Right Closed Multiplication sets in Prime Near-Ring (α,β)-Derivation

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Abstract

In this paper , we extended the concept given in [1] which they used a semi group ideal to get the result has been given in [2] . We introduce a new body which is call it a right closed multiplicative set with zero . This structure give use similar results , and any semi group ideal satisfy the conditions of right closed multiplicative set . We prove that for any prime near-ring N and a multiplicative set I , if I , is abelian , then N is abelian . These results depends on many papers for example [3] , [4] , [5] , [6] , [7] , [8] .

المجموعات المغلقة بالضرب من اليمين في الحلقة المقتربة الأولية مع الاشتقاق (α,β)

الخلاصة

في هذا البحث ، عملنا على تعميم الفكرة في البحث المنشور في [١] ، حيث استفادوا من تعريف المثاليات الأولية للحصول على النتائج التي ظهرت في [٢] بينما نحن استحدثنا تعريف ذو شروط اقل و يقوم بنفس المهمة في النتائج و برهنا تلك النتائج . إن التعريف الذي استدلينا علية و أسميناه المجموعة المغلقة على عملية الضرب من اليمين و تحتوي على الصفر . واهتدينا إلى النتيجة الرئيسية التي تنص على أن الحلقة المقترب الأولية N تكون أبدالية مع الجمع أذا احتوت على مجموعه مغلقة من اليمين تحتوي على الصفر و هي أبدالية مع الجمع . هذه النتائج تعتمد على بعض البحوث منها[٣] ، [٤] ، [٥] ، [٦] ، [٧] ، [٨] .

1.Introduction :-

In this section we introduce a necessary conditions and definitions to get our results .

Definition(1.1)[1]:-An additive mapping $D: N \rightarrow N$ is said to be derivation on N if $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$.

Notation(1.2)[1]:-In this paper N will be denoted a left near-ring with multiplicative center $Z(N)$, the symbol $[x,y]$ will denote the commutator $xy - yx$, the symbol (x,y) will denote the additive commutator $x + y - y - x$, $[x,y]_{\alpha,\beta}$ will denote the (α,β) −commutator $\beta(x)y - y\alpha(x)$ and a near-ring N is called a zero symmetric if $0x = 0$, for all $x \in N$.

Definition(1.3)[1]:- An additive mapping $D: N \rightarrow N$ is called a (α, β) −derivation if there exists "as automorphisms" $\alpha , \beta : N \rightarrow N$ such that $D(xy) = \beta(x)D(y) + D(x)\alpha(y)$ for all $x , y \in N$.

Definition(1.4)[1]:- The (α,β) −derivation D will be called (α,β) −commuting if $[x, D(x)]_{\alpha,\beta} = 0$ for all $x \in N$.

Definition(1.5)[1]:-A near-ring N is said to be prime if $aNb = 0$ implies that $a = 0$ or $b = 0$. Further an element $x \in N$ for which $D(x) = 0$ is called a constant.

Definition (1.6) :- A subset I of a near-ring N is called a right closed multiplication set contain zero , if $NI \subseteq I$. We will use right closed multiplication set contain zero (RCM) for this set .

2.Main result :-

In this section , we give some results which depend on section one .

Lemma(2.1) :- An additive endomorphism D on a near-ring N is $(\alpha,\beta)-$derivation if and only if $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$ for all $x,y \in N$.

Proof :- Let D be a $(\alpha,\beta)-$derivation on a near-ring N . Since $x(y+y) = xy + xy$ we obtain ,

$$D(x(y+y)) = \beta(x)D(y+y) + D(x)\alpha(y+y)$$

$$= \beta(x)D(y) + \beta(x)D(y) + D(x)\alpha(y) + D(x)\alpha(y) \text{....(2.1)}$$

for all $x,y \in N$, on the other hand , we have ; $D(xy+xy) = D(xy) + D(xy)$

$$= \beta(x)D(y) + D(x)\alpha(x) + \beta(x)D(y) + D(x)\alpha(y) \text{....(2.2)}$$

for all $x,y \in N$, combining (2.1) and (2.2) , we find $\beta(x)D(y) + D(x)\alpha(y) = D(x)\alpha(y) + \beta(x)D(y)$ for all $x,y \in N$. Thus , we have $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$.

Conversely , let $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$............(2.3) for all $x,y \in N$, then ;

$$D(x(y+y)) = D(x)\alpha(y+y) + \beta(x)D(y+y)$$

$= D(x)\alpha(y) + D(x)\alpha(y) + \beta(x)D(y) + \beta(x)D(y)$...(2.4) for all $x, y \in N$. Also ;

$D(xy + xy) = D(xy) + D(xy)$

$= D(x)\alpha(y) + \beta(x)D(y) + D(x)\alpha(y) + \beta(x)D(y)$.....(2.5) for all $x, y \in N$ combining (2.4) and (2.5) , we obtain .

$D(xy) = D(x)\alpha(y) + \beta(x)D(y) = \beta(x)D(y) + D(x)\alpha(y)$. Thus

$$D(xy) = \beta(x)D(y) + D(x)\alpha(y) \text{ for all } x, y \in N . \blacksquare$$

<u>Lemma(2.2)</u> :- Let N be a prime near-ring and I be a non zero RCM . If (I ,+) is a abelian then (N,+) is abelian .

<u>Proof</u> :-Let $x, y \in N$ and $a \in I$, then $xa, ya \in I$. So $xa + ya = ya + xa$. Then we get $(x + y - y - x)a = 0$ for all $a \in I$ and $x, y \in N$. This means that $(x + y - y - x)I = (x + y - y - x)NI = 0$ because I is a non zero RCM set . Since N is a prime near-ring we have $x + y - y - x = 0$ for all $x, y \in N$. Thus (N,+) is abelian .$\blacksquare$

<u>Lemma(2.3)</u> :- Let N be a prime near-ring and I be a RCM set of N .

(i)- If z is a non zero element in $Z(N)$, then z is not zero divisor.

(ii)- If there exist a non zero element z of $Z(N)$ such that $z + z \in Z(N)$, then $(I,+)$ is abelian .

Proof :- (i) – If $z \in Z(N)-\{0\}$, and $zx=0$ for some $x \in I$. Left multiplying this equation by b , where $b \in N$, we get $bzx = 0$. Since N is multiplicative with center $Z(N)$, we get $zbx = 0$, for all $b \in N, x \in I$. Hence , $zNx=0$. Since N is a prime near-ring and z is a non zero element , we get $x=0$. ■

(ii) - Let $z \in Z(N)-\{0\}$, such that $z+z \in Z(N)$, and let $x, y \in I$, then $(x+y)(z+z)=(z+z)(x+y)$. Hence $xz+xz+yz+yz=zx+zy+zx+zy$ since $z \in Z(N)$, we get $zx+zy=zy+zx$. Thus , $z(x+y-x-y)=0$ then be (i) $z \neq 0$ we get $(x,y)=0$ hence $(I,+)$ is abelian .■

Lemma (2.4) :- Let D be a non zero (α,β) –derivation on a prime near-ring N and I be a non zero RCM set of N. such that $\alpha(I)=I$ and $\beta(I)=I$, let $x \in I$ then :

(i)- If $\beta(x)D(I)=0$ then $x=0$. **(ii)**- If $xD(I)=0$ then $x=0$.

Proof :- **(i)**- For $a, b \in I$, we have $\beta(x)D(ab)=0$, so $\beta(x)\big(\beta(a)D(b)+D(a)\alpha(b)\big)=0$, to get $\beta(x)\beta(a)D(b)+\beta(a)D(a)\alpha(b)=0$, the second summand in this equation equal zero by the hypothesis , so get $\beta(x)\beta(a)D(b)=0$, for $a, b, x \in I$. $\beta(I)=I$, we get $\beta(x)ID(b)=0$, since I is a RCM set of N , we get $\beta(x)NID(b)=0$. Since N is a prime near-ring , I is a non zero RCM set of N , D is a non zero (α,β) –derivation of N , we get $\beta(x)=0$, for all $x \in I$. Since $\beta(I)=I$, thus get $x=0$. ■

(ii)- For all $a,b \in I$, we get $xD(ab)=0$. Thus $x\beta(a)D(b)+xD(a)\alpha(b)=0$, the second summand in this equation equal zero by the hypothesis and I is a RCM set of N , we get $x\beta(a)D(b)=0$, for all $x,a,b \in I$. But $\beta(I)=I$, then $xID(b)=0$, this means that $xNID(I)=0$. By the same way in (i) we get $x=0$. ■

<u>Lemma(2.5)</u> **:-** Let D be $(\alpha,\beta)-$derivation on a near-ring N and I be a RCM set of N such that $\alpha(I)=I$ and $\beta(I)=I$. Suppose $u \in I$ is a not a left zero divisor . If $[u,D(u)]=0$, then (x,u) is a constant for every $x \in I$.

<u>**Proof**</u> **:-** From $u(u+x)=u^2+ux$, apply D for both sided to have $D\big(u(u+x)\big)=D(u^2+ux)$. Expanding this equation , to have $D\big(u(u+x)\big)=\beta(u)D(u+x)+D(u)\alpha(u+x)$

$$=\beta(u)D(u)+\beta(u)D(u)+D(u)\alpha(u)+D(u)\alpha(x)$$. and

$D(u^2+ux)=D(u^2)+D(ux)$

$$=\beta(u)D(u)+D(u)\alpha(u)+\beta(u)D(u)+D(u)\alpha(u)$$, for all $u,x \in I$. since $D\big(u(u+x)\big)=D(u^2+ux)$ which reduces to $\beta(u)D(x)+D(u)\alpha(u)=D(u)\alpha(u)+\beta(u)D(x)$, for all $u,x \in I$. By using the hypothesis $[u,D(u)]=0$, this equation is expressible as $\beta(u)\Big(\big(D(x)+D(u)-D(x)-D(u)\big)\Big)=0=\beta(u)D((x,u))$. so $\beta(I)=I$, $uD((x,u))=0$. From u is not a left zero divisor , we get $D((x,u))=0$ .Thus , (x,u) is a constant for every $x \in I$.■

Proposition(2.6) :- Let N be a near-ring and I is a RCM set of N have no non zero divisors of zero . If N admits a non zero (α,β) –derivation on D which is commuting on I,then $(N,+)$ is abelian.

Proof :- Let c be any additive commutator in I . Then , by lemma (2.5) , yields that c is a constant . Now for any $x \in I$, cx is also an additive commutator in I . Hence , also a constant . Thus , $0 = D(cx) = \beta(c)D(x) + D(c)\alpha(x)$. Second summand in this equation equal zero, we get $\beta(c)D(x) = 0$, for all $x \in I$ and an additive commutator c in I . By lemma (2.4) (i) , we get $c = 0$ for all additive commutator c in I .Hence , $(I,+)$ is abelian . By lemma (2.2) , we get $(N,+)$ is abelian . ■

Lemma (2.7) :- Let N be a prime near-ring , I be a non zero RCM set of N and D be a non zero (α,β) –derivation on N , $\beta(I) = I$. If $D((x,y)) = 0$, for all $x , y \in I$, then $(I,+)$ is abelian .

Proof :- Suppose that $D\big((x,y)\big) = 0$, for all $x,y \in I$. Taking xu instead of x and yu instead of y , where $u \in I$ we get $0 = D((xu,yu)) = D((x,y)u) = \beta((x,y))D(u) + D((x,y))\alpha(u)$, for all $x , y , u \in I$. By the hypothesis have $\beta((x,y))D(u) = 0$, for all $x, y, u \in I$. Hence , $\beta((x,y))D(I) = 0$. Using lemma (2.4) (i) , to get $(x,y) = 0$, for all $x , y \in I$. Thus , $(I,+)$ is abelian . ■

lemma (2.8) :- Let N be a prime near-ring and I be a non zero RCM set of N . If I is a commutative then N is a commutative near-ring .

Proof :- For all $a , b \in I$, $a.b = b.a$ we get $a.b - b.a = 0$, since , $a.b - b.a = [a,b] = 0$ so $[a,b] = 0$. Taking xa instead of a and yb instead of b , where $x , y \in N$, to get $0 = [xa,yb] = xayb - ybxa = xyab - yxab = [x,y]ab$, for all $a , b \in I$ and $x , y \in N$. Thus , $[x,y]ab = [x,y]I^2 = 0$. Since

$NI \subseteq I$, we get $[x,y]NI^2 = 0$, for all $x, y \in N$. Since N is a prime near-ring and I is a non zero, we get $[x,y] = 0$, for all $x, y \in N$. Hence, N is a commutative near-ring. ■

<u>Lemma(2.9)</u> :- Let N be a prime near-ring admits, a non zero $(\alpha,\beta)-$derivation D and I be a RCM set of N. If $D(I) \subseteq Z(N)$ then $(I,+)$ is abelian.

<u>Proof</u> :- Since $D(I) \subseteq Z(N)$ and D is a non zero $(\alpha,\beta)-$derivation. There exists a non zero element x in I, such that $z = D(x) \in Z(N) - \{0\}$ so $z + z = D(x) + D(x) = D(x + x) \in Z(N)$. Hence $(I,+)$ is abelian by lemma((2.3) (ii)) .■

<u>Corollary (2.10)</u> :- Let N be a prime near-ring admits a non zero $(\alpha,\beta)-$derivation and I be a RCM set of N. If $D(I) \subseteq Z(N)$ then $(N,+)$ is abelian.

<u>Proof</u> :-Using lemma (2.9), to have $(I,+)$ abelian, then using lemma (2.2), we get $(N,+)$ is abelian .■

<u>Proposition (2.11)</u> :- Let N be a prime near-ring admitting a non zero $(\alpha,\beta)-$derivation D such that $\beta D = D\beta$, I be a RCM set of N such that $\beta(I) = I$ and β is homomoriphism on N. If $[D(I), D(I)] = 0$, then $(N,+)$ is abelian .

<u>Proof</u> :- By the hypothesis $[D(I), D(I)] = 0$, for all $x, y, t \in I$ we have $D(t+t)D(x+y) = D(x+y)D(t+t)$. Hence, $D(t)D(x) + D(t)D(y) - D(x)D(t) - D(y)D(t) = 0$. By application the hypothesis in this equation, we get,

$\big(D(x) + D(y) - D(x) - D(y)\big)D(t) = (D(x + y - x - y))D(t) = 0$ then $D((x,y))D(t) = 0$, for all $x , y, t \in I$. Since β is homomorphism on N, we get $\beta(D((x,y)))\beta(D(t)) = 0$. By using $\beta D = D\beta$, obtain $\beta(D((x,y)))D(\beta(t)) = 0$, for all $x , y , t \in I$. Using lemma ((2.4)(i)) . Obtain $D((x,y)) = 0$, for all $x , y \in I$. Then using lemma (2.7) , we get $(I, +)$ is abelian . So using lemma (2.2) , to get $(N, +)$ is abelian . ■

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