_h$ in problem (1.3) we get

$$(u_{h,t}, u_h) - n(u_h, u_h, u_h) - b(u_h, p_h) = (f, u_h) \tag{3.2}$$

note that

$$(u_{h,t}, u_h) = \int_\Omega \frac{d}{dt} u_h u_h \; dx = \frac{1}{2} \int_\Omega \frac{d}{dt} u_h^2 \; dx = \frac{1}{2} \frac{d}{dt} \|u_h\|^2 \quad ,$$

Now, by using Young's inequality with $\in = 2c, c_1$ and c_2 respectively we have

$$(f, u_h) \le \| f \| \|u_h\| \le \frac{1}{2c} \| f \|^2 + \frac{c}{2} \|u_h\|^2 \quad ,$$

$$n(u_h, u_h, u_h) \le \beta \|u_h\|^2 \, \|u_h\| \le \beta \left[\frac{1}{4c_1} \|u_h\|^4 + c_1 \|u_h\|^2 \right] \quad ,$$

$$b(u_h, \, p_h) \le S(h) \|u_h\| \, \|p_h\| \le S(h) \left[\frac{1}{2c_2} \|p_h\|^2 + \frac{c_2}{2} \|u_h\|^2 \right] \quad .$$

From the above inequalities we get

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + \beta\left[\frac{1}{4c_1}\|u_h\|^4 + c_1\|u_h\|^2\right] + S(h)\left[\frac{1}{2c_2}\|p_h\|^2 + \frac{c_2}{2}\|u_h\|^2\right] \le \frac{1}{2c}\|f\|^2 + \frac{c}{2}\|u_h\|^2 \,,$$

put $c = \beta c_1 = S(h) c_2$ in the above equation we get

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + c\,\|u_h\|^2 + \frac{\beta^2}{4c}\|u_h\|^4 + \frac{(S(h))^2}{2c}\|p_h\|^2 + \frac{c}{2}\|u_h\|^2 \le \frac{1}{2c}\|f\|^2 + \frac{c}{2}\|u_h\|^2 \,,$$

$$\frac{d}{dt}\|u_h\|^2 + 2c\,\|u_h\|^2 + \frac{\beta^2}{2c}\|u_h\|^4 + \frac{(S(h))^2}{c}\|p_h\|^2 \le \frac{1}{c}\|f\|^2 \,,$$

multiply both sides of the above inequality by the integral factor e^{-2ct} and then integrate from 0 to T, we have

$$e^{-2cT}\,\|u_h(T)\|^2 + \int_0^T e^{-2ct}\left[\frac{\beta^2}{2c}\|u_h\|^4 + \frac{(S(h))^2}{c}\|p_h\|^2\right]dt \le \int_0^T \frac{e^{-2ct}}{c}\|f\|^2\,dt + \|u_h(0)\|^2 \,,$$

since $e^{-2ct}\left[\frac{\beta^2}{2c}\|u_h\|^4 + \frac{(S(h))^2}{c}\|p_h\|^2\right] \ge 0$,we get

$$\|u_h(T)\|^2 \le e^{2cT}\|u_h^0\|^2 + \int_0^T \frac{e^{2c(T-t)}}{c}\|f\|^2\,dt \,,$$

$$\le \mu e^{2cT} - \frac{\gamma}{2c^2}\int_0^T - 2ce^{2c(T-t)}dt \quad ,$$

where $\mu = \| u_h^0 \|^2 \quad , \gamma = \| f \|^2$,

$$\| u_h(T) \|^2 \le \mu e^{2cT} - \frac{\gamma_1}{2c^2}$$

where $\gamma_1 = \gamma \int_0^T - 2ce^{2c(T-t)}dt \quad ,$

$$\therefore \| u_h(T) \| \le \delta \quad .$$

□

Lemma 3.2 Suppose that u_h is the discrete solution of problem (1.4). Then, there exist constants $\gamma, \beta > 0$ independent of h such that:

$$\| u_h \| \le \sqrt{\frac{\gamma}{\beta}} \quad . \tag{3.3}$$

Proof:

Put $v = u_h \in V_h$ and $q = p_h \in Q_h$ in equations (1.4a) and (1.4b) respectively, then, by subtracting the two resulting equations, we get

$$-n(u_h, u_h, u_h) = (f, u_h),$$

then we have [5],

$$\beta \| u_h \|^3 \le \gamma \| u_h \| ,$$

$$\beta \| u_h \|^2 \le \gamma \ ,$$

we have

$$\|u_h\| \le \sqrt{\frac{\gamma}{\beta}}\,.$$

□

Lemma3.3 Suppose that p_h are the discrete solutions of problem (1.4). Then, there exist constants $\gamma, \mu > 0$ independent of h such that:

$$\|p_h\| \le \frac{\gamma}{\mu}\,. \tag{3.4}$$

Proof: For the continuous equation (1.4a) using the test function $v = u_h \in V_h$ which gives

$$-n(u_h, u_h, u_h) - b(u_h, p_h) = (f, u_h),$$

we have [5]

$$\beta\|u_h\|^3 + \mu\|u_h\|\|p_h\| \le \gamma\,\|u_h\|,$$

$$\beta\|u_h\|^2 + \mu\|p_h\| \le \gamma\;,$$

since $\|u_h\|^2 \ge 0$, we get

$$\|p_h\| \le \frac{\gamma}{\mu}\,.$$

□

Lemma3.4 Suppose that u_h and p_h are the discrete solutions of problem (1.4). Then, there exist constants $\alpha > 0$ independent of h such that:

$$\|u_h\| + \|p_h\| \le \alpha\,. \tag{3.5}$$

Proof: We can prove this **Lemma** from equations (3.3) and (3.4), where $\alpha = \max\left\{\sqrt{\frac{\gamma}{\beta}}, \frac{\gamma}{\mu}\right\}$.

□

Theorem 3.1 Let $u_h^{k+1} \in V_h$ is the approximation solution of equation (1.5). Then

$$\left\| u_h^{k+1} \right\| \le \delta_2 \, n_k \quad . \qquad (3.6)$$

Proof: Put $u_{h,t} = \frac{u_h^{k+1} - u_h^k}{n_k}$ in problem (1.3), then by using forward Euler method, we get

$$\left(\frac{u_h^{k+1} - u_h^k}{n_k}, v \right) - n\left(u_h^k, u_h^k, v\right) - b\left(v, p_h^k\right) = (f, v) \quad ,$$

then,

$$\left(u_h^{k+1}, v\right) - \left(u_h^k, v\right) + n_k\left[-n\left(u_h^k, u_h^k, v\right) - b\left(v, p_h^k\right)\right] = n_k (f, v) \quad ,$$

by choosing $v = u_h^k$, we get

$$\left(u_h^{k+1}, u_h^k\right) - \left(u_h^k, u_h^k\right) + n_k\left[-n\left(u_h^k, u_h^k, u_h^k\right) - b\left(u_h^k, p_h^k\right)\right] = n_k \left(f, u_h^k\right) \quad ,$$

so,

$$\left\| u_h^{k+1} \right\| \left\| u_h^k \right\| + \left\| u_h^k \right\|^2 + \beta n_k \left\| u_h^k \right\|^3 + \mu n_k \left\| p_h^k \right\| \left\| u_h^k \right\| \le n_k \left\| f(t_k) \right\| \left\| u_h^k \right\| \quad ,$$

since $\left\| u_h^k \right\|^2 \ge 0$, we get

$$\left\| u_h^{k+1} \right\| + a_1 + a_2 \, n_k \le a_3 \, n_k \quad ,$$

$$\left\|u_h^{k+1}\right\| + \delta_1 n_k \le a_3 n_k \quad ,$$

where $\delta_1 n_k = \min\{a_1, a_2 n_k\}$

$$\left\|u_h^{k+1}\right\| \le \delta_2 n_k \quad .$$

where $\delta_2 n_k = a_3 n_k - \delta_1 n_k$.

□

Theorem 3.2 Let $u_h^{k+1} \in V_h$ and $p_h^{k+1} \in Q_h$ are approximation solutions of equation (1.6). Then

$$\left\|u_h^{k+1}\right\| + \left\|p_h^{k+1}\right\| \le O(n_k) \quad . \tag{3.7}$$

Proof:

Put $u_{h,t} = \dfrac{u_h^{k+1} - u_h^k}{n_k}, u_h = \dfrac{u_h^{k+1} + u_h^k}{2}$ and $p_h = \dfrac{p_h^{k+1} + p_h^k}{2}$ in problem (1.3),

then we get

$$\left(\frac{u_h^{k+1} - u_h^k}{n_k}, v\right) - n\left(\frac{u_h^{k+1} + u_h^k}{2}, \frac{u_h^{k+1} + u_h^k}{2}, v\right) - b\left(v, \frac{p_h^{k+1} + p_h^k}{2}\right) = \left(\frac{f(t_{k+1}) + f(t_k)}{2}, v\right) \quad ,$$

$$\left(u_h^{k+1}, v\right) - \left(u_h^k, v\right) + \frac{n_k}{2}\left[-n\left(u_h^{k+1}, u_h^{k+1}, v\right) - n\left(u_h^k, u_h^k, v\right) - 2n\left(u_h^{k+1}, u_h^k, v\right)\right] +$$
$$\frac{n_k}{2}\left[-b\left(v, p_h^{k+1}\right) - b\left(v, p_h^k\right)\right] = \frac{n_k}{2}\left[\left(f(t_{k+1}), v\right) + \left(f(t_k), v\right)\right] \quad ,$$

by choosing $v = u_h^k$, we get

$$\|u_h^{k+1}\|\,\|u_h^k\|+\|u_h^k\|^2+\frac{n_k}{2}\left[\beta\,\|u_h^{k+1}\|^2\,\|u_h^k\|+\beta\|u_h^k\|^3+2\beta\|u_h^{k+1}\|\,\|u_h^k\|^2\right]+$$
$$\frac{n_k}{2}\left[\mu\|p_h^{k+1}\|\,\|u_h^k\|+\mu\|p_h^k\|\,\|u_h^k\|\right]\le\frac{n_k}{2}\left[\|f(t_{k+1})\|\,\|u_h^k\|+\|f(t_k)\|\,\|u_h^k\|\right] \quad ,$$

since $\|u_h^{k+1}\|^2$ and $\|u_h^k\|^2\ge 0$

then,

$$\left(1+\beta\,n_k\|u_h^k\|\right)\|u_h^{k+1}\|+\|u_h^k\|+\frac{\mu\,n_k}{2}\|p_h^{k+1}\|+\frac{\mu\,n_k}{2}\|p_h^k\|\le\frac{n_k}{2}\left[\|f(t_{k+1})\|+\|f(t_k)\|\right] \quad ,$$

$$b_1\,n_k\,\|u_h^{k+1}\|+b_2\,n_k\,\|p_h^{k+1}\|\le b_3\,n_k+b_4\,n_k+b_5\,n_k \quad ,$$

where $b_1\,n_k=1+\beta\,n_k\|u_h^k\|,\ b_2\,n_k=\dfrac{\mu\,n_k}{2},\ b_3\,n_k=-\|u_h^k\|,\ b_4\,n_k=-\dfrac{\mu\,n_k}{2}\|p_h^k\|$

and $b_5\,n_k=\dfrac{n_k}{2}\left[\|f(t_{k+1})\|+\|f(t_k)\|\right]$

$$b_6\,n_k\left[\|u_h^{k+1}\|+\|p_h^{k+1}\|\right]\le b_3\,n_k+b_4\,n_k+b_5\,n_k \quad ,$$

where $b_6=\min\{b_1,b_2\}$,

$$\therefore\|u_h^{k+1}\|+\|p_h^{k+1}\|\le O(n_k) \quad .$$

□

4.Error Estimated

We shall now study the errors $u^k-u_h^k$ and $p^k-p_h^k$ where u^k and p^k are the solution of weak form and u_h^k and p_h^k are the solution of the mixed finite element problem $(V_h \text{ and } Q_h)$

Theorem 4.1 Let $u^k \in V$ be the solution of problem (1.8) and $u_h^k \in V_h$ is the approximation solution of problem (1.9). Then, there exists a constant $C>0$ independent of h *and* τ such that:

$$\left\|u^k - u_h^k\right\| \le C\left(h^r + \tau\right). \tag{4.1}$$

Proof: Let $u^k - u_h^k = \left(u - \Pi_h u^k\right) - \left(u_h^k - \Pi_h u^k\right) = \rho^k - \theta^k$

For each time step k and each norm, we apply the triangle inequality

$\left\|u^k - u_h^k\right\| \le \left\|\rho^k\right\| + \left\|\theta^k\right\|$

from Lemma 2.1 $\left\|\rho^k\right\| \le Ch^r \left\|u^k\right\|$,

To find a bound on θ^k term, note that

$$\frac{1}{\tau}\left(\theta^k - \theta^{k-1}, \varphi\right) - n\left(\hat{\theta}^k, \bar{\theta}^k, \varphi\right) - b\left(\varphi, \bar{p}_h^k - \bar{p}^k\right) =$$
$$\frac{1}{\tau}\left(u_h^k - u_h^{k-1}, \varphi\right) - n\left(\hat{u}_h^k, \bar{u}_h^k, \varphi\right) - b\left(\varphi, \bar{p}_h^k\right)$$
$$-\frac{1}{\tau}\left(\Pi_h u^k - \Pi_h u^{k-1}, \varphi\right) + n\left(\Pi_h \hat{u}^k, \Pi_h \bar{u}^k, \varphi\right) + b\left(\varphi, \bar{p}^k\right).$$

By the definition of interpolation, we have

$A\left(u^k - \Pi_h u^k, \varphi\right) = 0$,

also note that

$$\frac{1}{\tau}\left(u_h^k - u_h^{k-1}, \varphi\right) - n\left(\hat{u}_h^k, \bar{u}_h^k, \varphi\right) - b\left(\varphi, \bar{p}_h^k\right) = \left(\bar{f}^k, \varphi\right),$$

then, we have

$$\frac{1}{\tau}\left(\theta^{k}-\theta^{k-1},\varphi\right)-n\left(\hat{\theta}^{k},\bar{\theta}^{k},\varphi\right)-b\left(\varphi,\bar{p}_{h}^{k}-\bar{p}^{k}\right)=\left(\bar{f}^{k},\varphi\right)$$

$$-\frac{1}{\tau}\left(\Pi_{h}u^{k}-\Pi_{h}u^{k-1},\varphi\right)-n\left(\Pi_{h}\hat{u}^{k},\Pi_{h}\bar{u}^{k},\varphi\right)-b\left(\varphi,\bar{p}^{k}\right),$$

$$=\left(u_{t}^{k},\varphi\right)-\frac{1}{\tau}\left(\Pi_{h}u^{k}-\Pi_{h}u^{k-1},\varphi\right).$$

Adding and subtracting $\frac{1}{\tau}\left(u^{k}-u^{k-1},\varphi\right)$ gives,

$$\frac{1}{\tau}\left(\theta^{k}-\theta^{k-1},\varphi\right)-n\left(\hat{\theta}^{k},\bar{\theta}^{k},\varphi\right)-b\left(\varphi,\bar{p}_{h}^{k}-\bar{p}^{k}\right)=$$

$$\frac{1}{\tau}\left(u^{k}-u^{k-1},\varphi\right)-\frac{1}{\tau}\left(\Pi_{h}u^{k}-\Pi_{h}u^{k-1},\varphi\right)+\left(u_{t}^{k},\varphi\right)-\frac{1}{\tau}\left(u^{k}-u^{k-1},\varphi\right)$$

$$=\frac{1}{\tau}\left(\rho^{k}-\rho^{k-1},\varphi\right)+\left(\xi^{k},\varphi\right)\quad,$$

where

$$\xi^{k}=u_{t}^{k}-\frac{1}{\tau}\left(u^{k}-u^{k-1}\right)$$

choosing $\varphi=\hat{\theta}^{k}$ and $\bar{p}_{h}^{k}=q^{k}$

$$\frac{1}{\tau}\left(\theta^{k}-\theta^{k-1},\hat{\theta}^{k}\right)-n\left(\hat{\theta}^{k},\bar{\theta}^{k},\hat{\theta}^{k}\right)=$$

$$\frac{1}{\tau}\left(\rho^{k}-\rho^{k-1},\hat{\theta}^{k}\right)+\left(\xi^{k},\hat{\theta}^{k}\right)-b\left(\hat{\theta}^{k},\bar{p}^{k}-q^{k}\right).$$

By using (2.2) and (2.3), and multiplying by τ, we get

$$\left\|\theta^{k}\right\|\left\|\hat{\theta}^{k}\right\|+\beta\tau\left\|\bar{\theta}^{k}\right\|\left\|\hat{\theta}^{k}\right\|^{2}\le\left\|\theta^{k-1}\right\|\left\|\hat{\theta}^{k}\right\|+\left\|\rho^{k}-\rho^{k-1}\right\|\left\|\hat{\theta}^{k}\right\|+\tau\left\|\xi^{k}\right\|\left\|\hat{\theta}^{k}\right\|+S(h)\tau\left\|\hat{\theta}^{k}\right\|\left\|\bar{p}^{k}-q^{k}\right\|,$$

(4.2)

applying Young's inequality two sides gives,

$$\frac{1}{2}\left\|\theta^{k}\right\|^{2}+\frac{1}{2}\left\|\hat{\theta}^{k}\right\|^{2}+\beta\tau\left[\frac{1}{2}\left\|\hat{\theta}^{k}\right\|^{4}+\frac{1}{2}\left\|\bar{\theta}^{k}\right\|^{2}\right]\leq\frac{1}{2}\left\|\theta^{k-1}\right\|^{2}+\frac{1}{2}\left\|\hat{\theta}^{k}\right\|^{2}+$$
$$\frac{1}{2\tau}\left\|\rho^{k}-\rho^{k-1}\right\|^{2}-\frac{\tau}{2}\left\|\hat{\theta}^{k}\right\|^{2}+\tau\left[\left\|\xi^{k}\right\|^{2}+\frac{1}{4}\left\|\hat{\theta}^{k}\right\|^{2}\right]+$$
$$S(h)\,\tau\left[\left\|\bar{p}^{k}-q^{k}\right\|^{2}+\frac{1}{4}\left\|\hat{\theta}^{k}\right\|^{2}\right],$$

choosing $S(h)=1$ **,** and multiplying 2 and rearranging gives

$$\left\|\theta^{k}\right\|^{2}+2\beta\tau\left[\frac{1}{2}\left\|\hat{\theta}^{k}\right\|^{4}+\frac{1}{2}\left\|\bar{\theta}^{k}\right\|^{2}\right]\leq\left\|\theta^{k-1}\right\|^{2}+\frac{1}{\tau}\left\|\rho^{k}-\rho^{k-1}\right\|^{2}+2\tau\left\|\xi^{k}\right\|^{2}+2\tau\left\|\bar{p}^{k}-q^{k}\right\|^{2},$$

since, $\left\|\hat{\theta}^{k}\right\|^{4},\ \left\|\bar{\theta}^{k}\right\|^{2}\geq 0$,

then,

$$\left\|\theta^{k}\right\|^{2}\leq\left\|\theta^{k-1}\right\|^{2}+\frac{1}{\tau}\left\|\rho^{k}-\rho^{k-1}\right\|^{2}+2\tau\left\|\xi^{k}\right\|^{2}+2\tau\left\|\bar{p}^{k}-q^{k}\right\|^{2},$$

Summing both sides from $k=1$ to $k=N$, we get

$$\left\|\theta^{N}\right\|^{2}\leq\left\|\theta^{0}\right\|^{2}+\frac{1}{\tau}\sum_{k=1}^{N}\left\|\rho^{k}-\rho^{k-1}\right\|^{2}+2\tau\sum_{k=1}^{N}\left\|\xi^{k}\right\|^{2}+2\tau\sum_{k=1}^{N}\left\|\bar{p}^{k}-q^{k}\right\|^{2},\qquad(4.3)$$

For the second term note that

$$\rho^{k}-\rho^{k-1}=\int_{t_{k-1}}^{t_k}\rho_t\,dt\ ,$$

this implies

$$\left\|\rho^{k}-\rho^{k-1}\right\|=\int_{t_{k-1}}^{t_k}\left\|\rho_t\right\|dt\ ,$$

thus

$$\| \rho^k - \rho^{k-1} \|^2 \le \left(\int_{t_{k-1}}^{t_k} \| \rho_t \| \, dt \right)^2 = \tau^2 \left(\int_{t_{k-1}}^{t_k} \| \rho_t \| \frac{dt}{\tau} \right)^2 ,$$

applying Jensen's inequality (see [1]) to the right hand side

$$\| \rho^k - \rho^{k-1} \|^2 \le \tau^2 \int_{t_{k-1}}^{t_k} \| \rho_t \|^2 \frac{dt}{\tau} = \tau \int_{t_{k-1}}^{t_{kn}} \| \rho_t \|^2 \, dt ,$$

this implies

$$\frac{1}{\tau} \sum_{k=1}^{N} \left\| \rho^k - \rho^{k-1} \right\|^2 \le \int_0^T \| \rho_t \|^2 \, dt \le C_1 h_u^{2r} \int_0^T \| u_t \|_r^2 \, dt = C_1 h_u^{2r} \| u_t \|_r , \qquad (4.4)$$

$$\frac{1}{\tau} \sum_{k=1}^{N} \left\| \rho^k - \rho^{k-1} \right\|^2 \le C_1 h_u^{2r} \| u_t \|_r .$$

To bound the third term of (4.3), note that

$$\xi^k = u_t^k - \frac{1}{\tau} \left(u^k - u^{k-1} \right) ,$$

then, $\tau \xi^k = \tau u_t^k - \int_{t_{k-1}}^{t_k} u_t \, dt = \left(t_k - t_{k-1} \right) u_t^k - \int_{t_{k-1}}^{t_k} u_t \, dt$,

from [Theorem 3.5.1 (p 38) in [1]], we get $\| \xi^k \| = \int_{t_{k-1}}^{t_k} \| u_{tt} \| \, dt$,

and too $\tau \sum_{k=1}^{N} \left\| \xi^k \right\|^2 \le \tau \left\| u_{tt} \right\|_{L^2}^2$.

To bound the fourth term of (4.3), note that

$$\| \bar{p}^k - q^k \|^2 \le C_2 h_p^{2r} .$$

Applying these results to (4.3) gives,

$$\| \theta^N \|^2 \le \left(\| u_h^0 - u^0 \| + C_3 h_u^r \| u^0 \| \right)^2 + C_1 h_u^{2r} \| u_t \|_r + 2\tau \| u_{tt} \|_r^2 + 2\tau \sum_{k=1}^{N} C_2 h_p^{2r} \quad ,$$

suppose that $h_u = h_p = h$ in this paper, this implies we get

$$\| \theta^N \| \le \| u_h^0 - u^0 \| + C_4 \left[h^r \left\{ \| u^0 \|_r + \| u_t \|_r \right\} + 2\tau \left\{ \| u_{tt} \|_r + \mu \right\} \right], \tag{4.5}$$

hence, $\qquad \| u^k - u_h^k \| \le C(h^r + \tau).$

The proof is complete.

□

Theorem 4.2 Let $p^k \in Q$ be the solution of problem (1.8) and $p_h^k \in Q_h$ is the approximation solution of problem (1.9) then there exists a constant $C_5 > 0$ independent of h *and* τ such that:

$$\| \bar{p}^k - \bar{p}_h^k \| \le C_5 \left(h^r + \tau \right) \quad . \tag{4.6}$$

Proof: Put $v = \bar{u}_h^k$, $v = \bar{u}^k$ in equations (1.8a) , (1.9a) respectively, then subtracting the equations we find

$$\frac{1}{\tau}\left(\left(u^k - u_h^k \right) - \left(u^{k-1} - u_h^{k-1} \right), \bar{u}_h^k - \bar{u}^k \right) - n \left(\hat{u}^k - \hat{u}_h^k , \bar{u}^k - \bar{u}_h^k , \bar{u}_h^k - \bar{u}^k \right) - b \left(\bar{u}_h^k - \bar{u}^k , \bar{p}^k - \bar{p}_h^k \right) = \left(\bar{f}^k , \bar{u}_h^k - \bar{u}^k \right). \tag{4.7}$$

Let $\bar{p}^k - \bar{p}_h^k = \left(\bar{p}^k - \Pi_h \bar{p}^k \right) - \left(\bar{p}_h^k - \Pi_h \bar{p}^k \right) = \psi^k - \chi^k \quad ,$

by using triangle inequality, we have

$$\| \bar{p}^k - \bar{p}_h^k \| \le \|\psi^k\| + \|\chi^k\| \quad ,$$

from Lemma 2.1 $\|\psi^k\| \le C h^r \|\bar{p}^k\|$.

To estimate χ^k from equation (4.7), put $u^k - u_h^k = \rho^k - \theta^k$ and $\bar{p}^k - \bar{p}_h^k = \psi^k - \chi^k$

$$\frac{1}{\tau}\left(\left(\rho^k - \theta^k\right) - \left(\rho^{k-1} - \theta^{k-1}\right), \bar{\theta}^k - \bar{\rho}^k\right) - n\left(\hat{\rho}^k - \hat{\theta}^k, \bar{\theta}^k - \bar{\rho}^k, \bar{\theta}^k - \bar{\rho}^k\right) -$$
$$b\left(\bar{\theta}^k - \bar{\rho}^k, \psi^k - \chi^k\right) = \left(\bar{f}^k, \bar{\theta}^k - \bar{\rho}^k\right) \quad ,$$

by using the elliptic projection , we get

$$\frac{1}{\tau}\left(\theta^k - \theta^{k-1}, \bar{\theta}^k\right) - n\left(\hat{\theta}^k, \bar{\theta}^k, \bar{\theta}^k\right) - b\left(\bar{\theta}^k, \chi^k\right) = \frac{1}{\tau}\left(\rho^k - \rho^{k-1}, \bar{\theta}^k\right) - \left(\bar{f}^k, \bar{\theta}^k\right),$$

by using (2.2) and (2.3), and multiplying by τ , we get

$$\|\theta^k\|\|\bar{\theta}^k\| + \beta\tau\|\hat{\theta}^k\|^2 \|\bar{\theta}^k\| + S(h)\,\tau\|\bar{\theta}^k\|\|\chi^k\| \le \|\theta^{k-1}\|\|\bar{\theta}^k\| + \|\rho^k - \rho^{k-1}\|\|\bar{\theta}^k\| + \tau\|\bar{f}^k\|\|\bar{\theta}^k\|,$$

dividing by $\|\bar{\theta}^k\|$, we get

$$\|\theta^k\| + \beta\tau\|\hat{\theta}^k\|^2 + S(h)\,\tau\|\chi^k\| \le \|\theta^{k-1}\| + \|\rho^k - \rho^{k-1}\| + \tau\|\bar{f}^k\| \quad ,$$

since $\beta\tau\|\hat{\theta}^k\|^2 \ge 0$, we get

$$\|\chi^k\| \le \frac{1}{S(h)\tau}\left[\|\theta^{k-1}\| - \|\theta^k\| + \|\rho^k - \rho^{k-1}\| + \tau\|\bar{f}^k\|\right].$$

Summing both sides from $k=1$ to $k=N$, we have

$$\sum_{k=1}^{N}\|\chi^k\| \le \frac{1}{S(h)\tau}\left[\|\theta^0\| - \|\theta^N\| + \sum_{k=1}^{N}\|\rho^k - \rho^{k-1}\| + \tau\sum_{k=1}^{N}\|\bar{f}^k\|\right],$$

from equations (4.4) and (4.5), let $1 \le k^* \le N$, we get

$$\| \chi^{k^*} \| \le \frac{1}{S(h)} \left[C\left(h^r + \tau\right) + C_1 h^r \| u_t \| + \| \bar{f}^k \| \right].$$

Hence, $\| \chi^{k^*} \| \le C_6 \left(h^r + \tau \right).$

The proof is complete.

□

Theorem 4.3 Let $\left(u^k, p^k\right) \in V \times Q$ is the solution of problem (1.8) and $\left(u_h^k, p_h^k\right) \in V_h \times Q_h$ is the approximation solution of problem (1.9), then, there exists a constant $C_7 > 0$ independent of h and τ such that:

$$\| u^k - u_h^k \| + \| \bar{p}^k - \bar{p}_h^k \| \le C_7 \left(h^r + \tau \right). \qquad (4.8)$$

Proof: We can prove this theorem from equations (4.1) and (4.6). □

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الاستقرار و تحليل الخطأ لمعادلات اويلر بطريقة كرانك-نيكلسون–كاليركين الخطية

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مستخلص

في هذا البحث، درسنا معادلات اويلر غير المضغوطة غير الثابتة في مجال ثنائي البعد مستخدمين طريقة العنصر المنتهي المختلطة. بواسطة طريقة كرانك-نيكلسون–كاليركين الخطية وجدنا الصيغة الضعيفة للمسألة أعلاه وثم حسبنا الحل التقريبي . دَرسنَا ثلاث حالاتَ، الحالة الأولى لمعادلات اويلر الثابتةِ وغير الثابتة أثبتتْ بعضَ الممهدات و المبرهناتِ لاستقرار طريقةِ العنصرِ المحددةِ المُخْتَلَطةِ نصفِ المنفصلةِ والمنفصلةِ بالكامل، الحالة الثانية لمعادلات اويلر الثابتة أثبتتْ بعضَ الممهدات الإهليليجيةِ واستمرارية هذه الطرقِ والحالةِ الثالثةِ لمعادلات اويلر الثابتة وغير الثابتة تم تطبيق هذه التخمينات للحصول على الحالة شبه المثالية لتحليل الخطأ في معيار الطاقة للسرعة والضغط والسرعة مع الضغط.