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Regular Proper Mappings

Habeeb Kareem Abdullah *Fadhila Kadhum Radhy*
Department of Mathematics *College of Education for Girls*
University of Kufa

Abstract

The main goal of this work is to create a general type of proper mappings namely, regular proper mappings and we introduce the definition of a new type of compact and coercive mappings and give some properties and some equivalent statements of these concepts as well as explain the relationship among them .

Introduction

One of the very important concepts in topology is the concept of mapping . There are several types of mapping , in this work we study an important class of mappings , namely , regular proper mapping .

Proper mapping was introduced by Bourbaki in [1] .

Let A be a subset of topological space X . We denote to the closure and interior of A by $\overline{A}$ and A° respectively .

James Dugundji in [2] defined the regular open set as , a subset A of a space X such that called regular open set if A = $\overset{\circ}{\overline{A}}$. Stephen Willard in [8] defined the regular open set similarly with Dugundji's definition .

This work consists of three sections .

Section one includes the fundamental concepts in general topology , and the proves of some related results which are needed in the next section .

Section two contains the definitions of regular compact mapping and regular coercive mapping . So it will introduce the relationship among them and some results about this subjects are proved .

Section three introduces the definition of regular proper mapping and some of its related results are proved .

1- Basic concepts

Definition 1.1 , [2] : A subset B of a space X is called **regular open (r- open)** set if B = $\overset{\circ}{\overline{B}}$. The complement of regular open set is defined to be a **regular closed (r- closed)** set .

Proposition 1.2 , [2] : A subset B of a space X is r- closed if and only if B = $\overline{\overset{\circ}{B}}$.

Its clearly that every r- open set is an open set and every r- closed set is closed set , but the converse is not true in general as the following example shows :

Example 1.3 : Let X = {a, b, c, d} be a set and T = {Ø, X, {a}, {a, b}, {a, c, d}} be a topology on X . Notice that {a, b} is an open set in X , but its not r- open set and {b} is a closed set in X , but its not r- closed set .

Corollary 1.4 :

(i) A subset B of a space X is clopen (open and closed) if and only if B is r- clopen (r- open and r- closed) .

(ii) If A is an r- closed set in X and B is a clopen set in X , then $A \cap B$ is r- closed set in B . **Proposition 1.5 :** Let $A \subseteq Y \subseteq X$. Then :

(i) If A is an r- open set in Y and Y is an r- open set in X , then A is an r- open set in X .

(ii) If A is an r- closed set in Y and Y is an r- closed set in X, then A is an r- closed set in X .

Definition 1.6 : Let A be a subset of a space X . A point $x \in A$ is called **r- interior** point of A if there exists an r- open set U in X such that $x \in U \subseteq A$.

The set of all r- interior points of A is called **r- interior** set of A and its denoted by $A^{\circ r}$.

Proposition 1.7 : Let (X , T) be a space and $A \subseteq X$. Then :

(i) $A^{\circ r} \subseteq A^{\circ}$.

(ii) $(A^{\circ r})^{\circ} = (A^{\circ})^{\circ r}$.

(iii) A is r- open if and only if $A^{\circ r} = A$.

Definition 1.8 : Let A be a subset of a space X . A point x in X is said to be **r- limit** point of A if for each r- open set U contains x implies that $U \cap A \setminus \{x\} \neq Ø$.

The set of all r- limit points of A is called **r- derived** set of A and its denoted by $A^{'r}$.

Definition 1.9 : Let X be a space and $B \subseteq X$. The intersection of all r- closed sets containing B is called the **r- closure** of B and denotes by $\overline{A}^{r}$.

Proposition 1.10 : Let X be a space and A , B $\subseteq$ X . Then :

(i) $\overline{A}^r$ is an r- closed set .

(ii) A $\subseteq$ $\overline{A}^r$.

(iii) A is r- closed if and only if $\overline{A}^r$ = A .

(iv) x $\in$ $\overline{A}^r$ if and only if A$\cap$U $\neq$ θ , for any r- open set U containing x .

Proposition 1.11: Let X and Y be two spaces , and A $\subseteq$ X , B $\subseteq$ Y . Then :
(i) A , B are r- open subset of X and Y respectively if and only if A$\times$B is r- open in X$\times$Y .
(ii) A , B are r- closed subsets of X and Y respectively if and only if A$\times$B is r- closed in X$\times$Y .
(iii) A , B are clopen subsets of X and Y respectively if and only if A$\times$B is clopen in X$\times$Y .
(iv) A , B are r- clopen subsets of X and Y respectively if and only if A$\times$B is r- clopen in X$\times$Y .

Definition 1.12 , [3] : Let X be a space and B be any subset of X . **A neighborhood of B** is any subset of X which containing an open set containing B .

The neighborhoods of a subset {x} , consisting of a single point are also called **neighborhood of a point x** .

The collection of all neighborhoods of the subset B is denoted by **N(B)** . In particular the collection of all neighborhoods of x is denoted by **N(x)** .

Proposition 1.13 , [1] : Let X be a set . If to each element x of X , there corresponds a collection β(x) of subsets of X , such that the properties :

(i) Every subset of X which contains a set belongs to β(x) , itself belongs to β(x) .

(ii) Every finite intersection of sets of β(x) belongs to β(x) .

(iii) The element x is in every set of β(x) .

(iv) If V belongs to β(x) , then there is a set W belonging to β(x) such that for each y $\in$ W , V belongs to β(y).

Then there is a unique topological structure on X such that , for each x $\in$X , β(x) is the collection of neighborhoods of x in this topology .

Definition 1.14 : Let X be a space and B $\subseteq$ X . An **r- neighborhood of B** is any subset of X which contains an r- open set containing B . The r-neighborhoods of a subset {x} consisting of a single point are also called **r- neighborhoods** of the point x .

Let us denote the collection of all r- neighborhoods of the subset B of X by **Nr(B) .** In particular , we denote the collection of all r- neighborhoods of x by **Nr(x)** .

Definition 1.15 , [1] : Let $f : X \to Y$ be a mapping of spaces .Then :
(i) f is called continuous mapping if $f^{-1}(A)$ is an open set in X for every open set A in Y .
(ii) f is called open mapping if $f(A)$ is an open set in Y for every open set A in X .
(iii) f is called closed mapping if $f(A)$ is a closed set in Y for every closed set A in X .

Definition 1.16 : A mapping $f : X \to Y$ is called r- irresolute if $f^{-1}(A)$ is an r- open set in X for every r- open set A in Y .

Definition 1.17 , [1] : Let X and Y be spaces . Then the mapping $f : X \to Y$ is called **homeomorphism** if
(i) f is bijective .
(ii) f is continuous .
(iii) f is open (or closed) .

Also , X is called **homeomorphic** to the space Y (written $X \cong Y$) .

Definition 1.18
(i) A mapping $f : X \to Y$ is called an **r- open mapping** if the image of each open subset of X is an r- open set in Y .
(ii) A mapping $f : X \to Y$ is called an **r- closed mapping** if the image of each closed subset of X is an r- closed set in Y .

Remark 1.19 : Every r- open (r- closed) mapping is open (closed) mapping .

The converse of Remark (1.19) , is not true in general as the following examples show :

Example 1.20 : Let X = {a, b, c}, Y = {x, y, z} and let T = {θ , X, {a}, {a, b}} , τ = {θ ,Y, {x}} be topologies on X and Y respectively . Let $f : X \to Y$ be a mapping which is defined by : $f(a) = f(b) = x$, $f(c) = y$. Notice that f is an open mapping , but f is not r- open .

Example 1.21 : Let X = {a, b, c, d} , Y = {x, y, z} and let T = {θ, X, {a}, {b, c}, {a, b, c}} , τ = {θ, Y, {x}, {x, z}} are topologies on X and Y respectively . Let f : X → Y be a mapping which is defined by : f(a) = f(c) = z , f(b) = x , f(d) = y . Notice that f is closed mapping , but f is not r- closed mapping .

Proposition 1.22 : A mapping $f : X \to Y$ is r- closed if and only if $\overline{f(A)}^r \subseteq f(\overline{A})$, $\forall A \subseteq X$.

Proof : →) Let $f : X \to Y$ be an r- closed mapping and $A \subseteq X$. Since $\overline{A}$ is a closed set in X , then $f(\overline{A})$ is an r- closed subset of Y , and since $A \subseteq \overline{A}$ then $f(A) \subseteq f(\overline{A})$. Thus $\overline{f(A)}^r \subseteq \overline{f(\overline{A})}^r = f(\overline{A})$, hence $\overline{f(A)}^r \subseteq f(\overline{A})$.
←) Let $\overline{f(A)}^r \subseteq f(\overline{A})$, for all $A \subseteq X$. Let F be a closed subset of X , i.e , $F = \overline{F}$, thus by hypothesis $\overline{f(F)}^r \subseteq f(F)$. But $f(F) \subseteq \overline{f(F)}^r$, then $f(F) = \overline{f(F)}^r$. Hence f(F) is an r- closed set in Y , thus f : X → Y is an r- closed mapping .

Proposition 1.23 : Let X and Y be spaces , f : X → Y be an r- closed mapping of X into Y . Then $f_{\{y\}} : f^{-1}(\{y\}) \to \{y\}$ is r- closed mapping , for each $y \in Y$.

Proof : Let F be a closed subset of $f^{-1}(\{y\})$. Then there is a closed subset F_1 of X , such that $F = F_1 \cap f^{-1}(\{y\})$. Since $f_{\{y\}}(F) = f(F_1) \cap \{y\}$, then either $f_{\{y\}}(F) = \theta$ or $f_{\{y\}}(F) = \{y\}$, thus $f_{\{y\}}(F)$ is r- closed in {y} . Therefore $f_{\{y\}}$ is an r- closed mapping .

Proposition 1.24 : Let X and Y be spaces , f : X → Y be an r- closed mapping of X into Y . Then for each clopen subset T of Y , $f_T : f^{-1}(T) \to T$ is an r- closed mapping .

Proof : Let F be a closed subset of $f^{-1}(T)$. Then there is a closed subset F_1 of X , such that $F = F_1 \cap f^{-1}(T)$. Since $f_T(F) = f(F_1) \cap T$, and $f(F_1)$ is r- closed in Y and T is clopen in Y then by Corollary (1.4) , $f(F) \cap T$ is r- closed in T . Thus f_T is an r- closed mapping .

Corollary 1.25 : Let f : X → Y be an r- closed mapping of a space X into a discrete space Y. Then for any subset T of Y , $f_T : f^{-1}(T) \to T$ is an r- closed mapping .

Proposition 1.26 : Let X , Y and Z be spaces , $f : X \rightarrow Y$ be a closed mapping and $g : Y \rightarrow Z$ be an r- closed mapping , then $gof : X \rightarrow Z$ is an r- closed mapping .
Proof : Let F be a closed subset of X , then $f(F)$ is closed set in Y . But g is an r- closed mapping , then $g(f(F)) = (gof)(F)$ is an r- closed set in Z . Then $gof : X \rightarrow Y$ is an r- closed mapping .
Corollary 1.27 : Let X , Y and Z be spaces . If $f : X \rightarrow Y$, and $g : Y \rightarrow Z$ are r- closed mapping , then $gof : X \rightarrow Z$ is an r- closed mapping .
Proof : Since f is an r- closed mapping , then f is a closed mapping , thus by Proposition (1.26) , gof is an r- closed mapping .
Proposition 1.28 : Let $f : X \rightarrow Y$ be an r- closed mapping . If F is a closed subset of X , then the restriction mapping $f_{|F} : F \rightarrow Y$ is an r- closed mapping .
Proof : Since F is a closed set in X , then the inclusion mapping $i_F : F \rightarrow X$ is a closed . Since f is an r- closed , then by Proposition (1.26) , $foi_F : F \rightarrow Y$ is an r-closed mapping . But $foi_F \equiv f_{|F}$, thus the restriction mapping $f_{|F} : F \rightarrow Y$ is an closed mapping .

Proposition 1.29 : A bijective mapping $f : X \rightarrow Y$ is r- closed if and only if is r- open .

Proof : $\rightarrow$) Let $f : X \rightarrow Y$ be a bijective , r- closed mapping and U be an open subset of X , thus U^c is closed .Since f is r- closed then $f(U^c)$ is r- closed in Y, thus $(f(U^c))^c$ is r- open.

Since f is bijective mapping , then $(f(U^c))^c = f(U)$, hence $f(U)$ is r- open in Y . Therefore f is an r- open mapping .
$\leftarrow$) Let $f : X \rightarrow Y$ be a bijective , r- open mapping and F be a closed subset of X , thus F^c is open . Since f is r- open then $f(F^c)$ is r- open in Y , thus $(f(F^c))^c$ is r- closed . Since f is a bijective mapping , then $(f(F^c))^c = f(F)$, hence $f(F)$ is an r- closed in Y . Therefore f is an r- closed mapping .

Definition 1.30 : Let X and Y be spaces .Then the mapping $f : X \rightarrow Y$ is called **r- homeomorphism** if :
(i) f is bijective .
(ii) f is continuous .
(iii) f is r- open (r- closed) .

Remark 1.31 : Every r- homeomorphism mapping is homeomorphism .

The converse of Remark (1.31) , is not true in general as the following example shows :

Example 1.32 : Let X = {a, b, c} be a set and T = {θ, X, {a}, {c}, {a, b}, {a, c}} be a topology on X . Let f : X $\rightarrow$ X be the identity mapping . Notice that f is homeomorphism , but its not r- homeomorphism .

Theorem 1.33 , [9] : Let X be a space and A be a subset of X , x $\in$ X .Then x $\in \overline{A}$ if and only if there is a net in A which converges to x .

Lemma 1.34 , [5] : If (χ_d) is a net in a space X and for each $d_o \in$ D , A_{do} = {χ_d | d $\geq$ d_o} , then x $\in$ X is a cluster point of (χ_d) if and only if x $\in \overline{A_d}$, for all d $\in$ D .

Definition 1.35 : Let $(\chi_d)_{d\in D}$ be a net in a space X , x $\in$ X . Then $(\chi_d)_{d\in D}$ **r-converges** to x [written $\chi_d \xrightarrow{r}$ x], if $(\chi_d)_{d\in D}$ is eventually in every r- nbd of x . The point x is called **an r- limit point** of $(\chi_d)_{d\in D}$.

Definition 1. 36 : Let $(\chi_d)_{d\in D}$ be a net in a space X , x $\in$ X .Then $(\chi_d)_{d\in D}$ is said to have x as **an r- cluster point** [written $\chi_d \overset{r}{\propto}$ x] if $(\chi_d)_{d\in D}$ is frequently in every r- nbd of x .

Proposition 1.37 : Let (X , T) be a space and A $\subseteq$ X , x $\in$ X .Then x $\in \overline{A}^r$ if and only if
there exists a net $(\chi_d)_{d\in D}$ in A and $\chi_d \overset{r}{\propto}$ x .

Proof : $\rightarrow$) Let x $\in \overline{A}^r$, then U$\cap$A $\neq$ θ , for every r- open set U , x $\in$ U . Notice that (Nr(x) , $\subseteq$) is a directed set , such that for all U_1 , U_2 $\in$ Nr(x) , $U_1 \geq U_2$ if and only if $U_1 \subseteq U_2$. Since for all U $\in$ Nr(x) , U$\cap$A $\neq$ θ , then we can define a net χ : Nr(x) $\rightarrow$ X as follows : χ(U) = $\chi_U \in$ U$\cap$A , U $\in$ Nr(x) . To prove that $\chi_U \overset{r}{\propto}$ x . Let B $\in$ Nr(x) , thus B$\cap$U $\in$ Nr(x) . Since B$\cap$U $\subseteq$ U , then B$\cap$U $\geq$ U , χ(B$\cap$U) = $\chi_{B\cap U} \in$ B$\cap$U $\subseteq$ B . Hence $\chi_U \overset{r}{\propto}$ x .

$\leftarrow$) Let $(\chi_d)_{d\in D}$ be a net in A , such that $\chi_d \overset{r}{\propto} x$, and let U be an r- open set , $x \in U$. Since $\chi_d \overset{r}{\propto} x$, then $(\chi_d)_{d\in D}$ is frequently in U . Thus $U \cap A \neq \theta$, for all r- open set U , $x \in U$. Hence $x \in \overline{A}^r$.

Proposition 1.38 : Let X be a space and $(\chi_d)_{d\in D}$ be a net in X , for each $d_o \in D$, such that $A_{do} = \{\chi_d \mid d \geq d_o\}$, then a point x of X is r- cluster point of $(\chi_d)_{d\in D}$ if and only if $x \in \overline{A_{do}}^r$, for all $d_o \in D$.

Proof : $\rightarrow$) Let x be an r- cluster point of $(\chi_d)_{d\in D}$ and let N be an r- open set contain x , then $(\chi_d)_{d\in D}$ is frequently in N , thus $A_{do} \cap N \neq \theta$, $\forall\, d_o \in D$, then by Proposition (1.10) , $x \in \overline{A_{do}}^r$.

$\leftarrow$) Let $x \in \overline{A_{do}}^r$, $\forall\, d_o \in D$, and suppose that x is not r- cluster point of $(\chi_d)_{d\in D}$, then there exists r- nbd N of x , such that $A_{do} \cap N = \theta$, $\forall\, d_o \in D$, $\chi_d \notin D$, $d \geq d_o$ $d \geq d_o$, then $x \notin \overline{A_{do}}^r$. This is contradiction . Hence x is r- cluster point of $(\chi_d)_d$.

2- Regular compact and regular coercive mappings

Definition 2.1 , [6] : A space X is called **Hausdorff (T_2)** if for any two distinct points x , y of X there exists disjoint open subsets U and V of X such that $x \in U$, $y \in V$.

Theorem 2.2 , [6] : Each singletion subset of a Hausdorff space is closed .

Definition 2.3 , [7] : A space X is called **compact** if every open cover of X has a finite subcover .

Theorem 2.4 , [6] : A space X is compact if and only if every net in X has a cluster point in X .

Theorem 2.5 , [7] :
(i) A closed subset of compact space is compact .
(ii) In any space , the intersection of a compact set with a closed set is compact .
(iii) Every compact subset of T_2- space is closed .

Definition 2.6 : A space X is called **r- compact** if every r- open cover of X has a finite subcover .

Proposition 2.7 : Every compact space is r- compact space .

The converse of Proposition (2.7) , is not true in general as the following example shows :

Example 2.8 : Let $T = \{A \subseteq R \mid Z \subseteq A\} \cup \{\theta\}$, be a topology on R . Notice that the topological space (R,T) is r- compact , but its not compact .

Theorem 2.9 :
(i) An r- closed subset of compact space is r- compact .
(ii) Every r- compact subset of T_2- space is r- closed .
(iii) In any space , the intersection of an r- compact set with an r- closed set is r- compact .
(iv) In a T_2- space , the intersection of two r- compact sets is r- compact .

Theorem 2.10 : A space X is an r- compact if and only if every net in X has r- cluster point in X .

Proposition 2.11 : Let X be a space and Y be an r- open subspace of X , $K \subseteq Y$. Then K is an r- compact set in Y if and only if K is an r- compact set in X .

Proof : $\rightarrow$) Let K be an r- compact set in Y . To prove that K is an r- compact set in X . Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an r- open cover in X of K , let $V_\lambda = U_\lambda \cap Y$, $\forall \lambda \in \Lambda$. Then V_λ is r- open in X , $\forall \lambda \in \Lambda$. But $V_\lambda \subseteq Y$, thus V_λ is r- open in Y , $\forall \lambda \in \Lambda$. Since $K \subseteq \bigcup_{\lambda \in \Lambda} V_\lambda$, then $\{V_\lambda\}_{\lambda \in \Lambda}$ is an r- open cover in Y of K , and by hypothesis this cover has finite subcover $\{V_{\lambda_1}, V_{\lambda_2}, \ldots, V_{\lambda_n}\}$ of K , thus the cover $\{U_\lambda\}_{\lambda \in \Lambda}$ has a finite subcover of K . Hence K is an r- compact set in X .

$\leftarrow$) Let K be an r- compact set in X . To prove that K is an r- compact set in Y. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an r- open cover in Y of K . Since Y is an r- open subspace of X , then by Proposition (1.5) , $\{U_\lambda\}_{\lambda \in \Lambda}$ is an r- open cover in X of K . Then by hypothesis there exists $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, such that $K \subseteq \bigcup_{\lambda=1}^{m} U_\lambda$, thus the cover $\{U_\lambda\}_{\lambda \in \Lambda}$ has a finite subcover of K . Hence K is an r- compact set in Y .

Definition 2.12 : Let X be a space and $W \subseteq X$. We say that W is **compactly r- closed set** if $W \cap K$ is r- compact , for every r- compact set K in X .

Proposition 2.13 : Every r- closed subset of a space X is compactly r- closed .

The converse of Proposition (2.13), is not true in general as the following example shows .

Example 2.14 : Let X = {a, b, c} be a space and T = {X, θ, {a, b}} be a topology on X . Notice that the set A = {a, b} is compactly r- closed , but its not r- closed set .

Theorem 2.15 : Let X be a T_2 - space .A subset A of X is compactly r- closed if and only if A is r- closed .

Remark 2.16: Let X be a compact , T_2 - space and A $\subseteq$ X . Then :
(i) A is closed if and only if A is r- closed .
(ii) A is compact if and only if A is r- compact .

Definition 2.17 , [6] : Let X and Y be space . A mapping f : X $\rightarrow$ Y is called **compact mapping** if the inverse image of each compact set in Y , is a compact set in X .

Definition 2.18 : Let X and Y be space . We say that the mapping f : X $\rightarrow$ Y is **an r- compact mapping** if the inverse image of each r- compact set in Y , is a compact set in X .

Example 2.19 : Let (X,T) and (Y,τ) be topological spaces , such that X is finite set , then the mapping f : X $\rightarrow$ Y is r- compact .

Remark 2.20 : Every r- compact mapping is compact mapping .

The converse of Remark (2.20) , is not true in general as the following example shows :

Example 2.21 : Let T = {A $\subseteq$ R | Z $\subseteq$ A}$\cup${θ} be a topology on R , and f : (R,T) $\rightarrow$ (R,T) be a mapping which is defined as f(x) = x , $\forall$ x $\in$ R . Notice that f is a compact mapping , but its not r- compact .

Proposition 2.22 : Let X and Y be spaces , and f : X $\rightarrow$ Y be an r- compact , continuous , mapping . If T is a clopen subset of Y , then $f_T : f^{-1}(T) \rightarrow T$ is an r- compact mapping .

Proof : Let K be an r- compact subset of T . Since T is clopen set in Y then by Corollary (1.4) , T is an r- open , and then by Proposition (2.11) , K is an r-compact set in Y . Since f is an r- compact mapping , then f^{-1}(K) is compact in X .

Now , since T is a closed set in Y , and f is a continuous mapping , then f^{-1}(T) is a closed set in X , thus by Theorem (2.5), $f^{-1}(T)\cap f^{-1}(K)$ is a compact set .But $f_T^{-1}(K) = f^{-1}(T)\cap f^{-1}(K)$, then $f_T^{-1}(K)$ is a compact set in f^{-1}(T) . Therefore f_T is an r- compact mapping .

Proposition 2.23 : Let X , Y and Z be spaces . If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ are continuous mapping . Then :
(i) If f is a compact mapping and g is an r- compact mapping , then $gof : X \rightarrow Z$ is an r-
compact mapping .
(ii) If f and g are r- compact mappings, then gof is an r- compact mapping .

Proof :
(i) Let K be an r- compact set in Z , then g^{-1}(K) is a compact set in Y , and then $f^{-1}(g^{-1}(K)) = (gof)^{-1}(K)$ is a compact set in X . Hence $gof : X \rightarrow Z$ is r- compact mapping .
(ii) By Remark (2.18) , and (i) .

Proposition 2.24 , [2] : For any closed subset of a space X , the inclusion mapping $i_F : F \rightarrow X$ is a compact mapping .
Proposition 2.25 : Let X and Y be spaces . If $f : X \rightarrow Y$ is an r- compact mapping and F is a closed subset of X , then $f_{|F} : F \rightarrow X$ is an r- compact mapping .

Proof : Since F is a closed subset of X , then by Proposition (2.24) , the inclusion $i_F : F \rightarrow X$ is a compact mapping . But $f_{|F} \equiv f o i_F$, then by Proposition (2.23) , $f_{|F}$ is an r- compact mapping .

Definition 2.26 , [4] : Let X and Y be spaces . A mapping $f : X \rightarrow Y$ is called **coercive** if for every compact set $J \subseteq Y$, there exists a compact set $K \subseteq X$ such that $f(X \setminus K) \subseteq Y \setminus J$.

Definition 2.27 : Let X and Y be spaces . We say that the mapping $f : X \rightarrow Y$ is **r- coercive** if for every r- compact set $J \subseteq Y$, there exists a compact set $K \subseteq X$ such that $f(X \setminus K) \subseteq Y \setminus J$.

Examples 2.28 :
(i) If f : (X,T) $\rightarrow$ (Y, τ) is a mapping , such that X is compact space , then f is r- coercive .
(ii) Every identity mapping on regular space is r- coercive .
Proposition 2.29 : Every r- coercive mapping is a coercive mapping .
Proof : Let $f : X \rightarrow Y$ be an r- coercive mapping , and J be a compact set in Y, so its r- compact , since f is r- coercive , then there exists a compact set K in X , such that $f(X \setminus K) \subseteq Y \setminus J$. Hence f is a coercive mapping .
The converse of Proposition (2.29) is not true in general as the Example (2.19) .
Proposition 2.30 : Let X and Y be spaces such that Y is a compact , T_2 - space . Then a mapping $f : X \rightarrow Y$ is r- coercive if and only if its a coercive mapping .
Proof : $\rightarrow$) By Proposition (2.29) .
$\leftarrow$) Let J is an r- compact set in Y . Since Y is a compact , T_2 - space , then by Proposition (2.16) , J is a compact set in Y , since f is a coercive mapping , then there exists a compact set K in X , such that $f(X \setminus K) \subseteq Y \setminus J$. Hence f is r- coercive .
Proposition 2.31 : Every r- compact mapping is an r- coercive .

Proof : Let $f : X \rightarrow Y$ be an r- compact mapping . To prove that f is an r- coercive . Let J be an r- compact set in Y . Since f is an r- compact mapping , then $f^{-1}(J)$ is a compact set in X . Thus $f(X \setminus f^{-1}(J)) \subseteq Y \setminus J$. Hence $f : X \rightarrow Y$ is an r- coercive mapping .
The converse of Proposition (2.31) , is not true in general as the following example shows .
Example 2.32 : Let Y = {x, y} be a set and T is the discrete topology on Y . Then a mapping f : ([o,1],U) $\rightarrow$ (Y,T) which is defined by :

$$f(t) = \begin{cases} x & \forall\, t \in (0,1) \\ y & \forall\, t \in \{0,1\} \end{cases}$$

is a coercive mapping , but its not compact mapping .
Proposition 2.33 : Let X and Y be spaces , such that Y is a T_2 – space , and $f : X \rightarrow Y$ is a continuous mapping . Then f is an r- coercive if and only if f is an r- compact .
Proof : $\rightarrow$) Let J be an r- compact set in Y . To prove that $f^{-1}(J)$ is a compact set in X . Since Y is a T_2 – space , and J is an r- compact set in Y , so it's a closed set , then $f^{-1}(J)$ is a closed set in X . Since f is an r- coercive mapping , then there exists a compact set K in X , such that $f(X \setminus K) \subseteq Y \setminus J$. Then $f(K^c) \subseteq J^c$, therefore $f^{-1}(J) \subseteq K$, and thus $f^{-1}(J)$ is a compact set in X . Hence f is an r- compact mapping .

$\leftarrow$) By Proposition (2.31) .

Proposition 2.34 : Let X , Y and Z be spaces and f : X$\to$Y, g : Y$\to$Z be mappings . Then :

(i) If f is coercive and g is r- coercive , then g$\circ f$: X $\to$ Z is an r- coercive mapping .

(ii) If f and g are r- coercive , then g$\circ f$: X $\to$ Z is an r- coercive mapping .

Proof :

(i) Let J be an r- compact set in Z . Since g : Y $\to$ Z is r-coercive mapping , then there exists a compact set K in Y , such that g(Y \ K) $\subseteq$ Z \ J . Since f : X $\to$ Y is a coercive mapping , then there exists a compact set H in X , such that f(X \ H) $\subseteq$Y \ K$\to$ g(f(X \ H) $\subseteq$ g(Y \ K) $\subseteq$ Z \ J $\to$(g$\circ f$)(X \ H)$\subseteq$Z \ J .Hence g$\circ f$ is an r- coercive mapping .

(ii) By Proposition (2.29) , and (i) .

Proposition 2.35 : Let X and Y be spaces , and f : X $\to$ Y be an r- coercive mapping . If F is a closed subset of X , then the restriction mapping $f_{|F}$: F $\to$ Y is an r- coercive mapping .

Proof: Since F is a closed subset of X , then by Proposition (2.24) , and Proposition (2.31) , the inclusion mapping i_F : F $\to$ X is a coercive mapping . But $f_{|F} \equiv f \circ i_F$, then by Proposition (2.34) , $f_{|F}$ is an r- coercive mapping .

Theorem 2.36 : Let X and Y be spaces , such that Y is a compact , T_2 - space , then for a continuous mapping f : X $\to$ Y , the following statements are equivalent **:**

(i) f is r- coercive .

(ii) f is r- compact .

(iii) f is compact .

(iv) f is coercive .

Proof :

(i $\to$ ii). By Proposition (2.33) .

(ii $\to$ iii). By Remark (2.20) .

(iii $\to$ iv). Let J be a compact set in Y . Since f is compact mapping , then f^{-1}(J) is compact set in X . Thus f(X \ f^{-1}(J)) $\subseteq$Y \ J . Hence f is a coercive mapping .

(iv $\to$ i). By Proposition (2.30) .

3- Regular Proper Mapping :

Definition 3.1 , [1] : Let X and Y be spaces , and f : X $\to$ Y be a mapping . We say that f is **a proper mapping** if :

(i) f is continuous .

(ii) $f \times I_Z$: X $\times$ Z $\to$ Y $\times$ Z is closed , for every space Z .

Definition 3.2 : Let X and Y be spaces , and f : X $\to$ Y be a mapping . We say that f is **a regular proper (r- proper) mapping** if :

(i) f is continuous .

(ii) $f \times I_Z : X \times Z \rightarrow Y \times Z$ is r- closed , for every space Z .

Example 3.3 : Let X = {a, b, c} , Y = {x, y} be spaces and T = {X, θ , {a}, {b}, {c}, {a, b}, {a, c}, {b, c}} , τ = {Y, θ , {x}, {y}} are topologies on X and Y respectively . The mapping $f : X \rightarrow Y$ which is defined as $f(a) = f(b) = x$, $f(c) = y$ is an r- proper mapping .

The following example shows that not every mapping is r- proper .

Example 3.4 : Let $f : (R , U) \rightarrow (R , U)$ be the mapping which is defined by $f(x) = 0$, for every x ∈ R . Notice that f is not r- proper mapping , since for the usual space (R , U) the mapping $f \times I_R : R\times R \rightarrow R\times R$, such that $(f\times I_R)(x,y) = (0,y)$, for every (x,y) ∈ R is not r- closed mapping .

Remarks 3.5 :

(i) Every r- proper mapping is r- closed .

(ii) Every r- proper mapping is proper .

(iii) Every r- homeomorphism is r- proper .

The converse of Remark (3.5.i) , is not true in general as the Example (3.4) . Also the converse of Remark (3.5.ii) , is not true as the following example shows :

Example 3.6 :

Let T be a cofinite topology on N , and let $f : N \rightarrow N$ be a mapping which is defined by : $f(x) = x$, ∀ x ∈ N . Notice that f is a proper mapping , but f is not r- proper mapping , since f is not r- closed mapping .

The converse of Remark (3.5.iii) , is not true in general as the following example shows :

Example 3.7 : Let X = {a, b} , Y = {x, y} be sets and T = {θ ,X,{a},{b}} ,τ = {θ ,Y,{x}, {y}} be topologies on X and Y respectively . Let $f: X \rightarrow Y$ be a mapping which is defined by : $f(a) = f(b) = x$. Notice that f is an r- proper mapping , but f is not r- homeomorphism , since f is not onto .

Proposition 3.8 : Let X and Y be spaces , and $f : X \rightarrow Y$ be an r- proper mapping . If T is a clopen subset of Y , then $f_T : f^{-1}(T) \rightarrow T$ is an r- proper mapping .

Proof : Since $f : X \rightarrow Y$ is a continuous mapping , then f_T is a continuous mapping . To prove that $f_T\times I_Z : f^{-1}(T) \times Z \rightarrow T\times Z$ is an r- closed mapping , for every space Z . Notice that $f_T\times I_Z \equiv (f\times I_Z)_{T\times Z}$. Since T is a clopen subset of Y, then by Proposition (1.11) , $T\times Z$ is a clopen subset of $Y\times Z$, thus by Proposition (1.24) , $(f\times I_Z)_{T\times Z} \equiv (f_T\times I_Z)$ is an r- closed mapping , hence $f_T : f^{-1}(T) \rightarrow T$ is an r- proper mapping .

Theorem 3.9 : Let $f : X \to P = \{w\}$ be a mapping on a space X . If f is an r-proper mapping , then X is a compact space , where w is any point which does not belong to X .
Proof : Since f is r- proper mapping , then by Remark (3.5.ii) , f is proper mapping . Thus by [1.Lemma (2.1) P.101] , X is compact space .
Theorem 3.10 : Let X and Y be spaces , and $f : X \to Y$ be a continuous mapping . Then the following statements are equivalent :
(i) f is an r- proper mapping .
(ii) f is an r- closed mapping and $f^{-1}(\{y\})$ is compact for each $y \in Y$.
(iii) If $(\chi_d)_{d \in D}$ is a net in X and $y \in Y$ is an r- cluster point of $f(\chi_d)$, then there is a cluster point $x \in X$ of $(\chi_d)_{d \in D}$, such that $f(x) = y$.

Proof :
(i→ii). Let $f : X \to Y$ be an r- proper mapping , then $f \times I_Z : X \times Z \to Y \times Z$ is an r- closed for every space Z . Let Z = {t}, then $X \times Z = X \times \{t\} \cong X$ and $Y \times Z = Y \times \{t\} \cong Y$, and we can replace $f \times I_Z$ by f , thus f is r- closed . Now , let $y \in Y$. Since f is an r- proper , then by Remarks (3.5) , f is proper mapping , so by [1, Theorem (3.1.5)] , $f^{-1}(\{y\})$ is compact for each $y \in Y$.
(ii → iii). Let $(\chi_d)_{d \in D}$ be a net in X and $y \in Y$ be an r- cluster point of a net $f(\chi_d)$ in Y . Assume that $f^{-1}(y) \neq \theta$, if $f^{-1}(y) = \theta$, then $y \notin f(X) \to y \in (f(X))^c$, since X is a closed set in X and f is an r- closed mapping , then $f(X)$ is an r- closed set in Y . Thus $(f(X))^c$ is an r- open set in Y. Therefore $(f(\chi_d))$ is frequently in $(f(X))^c$.
But $f(\chi_d) \in f(X)$, $d \in D$, then $f(X) \cap (f(X))^c \neq \theta$, and this is a contradiction . Thus $f^{-1}(y) \neq \theta$.
Now , suppose that the statement (iii) , is not true , that means , for all $x \in f^{-1}(y)$ there exists an open set U_X in X contains x , such that (χ_d) is not frequently in U_X . Notice that $f^{-1}(y) = \bigcup_{x \in f^{-1}(y)} \{x\}$. Therefore the family $\{U_X \mid x \in f^{-1}(y)\}$ is an open cover for $f^{-1}(y)$. But $f^{-1}(y)$ is a compact set , then there exists $x_1, x_2, \ldots, x_n \in f^{-1}(y)$, such that $f^{-1}(y) \subseteq Ux_1 \bigcup Ux_2 \ldots \bigcup Ux_n$, then $f^{-1}(y) \cap [\bigcup_{i=1}^{n} U_{xi}]^c = \theta \to f^{-1}(y) \cap [\bigcap_{i=1}^{n} U_{xi}^c] = \theta$. But $(x_i)_{i \in \Lambda}$ is not frequently in Ux_i , $\forall\, i = 1, \ldots, n$. Thus (χ_d) is not frequently in $\bigcup_{i=1}^{n} U_{xi}$, but $\bigcup_{i=1}^{n} U_{xi}$ is an open set in X , then $\bigcap_{i=1}^{n} U_{xi}^c$ is a closed set in X . Thus $f(\bigcap_{i=1}^{n} U_{xi}^c)$ is an r- closed set in Y .

Claim $y \notin f(\bigcap_{i=1}^{n} U_{xi}^{c})$, if $y \in f(\bigcap_{i=1}^{n} U_{xi}^{c})$, then there exists $x \in \bigcap_{i=1}^{n} U_{xi}^{c}$, such that $f(x) = y$, thus $x \notin \bigcup_{i=1}^{n} U_{xi}$, but $x \in f^{-1}(y)$, therefore $f^{-1}(y)$ is not a subset of $\bigcup_{i=1}^{n} U_{xi}$, and this is a contradiction . Hence there is an r- open set A in Y , such that $y \in A$ and $A \cap f(\bigcap_{i=1}^{n} U_{xi}^{c}) = \theta \rightarrow f^{-1}(A) \cap f^{-1}(f(\bigcap_{i=1}^{n} U_{xi}^{c})) = \theta \rightarrow f^{-1}(A) \cap [\bigcap_{i=1}^{n} U_{xi}^{c}] = \theta \rightarrow f^{-1}(A) \subseteq \bigcup_{i=1}^{n} U_{xi}$. But $(f(\chi_d))$ is frequently in A , then (χ_d) is frequently in $f^{-1}(A)$, and then (χ_d) is frequently in $\bigcup_{i=1}^{n} U_{xi}$. This is contradiction , and this is complete the proof .

(iii → i). Let Z be any space . To prove that $f : X \rightarrow Y$ is an r- proper mapping , i.e , to prove that $f \times I_Z : X \times Z \rightarrow Y \times Z$ is an r- closed mapping . Let F be a closed set in $X \times Z$. To prove that $(f \times I_Z)(F)$ is an r- closed set in $Y \times Z$. Let $(y,z) \in \overline{(f \times I_Z)(F)}^{r}$, then by Proposition (1.38) , there exists a net $\{(y_d , z_d)\}_{d \in D}$ in $(f \times I_Z)(F)$ such that $(y_d , z_d) \overset{r}{\propto} (y,z)$, then $(y_d , z_d) = ((f \times I_Z)(x_d , y_d))$, where $\{(x_d , y_d)\}_{d \in D}$ is a net in F . Thus $(f(x_d) , I_Z(z_d)) \overset{r}{\propto} (y,z)$, so $f(x_d) \overset{r}{\propto} y$ and $z_d \overset{r}{\propto} z$. Then by (iii) ,. $x \in X$, such that $x_d \propto x$ and $f(x) = y$, Since $(x_d , z_d) \propto (x,z)$ and $\{(x_d , z_d)\}_{d \in D}$ is a net in F , thus $(x,y) \in \overline{F}$.

Since $F = \overline{F}$, then $(x,y) \in F \rightarrow (y,z) = ((f \times I_Z)(x,y)) \rightarrow (y,z) \in (f \times I_Z)(F)$, and then $\overline{(f \times I_Z)(F)}^{r} = (f \times I_Z)(F)$, thus $(f \times I_Z)(F)$ is an r- closed set in $Y \times Z$. Hence $f \times I_Z : X \times Z \rightarrow Y \times Z$ is an r- closed mapping , hence $f : X \rightarrow Y$ is an r- proper mapping .

Corollary 3.11 : If X is a compact space , then the mapping $f : X \rightarrow P = \{w\}$ on a space X is r- proper , where w is any point which does not belongs to X .

Proof : Let X be a compact space . Since P is a single point , then f is a continuous mapping . To prove that $f : X \rightarrow P = \{w\}$ is an r- proper mapping :

(i) Since $f^{-1}(P) = X$, then $f^{-1}(P)$ is a compact set .

(ii) Let F is a closed subset of X , then either : $f(F) = \theta$ or $f(F) = \{w\}$. So $f(F)$ is r- closed in P , then f is r- closed mapping . Thus by Theorem (3.10) , f is an r- proper mapping .

Proposition 3.12 : Let X and Y be spaces . If $f : X \rightarrow Y$ is an r- proper mapping , then $f_{\{y\}} : f^{-1}(\{y\}) \rightarrow \{y\}$ is an r- proper mapping , for all $y \in Y$.

Proof : Since $f : X \to Y$ is an r- proper mapping , then $f^{-1}(\{y\})$ is compact for each $y \in Y$. Since $\{y\}$ is a single point , then by Corollary (3.11) , $f_{\{y\}} : f^{-1}(\{y\}) \to \{y\}$ is an r- proper mapping .

Proposition 3.13 : Let X and Y be spaces , such that X is a compact , T_2- space and $f : X \to Y$ be a homeomorphism mapping , then $f^{-1} : Y \to X$ is an r- proper mapping .

Proof : Since f is an open mapping , then f^{-1} is continuous mapping . To prove that f^{-1} is r- proper :

(i) Let F be a closed subset of Y , since f is continuous , then $f^{-1}(F)$ is closed in X , since X is compact , T_2- space , then by Remark (2.16) , $f^{-1}(F)$ is r- closed in X . Hence f^{-1} is an r- closed mapping .

(ii) Let $x \in X$, then $\{x\}$ is compact set in . Since f is continuous , then $f(\{x\}) = (f^{-1})^{-1}(\{x\})$ is compact set in Y , therefore by Theorem (3.10) , f^{-1} is r- proper mapping .

Proposition 3.14 : Let X and Y be spaces , and $f : X \to Y$ be a continuous , one to one , mapping , then the following statements are equivalent :

(i) f is r- proper mapping .

(ii) f is r- closed mapping .

(iii) f is r- homeomorphism of X onto an r- closed subset of Y .

Proof :

(i → ii). By Remark (3.5) .

(ii → iii). Let $f : X \to Y$ be an r- closed mapping . Since X is a closed set in X , then $f(X)$ is an r- closed set in Y . Since f is continuous and one to one , then f is an r- homeomorphism of X onto r- closed subset $f(X)$ of Y .

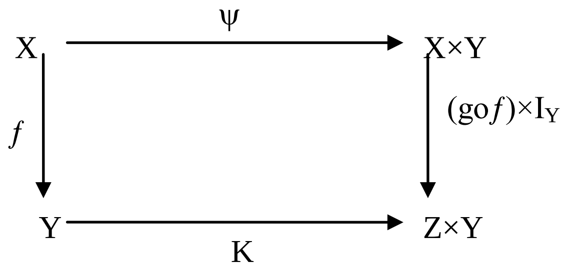
(iii → i). Let f be an r- homeomorphism of X onto an r- closed subset U of Y . Now , let Z be any space , and W be a basic open set in $X \times Z$, then $W = W_1 \times W_2$, where W_1 is an open set in X and W_2 is an open set in Z . Since $(f \times I_Z)(W_1 \times W_2) = f(W_1) \times W_2$, and $f : X \to U$ is an r- homeomorphism , then $f : X \to U$ is an r- open mapping and then $f(W_1)$ is an r- open set in U, thus $f(W_1) \times W_2$ is r- open in $U \times Z$, so $f \times I_Z$ is an r- open mapping . Since $f \times I_Z : X \times Z \to U \times Z$ is bijective , then by Proposition (1.29) , the mapping $f \times I_Z$ is r- closed . Now , let F be a closed subset of $X \times Z$, then $(f \times I_Z)(F)$ is an r- closed set in $U \times Z$, since $U \times Z$ is an r- closed set in $Y \times Z$, then by Proposition (1.5) , $(f \times I_Z)(F)$ is r- closed in $Y \times Z$. Hence $f \times I_Z : X \times Z \to Y \times Z$ is an r- closed mapping , thus $f : X \to Y$ is an r- proper mapping .

Proposition 3.15 : Let X , Y and Z be spaces . If f : X $\to$ Y is proper and g : Y $\to$ Z is an r- proper mapping , then go$f$: X $\to$ Y is an r- proper mapping .
Proof : To prove that go$f$: X $\to$ Z is an r- proper mapping :
(i) Since f :X $\to$ Y is a proper mapping , then f is closed . Similarly , since g : Y $\to$ Z is an r- proper mapping , then g is r- closed . Thus by Proposition (1.26) , go$f$: X $\to$ Z is an r- closed mapping .
(ii) Let z $\in$ Z , then $g^{-1}(\{z\})$ is a compact set in Y , and then $f^{-1}(g^{-1}(\{z\}) = (gof)^{-1}(\{z\})$ is a compact set in X . Therefore by (i) , (ii) and since go$f$ is continuous then by using Theorem (3.10) , go$f$ is an r- proper mapping .
Proposition 3.16 : Let X , Y and Z be spaces , and f : X $\to$ Y and g : Y $\to$ Z are r- proper maps , then go$f$: X $\to$ Z is an r- proper mapping .
Proof : Since f and g are r- proper maps , then $f\times I_W$ and $g\times I_W$ are r- closed , for every space W , then by Corollary (1.27) , $(g\times I_W)o(f\times I_W)$ is r- closed mapping . But $(g\times I_W)o(f\times I_W) = (gof)\times I_W$, then $(gof)\times I_W$ is r- closed , and since go$f$ is continuous . Hence go$f$ is an r- proper mapping .

Proposition 3.17 : Let X , Y and Z be spaces , and f : X $\to$ Y and g : Y $\to$ Z be continuous maps , such that go$f$: X $\to$ Z is an r- proper mapping . If f is onto , then g is an r- proper mapping .
Proof :
(i) Let F be a closed subset of Y , since f is continuous , then $f^{-1}(F)$ is closed in X . Since go$f$ is an r- proper mapping , then $gof(f^{-1}(F))$ is r- closed in Z . But f is onto , then $gof(f^{-1}(F)) = g(F)$. Hence g(F) is an r- closed set in Z . Thus g is r- closed mapping .
(ii) Let z $\in$ Z , since go$f$ is r- proper mapping , then by Theorem (3.10) , the set $(gof)^{-1}(\{z\}) = f^{-1}(g^{-1}(\{z\}))$ is compact . Now , since f is continuous , then $f(f^{-1}(g^{-1}(\{z\})))$ is compact set , but f is onto , then $f(f^{-1}(g^{-1}(\{z\}))) = g^{-1}(\{z\})$ is compact for every z $\in$ Z . So by Theorem (3.10) , the mapping go$f$ is r- proper .
Proposition 3.18 : Let X , Y and Z be spaces , and f : X $\to$ Y , g : Y $\to$ Z be continuous maps , such that go$f$: X $\to$ Z is an r- proper mapping . If g is one to one , r- irresolute mapping then f is an r- proper mapping .
Proof :
(i) Let F be a closed subset of X . Then $(gof)(F)$ is an r- closed set in Z . Since g : Y $\to$ Z is one to one , r- irresolute , mapping , then $g^{-1}(g(f(F))) = f(F)$ is r- closed in Y . Hence the mapping f : X $\to$ Y is r- closed .
(ii) Let y $\in$ Y , then g(y) $\in$ Z . Now , since go$f$:X $\to$ Z is r- proper and g is one to one , then the set $(gof)^{-1}(g(\{y\}) = f^{-1}(g^{-1}(g(\{y\}))) = f^{-1}(\{y\})$ is compact , for every y $\in$ Y . Therefore by Theorem (3.10) , the mapping f : X $\to$ Y is r- proper .

Proposition 3.19 : Let X , Y and Z be spaces , f : X $\rightarrow$ Y be a continuous mapping and g : Y $\rightarrow$ Z be an r- irresolute mapping , such that go$f$: X $\rightarrow$ Y is an r- proper mapping . If Y is a T_2 - space , then f is r- proper .

Proof : Consider the commutative diagram :



□ (x) = (x, f(x)) and K(y) = (g(y), y) . Since X is T_2 - space , then the graph of □ is closed in X×Y [1, Proposition .5.P.99] , and since □ is one to one , then by [1, Proposition .2.P.98] , □ is a proper mapping . We have (go$f$)×I_Z is r- proper , then by Proposition (3.15) , ((go$f$)×I_Z)o□ is r- proper . But ((go$f$)×I_Z)o□ = Kof , so that Kof is r- proper . Since g is an r- irresolute mapping , then K is r- irresolute . Therefore by Proposition (3.18) , f is an r- proper mapping .

Corollary 3.20 : Every continuous mapping of a compact space X into a T_2- space Y is r- proper .

Proof : Let f : X $\rightarrow$ Y be a continuous mapping .To prove that f is r- proper . Let g : Y $\rightarrow$ P be a mapping (where P is a singleton set) , since X is a compact space , then go$f$: X $\rightarrow$ P is r- proper . Since Y is a T_2- space , then by Proposition (3.19) , f is r- proper mapping

Proposition 3.21 : Let X ,Y and Z be spaces . If f : X $\rightarrow$ Y is an r- proper mapping and h : Y $\rightarrow$ Z is homeomorphism mapping , then ho$f$: X $\rightarrow$ Z is an r- proper mapping .

Proof :

(i) Let F be a closed subset of X , then f(F) is an r- closed set in Y , since h is homeomorphism , then ho$f$(F) is an r- closed set in Z . Hence the mapping ho$f$: X $\rightarrow$ Z is r- closed .

(ii) Let z $\in$ Z , then h^{-1}({z}) is a compact set in Y (since every homeomorphism mapping is proper) . So $(f^{-1}(h^{-1}))(\{z\}) = (hof)^{-1}(\{z\})$ is a compact set in X . Therefore by Theorem (3.10) , and since ho$f$ is continuous , the mapping ho$f$: X $\rightarrow$ Z is an r- proper .

Proposition 3.22 : Let f_1 : $X_1 \rightarrow Y_1$ and f_2 : $X_2 \rightarrow Y_2$ be maps . Then $f_1 \times f_2$: $X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is an r- proper mapping if and only if f_1 and f_2 are r- proper .

Proof : $\rightarrow$) To prove that f_2 is an r- proper . Since $f_1 \times f_2$ is continuous , then both f_1 and f_2 are continuous . To prove that $f_2 \times I_Z$: $X_2 \times Z \rightarrow Y_2 \times Z$ is r- closed , for every space Z . Let F be a closed subset of $X_2 \times Z$, since X_1 is a closed set in X_1 , then $X_1 \times F$ is a closed set in $X_1 \times X_2 \times Z$. Since $f_1 \times f_2$ is r- proper , then $(f_1 \times f_2 \times I_Z)(X_1 \times F)$ is an r- closed set in $Y_1 \times Y_2 \times Z$. But $(f_1 \times f_2 \times I_Z)(X_1 \times F) = f_1(X_1) \times (f_2 \times I_Z)(F)$, thus $(f_2 \times I_Z)(F)$ is an r- closed set in $Y_2 \times Z$, then $f_2 \times I_Z : X_2 \times Z \to Y_2 \times Z$ is an r- closed mapping . Therefore $f_2 : X_2 \to Y_2$ is an r- proper mapping .

Similarly , we can prove that $f_1 : X_1 \to Y_1$ is an r- proper mapping .

$\leftarrow$) To prove that $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is r- proper . Since f_1 and f_2 are continuous , then $f_1 \times f_2$ is a continuous mapping . Let Z be any space . Notice that :

$f_1 \times f_2 \times I_Z = (Iy_1 \times f_2 \times I_Z)o(f_1 \times Ix_2 \times I_Z)$, since f_1 and f_2 are r- proper maps , then $(Iy_1 \times f_2 \times I_Z)$

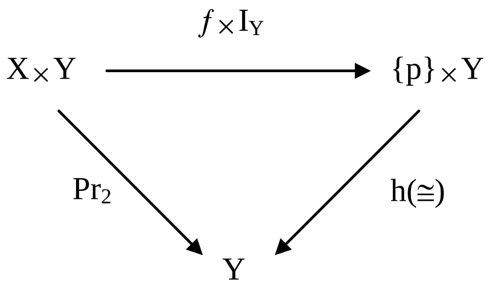
and $(f_1 \times Ix_2 \times I_Z) = f_1 \times Ix_{2 \times Z}$ are r- closed maps . Therefore by Corollary (1.27) , the mapping $f_1 \times f_2 \times I_Z$ is an r- closed . Hence $f_1 \times f_2$ is an r- proper mapping .

Proposition 3.23 : Let $f : X \to Y$ be an r- proper mapping , then $f \times I_Z : X \times Z \to Y \times Z$ is an r- proper mapping , for every space Z .

Proof : Since f is r- proper , then $f \times I_W$ is an r- closed mapping , for every space W . Notice that $f \times I_Z \times I_W = f \times I_{Z \times W}$, but $f \times I_{Z \times W}$ is an r- closed mapping , then $f \times I_Z \times I_W$ is r- closed , for every space W . Hence $f \times I_Z$ is r- proper .

Proposition 3.24 : Let X be a compact space and Y be any topological space , then the projection mapping $Pr_2 : X \times Y \to Y$ is r- proper .

Proof : Consider the commutative diagram :



Where $h : \{p\} \times Y \to Y$ is the homeomorphism of $\{p\} \times Y$ onto Y and $Pr_2 : X \times Y \to Y$ is the projection of $X \times Y$ into Y . Since X is a compact space , then by Corollary (3.11) , $f : X \to \{p\}$ is r- proper and $I_Y : Y \to Y$ is a proper mapping , then $f \times I_Y$ is an r- proper mapping . Hence $ho(f \times I_Y)$ is an r- proper mapping , but $Pr_2 = ho(f \times I_Y)$, then Pr_2 is an r- proper mapping .

Proposition 3.25 : Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be continuous maps , such that $f_1 \times f_2$ is a compact mapping and f_2 (f_1) is r- closed mapping , then f_2 (f_1) is an r- proper .

Proof : Let $y_2 \in Y_2$. Take any compact set K in Y_1 . Then $K \times \{y_2\}$ is compact in $Y_1 \times Y_2$. So that $(f_1 \times f_2)^{-1}(K \times \{y_2\})$ is compact in $X_1 \times X_2$. But $(f_1 \times f_2)^{-1}(K \times \{y_2\}) = f_1^{-1}(K) \times f_2^{-1}(\{y_2\})$, then $f_1^{-1}(K)$ and $f_2^{-1}(\{y_2\})$ are compact in X_1 and X_2 respectively . Since f_2 is an r- closed mapping , then by Theorem (3.10) , f_2 is an r- proper .

Proposition 3.26 : Let X and Y be spaces , and $f : X \rightarrow Y$ be an r- proper mapping . If F is a clopen subset of X , then the restriction map $f_{|F} : F \rightarrow Y$ is an r- proper mapping .

Proof : To prove that $f_{|F} \times I_Z : F \times Z \rightarrow Y \times Z$ is an r- closed mapping for every space Z . Since F is a clopen subset of X , then $F \times Z$ is a clopen subset of $X \times Z$. Since $f \times I_Z$ is an r- closed mapping , then by Proposition (1.24) , $(f \times I_Z)_{F \times Z}$ is an r- closed mapping . But $f_{|F} \times I_Z = (f \times I_Z)_{F \times Z}$, thus $f_{|F} \times I_Z$ is an r- closed mapping . Hence $f_{|F} : F \rightarrow Y$ is an r- proper .

Proposition 3.27 : Let X and Y be spaces . If $f : X \rightarrow Y$ is an r- proper mapping , then f is an r- compact .

Proof : Let A be an r- compact subset of Y . To prove that $f^{-1}(A)$ is a compact set in X , let $(\chi_d)_{d \in D}$ be a net in $f^{-1}(A)$, then $f(\chi_d)$ is a net in A . Since A is an r- compact set in Y , then by Proposition (2.10) , there exists y $\in$ A , such that y is an r- cluster point of $f(\chi_d)$. Since f is r- proper , then by Theorem (3.10) , there exists x $\in$ X , such that x is a cluster point of (χ_d) , such that $f(x) = y$. Then $x \in f^{-1}(A)$. Thus every net in $f^{-1}(A)$ has cluster point in itself , then by Proposition (2.4) , $f^{-1}(A)$ is a compact set in X . Therefore $f : X \rightarrow Y$ is an r- compact mapping .

The converse of Proposition (3.27), is not true in general as the following example shows :

Example 3.28 : Let X = {a, b, c, d} ,Y = {x, y, z} be sets and T = {θ , X, {a, b}, {d}, {a, b, d}}, τ = {θ , Y, {z}} be topologies on X and Y respectively . Let $f : X \rightarrow Y$ be a mapping which is defined by : $f(a) = f(b) = f(c) = y$, $f(d) = z$.

Notice that f is an r- compact mapping , but f is not r- proper mapping . Since {c, d} is a closed set in X , and $f(\{c, d\}) = \{y, z\}$ is not r- closed set in Y , then f is not r- closed mapping . Hence f is not r- proper mapping .

Theorem 3.29 : Let X and Y be spaces , such that Y is a T_2- space . If $f : X \rightarrow Y$ is a continuous mapping , then f is an r- proper mapping if and only if f is an r- compact mapping .

Proof : $\rightarrow$) By Proposition (3.27) .

$\leftarrow$) To prove that f is an r- proper mapping :

(i) Let F be a closed subset of X . To prove that $f(F)$ is an r- closed set in Y, let K be an r- compact set in Y , then $f^{-1}(K)$ is a compact set in X , then by Theorem (2.5) , $F \cap f^{-1}(K)$ is compact in X . Since f is continuous , then $f(F \cap f^{-}$

^{1}(K)) is compact set in Y , and then its r- compact . But $f(F \cap f^{-1}(K)) = f(F) \cap K$, then $f(F) \cap K$ is r- compact , thus $f(F)$ is compactly r- closed set in Y . Since Y is a T_2-space , then by Theorem (2.15) , $f(F)$ is an r- closed set in Y. Hence f is an r- closed mapping .
(ii) Let $y \in Y$, then {y} is r- compact in Y . Since f is an r- compact mapping , then $f^{-1}(\{y\})$ is compact in X , therefore by Theorem (3.10) , f is an r- proper mapping .

Theorem 3.30 : Let $f : X \to P = \{w\}$ be a mapping on a space X , where w is any point which does not belong to X , then the following statements are equivalent :
(i) f is an r- compact mapping .
(ii) f is an r- proper mapping .
(iii) f is a proper mapping .
(iv) X is a compact space .

Proof :
(i → ii). By Theorem (3.29) .
(ii → iii). By Remark (3.5) .
(iii → iv). See [1] .
(iv → i). Since $f^{-1}(P) = X$ and X is a compact space , then f is an r- compact mapping .
Theorem 3.31 : Let X and Y be spaces , such that Y is a compact , T_2- space and $f : X \to Y$ be a continuous mapping , then the following statements are equivalent :
(i) f is a proper mapping .
(ii) f is a compact mapping .
(iii) f is an r- compact mapping .
(iv) f is an r- proper mapping .
Proof :
(i → ii). See [1] .
(ii → iii). Let H be an r- compact set in Y . To prove that $f^{-1}(H)$ is compact in X . Since Y is a compact , T_2- space , then by Proposition (2.15) , H is a compact set in Y , then by (ii) , $f^{-1}(H)$ is a compact set in X . Hence f is an r- compact mapping .
(iii → iv). Theorem (3.29) .

(iv → i). By Remark (3.5) .

Proposition 3.32 : Let X and Y be spaces , such that Y is a T_2- space and f : X → Y be a continuous mapping . Then the following statements are equivalent :

(i) f is an r- coercive mapping .

(ii) f is an r- compact mapping .

(iii) f is an r- proper mapping .

Proof :

(i → ii). By Proposition (2.33) .

(ii → iii). By Proposition (3.29) .

(iii → i). Let J be an r- compact set in Y . Since f is r- proper , then by Proposition (3.29) , f is an r- compact mapping , then $f^{-1}(J)$ is a compact set in X . Since $f(X \setminus f^{-1}(J)) \subseteq Y \setminus J$. Hence $f : X \to Y$ is an r- coercive mapping .

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التطبيقات السديدة المنتظمة

الخلاصة

الهدف الأساسي من هذا العمل هو تقديم نوع عام و جديد للتطبيق السديد هو التطبيق السديد المنتظم . كما قدمنا تعريف جديد للتطبيق المتراص و التطبيق الأضطراري . كما تضمن البحث بعض الخواص و العبارات المتكافئة و كذلك شرحنا العلاقة بين هذه التعريفات .