On Solution Of Cylindrical Equation By New Assumption

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المستخلص

هذا البحث تناول استخدام تعويض موسع جديد لإيجاد الحل الكامل والعام للمعادلة التفاضلية الجزئية ذات الصيغة العامة

$$u_{yy} - \beta^2 (u_{xx} + u_{tt}) = H(x, t, y)$$

حيث أن β^2 ثابت اختياري و $H(x,t,y)$ دالة إلى (x,t,y).

ABSTRACT

This research presents the use of a new extension assumption for finding the complete (c.s) and general(g.s) solution of partial differential equation which have the general form

$u_{yy} - \beta^2 (u_{xx} + u_{tt}) = H(x, t, y)$

where β^2 arbitrary constant and $H(x,t,y)$ is function of x,t,y .

INTRODUCTION

Dalembert transformation is one of the assumptions which is known to solve the wave equation in one dimension without out force with boundary condition and without initial condition which has the general form

$u_{tt} = c^2 u_{xx}$.

In this paper, it is found that the extension for Dalembert assumption of three variables x,t,y for solving cylindrical equation and wave equation in two diminutions without out force and without any conditions.

The extension assumption is transformation the new equation (cylindrical and wave equations) to equation with homogenous terms which has the general form $F(D_z^2, D_z D_w, D_w^2)u = \frac{-1}{\beta^2} H(z,w)$

and after solving the homogenous part it is found (u_c) and non homogenous part it is found (u_p), for the last equation, and by substitute $Z = r + \beta y$, $W = r - \beta y$ where $r = x + t$, hence the complete and general solutions of cylindrical and wave equation is obtained.

Definition:[3]

Cylindrical equation is partial differential equation of the horizontal displacement of the fluctuated cover which has the general form

$u_{yy} - \beta^2 (u_{xx} + u_{tt}) = H(x,t,y)$,

where $\beta^2 = \frac{T}{\rho} =$(constant force / surface density) and $H(x,t,y)$ displacement (function of x,t,y) represent out force.

and if $H(x,t,y) = 0$ then the above equation becomes wave equation in two dimension without out force .

The Dalembert Solution Of the Wave Equation:[2]

Dalembert solved in 1750 the wave equation in one dimension without out effect, which have the general form :

$u_{tt} = \alpha^2 u_{xx}, -\infty < x < \infty \ ,0 < t < \infty$

where α^2 is arbitrary constant and without boundary condition and with initial condition $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

by using assumption

$Z = x + \alpha t$, $W = x - \alpha t$

Now , by found

$\frac{\partial Z}{\partial x} = 1$, $\frac{\partial Z}{\partial t} = \alpha$, $\frac{\partial w}{\partial x} = 1$, $\frac{\partial w}{\partial t} = -\alpha$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial x} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial x} \quad \Rightarrow \quad u_x = u_z + u_w$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial z^2}.\frac{\partial Z}{\partial x} + \frac{\partial^2 u}{\partial z \partial w}.\frac{\partial W}{\partial x} + \frac{\partial^2 u}{\partial w^2}.\frac{\partial W}{\partial x} + \frac{\partial^2 u}{\partial w \partial z}.\frac{\partial Z}{\partial x}$$

$$u_{xx} = u_{zz} + 2u_{zw} + u_{ww} \quad ... \ (1$$

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial t} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial t} \quad \Rightarrow \quad u_t = \alpha\, u_z - \alpha\, u_w$$

$$u_{tt} = \alpha[\frac{\partial^2 u}{\partial t^2}] = \alpha[\frac{\partial^2 u}{\partial z^2}.\frac{\partial z}{\partial t} + \frac{\partial^2 u}{\partial z \partial w}.\frac{\partial w}{\partial t} - \frac{\partial^2 u}{\partial w^2}.\frac{\partial w}{\partial t} - \frac{\partial^2 u}{\partial w \partial z}.\frac{\partial z}{\partial t}]$$

$$u_{tt} = \alpha^2 [u_{zz} - 2\, u_{zw} + u_{ww}] \quad \text{... (2}$$

since ,

$u_{tt} = \alpha^2 u_{xx}$, then

$$\alpha^2 [u_{zz} - 2\, u_{zw} + u_{ww}] = \alpha^2 [u_{zz} + 2\, u_{zw} + u_{ww}]$$

$$u_{zz} - 2\, u_{zw} + u_{ww} = u_{zz} + 2\, u_{zw} + u_{ww}$$

$u_{zw} = 0$, which is partial differential equation , can be solved by direct integral , hence it is obtained that:

$u(z, w) = f(z) + g(w)$, where $f(z), g(w)$ arbitrary function

since ,

$Z = x + \alpha t$, $W = x - \alpha t$

then ,

$$u(x,t) = F(x+\alpha t) + G(x-\alpha t)$$

Now by using initial condition $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

it is obtained that:

$$f(x) = F(x) + G(x) \text{...(3}$$

$$u_t(x,t) = \alpha F'(x+\alpha t) - \alpha G'(x-\alpha t)$$

$$g(x) = \alpha F'(x) - \alpha G'(x) \text{ ...(4}$$

By integral (4) with respect to x it is obtained that :

$$\int_{x_0}^{x} g(v)dv + c = \alpha F(x) - \alpha G(x) \text{ ...(5}$$

by solve (3) and (5) , we get

$$F(x) = \frac{1}{2\alpha}\int_{x_0}^{x} g(v)dv + \frac{c}{2\alpha} + \frac{1}{2}\, f(x)$$

$$G(x) = \frac{-1}{2\alpha}\int_{x_0}^{x} g(v)dv - \frac{c}{2\alpha} + \frac{1}{2}\, f(x)$$

$$F(x+\alpha t) = \frac{1}{2\alpha}\int_{x_0}^{x+\alpha t} g(v)dv + \frac{c}{2\alpha} + \frac{1}{2}\, f(x+\alpha t)$$

$$G(x-\alpha t) = \frac{-1}{2\alpha}\int_{x_0}^{x-\alpha t} g(v)dv - \frac{c}{2\alpha} + \frac{1}{2}\, f(x-\alpha t)$$

since

$$u(x,t)=F(x+\alpha t)+G(x-\alpha t)$$

then

$$+\frac{1}{2\alpha}\int_{x_0}^{x-\alpha t} g(v)dv-\frac{c}{2\alpha}+\frac{1}{2}f(x-\alpha t)\quad u(x,t)=\frac{1}{2\alpha}\int_{x_0}^{x+\alpha t} g(v)dv+\frac{c}{2\alpha}+\frac{1}{2}f(x+\alpha t)$$

$$u(x,t)=\frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(v)dv+\frac{1}{2}[f(x+\alpha t)+f(x-\alpha t)]$$

the above solution is Dalembert solution of the wave equation.

Example:- To find the solution of the equation

$$u_{tt}=u_{xx},-\infty<x<\infty\quad,0<t<\infty$$

$f(x)=0$, $g(x)=\sin x$ when

since

$$u(x,t)=\frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(v)dv+\frac{1}{2}[f(x+\alpha t)+f(x-\alpha t)]$$

then

$$u(x,t)=\frac{1}{2}\int_{x-t}^{x+t}\sin(v)dv$$

$$u(x,t)=\frac{-1}{2}[\cos(v)\Big|_{x-t}^{x+t}]$$

$$u(x,t)=\frac{-1}{2}[\cos(x+t)-\cos(x-t)]$$

Extension of Dalemberts Assumption

It comes to be known by the us of the above method how Dalembert has used assumption for solving the wave equation. Now the Dalembert assumption is taken but in an extension form.

let :

$Z=r+\beta y$, $W=r-\beta y$, where $r=x+t$

$$\frac{\partial Z}{\partial r}=1\ ,\ \frac{\partial Z}{\partial y}=\beta\ ,\ \frac{\partial w}{\partial r}=1\ ,\ \frac{\partial w}{\partial y}=-\beta\ ,\ \frac{\partial r}{\partial x}=1\ ,\ \frac{\partial r}{\partial t}=1$$

now it is found:

1) $u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial r}.\frac{\partial r}{\partial x} \quad \Rightarrow \; u_x = u_z + u_w$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial z^2}.\frac{\partial z}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial z \partial w}.\frac{\partial w}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial w^2}.\frac{\partial w}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial w \partial z}.\frac{\partial z}{\partial r}.\frac{\partial r}{\partial x}$$

$$u_{xx} = u_{zz} + 2u_{zw} + u_{ww}$$

2) $u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial r}.\frac{\partial r}{\partial t} \quad \Rightarrow \; u_t = u_z + u_w$

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial z^2}.\frac{\partial z}{\partial r}.\frac{\partial r}{\partial t} + \frac{\partial^2 u}{\partial z \partial w}.\frac{\partial w}{\partial r}.\frac{\partial r}{\partial t} + \frac{\partial^2 u}{\partial w^2}.\frac{\partial w}{\partial r}.\frac{\partial r}{\partial t} + \frac{\partial^2 u}{\partial w \partial z}.\frac{\partial z}{\partial r}.\frac{\partial r}{\partial t}$$

$$u_{tt} = u_{zz} + 2u_{zw} + u_{ww}$$

3) $u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial y} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial y} \quad \Rightarrow \; u_y = \beta u_z - \beta u_w$

$$u_{yyt} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial z^2}.\frac{\partial z}{\partial y} + \frac{\partial^2 u}{\partial z \partial w}.\frac{\partial w}{\partial y} + \frac{\partial^2 u}{\partial w^2}.\frac{\partial w}{\partial y} + \frac{\partial^2 u}{\partial w \partial z}.\frac{\partial z}{\partial y}$$

$$u_{yy} = \beta^2 [u_{zz} - 2u_{zw} + u_{ww}]$$

Solution of Cylindrical Equation by a New Method

It has been explained how to use a new method for solving cylindrical equation by using new extension
let the cylindrical equation when $H(x,t,y) \neq 0$ in general form

$$u_{yy} - \beta^2 (u_{xx} + u_{tt}) = H(x,t,y) \quad ... (6$$

by substation

$u_{xx} = u_{zz} + 2u_{zw} + u_{ww}$, $u_{tt} = u_{zz} + 2u_{zw} + u_{ww}$ and

$u_{yy} = \beta^2 [u_{zz} - 2u_{zw} + u_{ww}]$

in the equation (6), it is obtained

$$\beta^2 [u_{zz} - 2u_{zw} + u_{ww}] - \beta^2 [2u_{zz} + 4u_{zw} + 2u_{ww}] = H(z,w)$$

$$u_{zz} + 6u_{zw} + u_{ww} = \frac{-1}{\beta^2} H(z,w)$$

let $\overline{H}(z,w)=\frac{-1}{\beta^2}H(z,w)$

$$u_{zz}+6u_{zw}+u_{ww}=\overline{H}(z,w) \quad \ldots (7$$

the above equation is homogenous terms and non homogenous with constant coefficient , therefore the equation (7)can be solved by finding the general solution for the non homogenous part (u_p),

let
$$u_p=\frac{1}{D_z^2(1+(\frac{6D_w}{D_z}+\frac{D_w^2}{D_z^2}))}\overline{H}(z,w)$$

$$u_p=\frac{1}{D_z^2}(1-(\frac{6D_w}{D_z}+\frac{D_w^2}{D_z^2})+\ldots)\overline{H}(z,w)$$

and finding the general solution for the homogenous part (u_c) by two methods:

1) The first one is by the use of [1],and by this way the general solution of cylindrical equation is obtained in form

$$u_c(z,w)=\sum_{i=1}^{2}\phi_i(m_i z+w)$$

and since $Z=r+\beta y$, $W=r-\beta y$, where $r=x+t$

then the general solution is obtained:

$$u(x,t,y)=\sum_{i=1}^{2}\phi_i(m_i(x+t+\beta y)+(x+t-\beta y))+\varphi\ (x,t,y)$$

where $m_1=-0.17$, $m_2=-5.8$ and $\phi_i\ i=1,2$ arbitrary functions.

2)The second method is by the use of $u(z,w)=e^{\int n(z)dz+\int m(w)dw}$, which transforms the homogenous part in the equation (7) to first ordinary differential equation and let $m(w)=\lambda$, by this way the complete solution of cylindrical equation is obtained in form

$$u_c(z,w)=e^{(w-3z)\lambda}\left(d\cos 2\sqrt{2}\lambda iz+b\sin 2\sqrt{2}\lambda iz\right)$$

$$u_c(z,w)=e^{(w-3z)\lambda}\left(d\frac{e^{-2\sqrt{2}\lambda z}+e^{2\sqrt{2}\lambda z}}{2}+b\frac{e^{-2\sqrt{2}\lambda z}+e^{2\sqrt{2}\lambda z}}{2i}\right)$$

$$u_c(z,w)=e^{(w-3z)\lambda}\left(c_1e^{2\sqrt{2}\lambda z}+c_2e^{-2\sqrt{2}\lambda z}\right)$$

and since $Z=r+\beta y$, $W=r-\beta y$, where $r=x+t$

then the complete solution is obtained

$$u(x,t,y)=e^{(x+t-\beta y-3(x+t+\beta y))\lambda}\left(c_1e^{2\sqrt{2}\lambda(x+t+\beta y)}+c_2e^{-2\sqrt{2}\lambda(x+t+\beta y)}\right)+\varphi(x,t,y)$$

where β, c_1, c_2 and λ arbitrary constants.

IF $H(x,t,y)=0$ then the cylindrical equation becomes in form

$$u_{yy}-\beta^2(u_{xx}+u_{tt})=0$$

then the complete solution and general solution by the same method can be found as in the above.

hence, the complete solution is

$$u(x,t,y)=e^{(x+t-\beta y-3(x+t+\beta y))\lambda}\left(c_1 e^{2\sqrt{2}\lambda(x+t+\beta y)}+c_2 e^{-2\sqrt{2}\lambda(x+t+\beta y)}\right)$$

where β, c_1, c_2 and λ arbitrary constants.

and the general solution is

$$u(x,t,y)=\sum_{i=1}^{2}\phi_i(m_i(x+t+\beta y)+(x+t-\beta y))$$

where $m_1=-0.17$, $m_2=-5.8$ and $\phi_i\, i=1,2$ arbitrary functions.

Example:

Example(1): To solve the equation $u_{yy}-\frac{1}{4}(u_{xx}+u_{tt})=2(x+t)+y$

$$\beta^2=\frac{1}{4}, Z=r+\frac{1}{2}y\ ,\quad W=r-\frac{1}{2}y,\ r=x+t\Rightarrow r=\frac{z+w}{2}\ ,\ y=z-w$$

$$H(x,t,y)=2(x+t)+y$$

$$\Rightarrow H(r,y)=2r+y\Rightarrow \overline{H}(z,w)=-4[\left(2\frac{z+w}{2}\right)+z-w]\Rightarrow \overline{H}(z,w)=-8z$$

hence

$$u_c(z,w)=\varphi_1(-0.17z+w)+\varphi_1(-5.8z+w)$$

and $u_p=\frac{-8}{D_z^2}(1-(\frac{6D_w}{D_z}+\frac{D_w^2}{D_z^2})+\ldots)z \Rightarrow u_p=\frac{-4}{3}z^3$

then the general solution is

$$u(x,t,y)=\sum_{i=1}^{2}\phi_i(m_i(x+t+\beta y)+(x+t-\beta y))+\varphi\,(x,t,y)$$

$$u\,(x,t,y)=\varphi_1(-0.17(x+t+\frac{1}{2}y)+(x+t-\frac{1}{2}y))+\varphi_1(-5.8(x+t+\frac{1}{2}y)+(x+t-\frac{1}{2}y))-\frac{4}{3}\left(x+t+\frac{1}{2}y\right)^3$$

when $\phi_i\, i=1,2$ arbitrary functions.

and the complete solution

$$u(x,t,y)=e^{(x+t-\beta y-3(x+t+\beta y))\lambda}\left(c_1e^{2\sqrt{2}\lambda(x+t+\beta y)}+c_2e^{-2\sqrt{2}\lambda(x+t+\beta y)}\right)$$

$$u(x,t,y)=e^{\left(x+t-\frac{1}{2}y-3(x+t+\frac{1}{2}y)\right)\lambda}\left(c_1e^{2\sqrt{2}\lambda(x+t+\frac{1}{2}y)}+c_2e^{-2\sqrt{2}\lambda(x+t+\frac{1}{2}y)}\right)-\frac{4}{3}\left(x+t+\frac{1}{2}y\right)^3$$

where c_1, c_2 and λ arbitrary constants

Example(2): To solve the equation $u_{yy}-(u_{xx}+u_{tt})=\frac{1}{2}(x+t)-y$

$\beta^2=1, Z=r+y$, $W=r-y$, $r=x+t\Rightarrow r=\frac{z+w}{2}$, $y=\frac{z-w}{2}$

$$H(x,t,y)=\frac{1}{2}(x+t)-y\Rightarrow H(r,y)=\frac{1}{2}r-y\Rightarrow \overline{H}(z,w)=-[\frac{1}{2}\left(\frac{z+w}{2}\right)-\left(\frac{z-w}{2}\right)]\Rightarrow$$

$$\overline{H}(z,w)=\frac{-3}{4}w+\frac{1}{4}z$$

hence ,

$$u_c(z,w)=\varphi_1(-0.17z+w)+\varphi_1(-5.8z+w)$$

and $u_p=\frac{1}{D_z^2}(1-(\frac{6D_w}{D_z}+\frac{D_w^2}{D_z^2})+\ldots)\left(\frac{1}{4}z-\frac{3}{4}w\right)\Rightarrow u_p=\frac{1}{8}(\frac{19}{3}z-3w)z^2$

then the general solution is

$$u(x,t,y)=\sum_{i=1}^{2}\phi_i(m_i(x+t+\beta y)+(x+t-\beta y))+\varphi(x,t,y)$$

$$u(x,t,y)=\varphi_1(-0.17(x+t+y)+(x+t-y))+\varphi_1(-5.8(x+t+y)+(x+t-y))+\frac{1}{8}(\frac{19}{3}(-2x-2t+4y))\left(x+t+y\right)^2$$

when $\phi_i\, i=1,2$ arbitrary functions.

and the complete solution

$$u(x,t,y)=e^{(-2x-2t+4y)\lambda}\left(c_1e^{2\sqrt{2}\lambda(x+t+y)}+c_2e^{-2\sqrt{2}\lambda(x+t+y)}\right)+\frac{1}{8}(\frac{19}{3}(-2x-2t+4y))\left(x+t+y\right)^2$$

where c_1, c_2 and λ arbitrary constants.

Example(3): To solve the equation $u_{yy} = 4(u_{xx} + u_{tt})$

since the complete solution is

$$u(x,t,y) = e^{(x+t-\beta y-3(x+t+\beta y))\lambda}\left(c_1 e^{2\sqrt{2}\lambda(x+t+\beta y)} + c_2 e^{-2\sqrt{2}\lambda(x+t+\beta y)}\right)$$

hence

$$u(x,t,y) = e^{(-2x-2t-8y)\lambda}\left(c_1 e^{2\sqrt{2}\lambda(x+t+2y)} + c_2 e^{-2\sqrt{2}\lambda(x+t+2y)}\right)$$

where c_1, c_2 and λ arbitrary constants.

and the general solution is

$$u(x,t,y) = \sum_{i=1}^{2} \phi_i(m_i(x+t+\beta y) + (x+t-\beta y))$$

hence

$$u(x,t,y) = \varphi_1(-0.17(x+t+2y) + (x+t-2y)) + \varphi_1(-5.8(x+t+2y) + (x+t-2y))$$

where $\phi_i\ i = 1,2$ arbitrary functions.

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