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But R have only idempotents are 0 and 1, (11).

Then only only idempotents are 0_1 and 1_1.

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implies that $\ker f^{-1} \subseteq B_*$.

Hence $x \in B_*$, B_* is maximal and $\ker f^{-1} \subseteq B_*$

We have that $f^{-1}(B_*)$ is maximal ideal of R by (1 i) ...(1)

The chain of level ideals of B is $B_* \subseteq R'$ by lemma (7)

The chain of level ideals of $f^{-1}(B)$ is $f^{-1}(B_*) \subseteq f^{-1}(R) = R$

$$f^{-1}(B)(0) = \sup \{B(x) \mid x \in f(0)\}$$

$$\geq \{B(0) = 1\} \text{ implies that } 1 \in Im(f^{-1}(B)) \qquad \ldots(2)$$

From (1) and (2), $f^{-1}(B)$ is a maximal fuzzy ideal of R by proposition (2.1.3, (2)).

THEOREM 2.1.12:

In a fuzzy ring X of R having fuzzy maximal ideal A of X, then only idempotents are 0_t and 1_t, $t \in [0,1)$.

PROOF:

Let X be a fuzzy ring of R, A be a fuzzy ideal of X

A is fuzzy maximal ideal then

1. $A(0) = 1$, $|Im(A)| = 2$, A_* is a maximal fuzzy ideal of R [9]
2. $Im(A) = \{t,1\}$, $t \in [0,1]$ and A_t is maximal ideal (10)

To prove R has only one maximal ideal, we prove $A_* = A_t$

Let $x \in A_t \Rightarrow A_t = \{x : x \in R, A(x) \geq t\}$, $A(x) \geq t \geq 1 = A(0)$

$\forall x \in A_t \Rightarrow A(x) \geq A(0) \Rightarrow x \in A_*$

$$\therefore A_t \subseteq A_* \text{ so}$$

But A_t is maximal ideal of R then $A_* = R$

This leads at once to the contradiecation since A_* is maximal ideal, then $A_t = A_*$

PROPOSITION 2.1.11:

Let $X : R \to [0,1]$, $Y : R' \to [0,1]$ are fuzzy rings, $f : R \to R'$ be a homomorphism from R onto R'. B is a maximal fuzzy ideal of R' then $f^{-1}(B)$ is a maximal fuzzy ideal of X.

PROOF: We must prove that $f^{-1}(B)$ is a fuzzy ideal of X

Let $x, y \in R$ such that $f(a) = x$, $f(b) = y$ where $a, b \in R'$

$$f^{-1}(B)(a - b) = \sup\{\min\{B(x), B(y)\} \mid x = f(a), y = f(b)\}$$

$$\geq \sup\{\min\{Y(x), B(y)\} \mid x = f(a), y = f(b)\}$$

$$= \sup\{\min\{X(a), f^{-1}(B)(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}$$

$$\geq \sup\{\min\{f^{-1}(B)(a), f^{-1}(B)(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}$$

$$\geq \min\{f^{-1}(B)(a), f^{-1}(B)(b)\}$$

$$f^{-1}(B)(a \cdot b) = \sup\{\min\{B(x), B(y)\} \mid x = f(a), y = f(b)\}$$

$$\geq \sup\{\min\{Y(x), B(y)\} \mid x = f(a), y = f(b)\}$$

$$= \sup\{\min\{X(a), f^{-1}(B)(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}$$

$$\geq \min\{X(a), f^{-1}(B)(b)\}$$

Similerty $f^{-1}(B)(a \cdot b) \geq \min\{X(b), f^{-1}(B)(a)\}$

Hence $f^{-1}(B)(a \cdot b) \geq \max\{f^{-1}(B)(a), f^{-1}(B)(b)\}$

Therefore $f^{-1}(B)$ is a fuzzy ideal of X.

Since B is maximal, then B_* is maximal ideal of R', $B(0) = 1$ and $1 \in Im(B) = \{t,1\}$ by proposition (2.1.2) and proposition (2.1.5).

Let $x \in \ker f^{-1}$, then $f^{-1}(x) = f^{-1}(0)$, but f is invariant and onto

$B(x) = B(0) = 1$ ($x \in B_*$)

$f(A)(x \cdot y) = \sup \{ \min \{ A(a), A(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}$

$\geq \sup \{ \min \{ X(a), A(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}$

$= \sup \{ \min \{ Y(x), f(A)(y) \} \mid x = f(a), y = f(b)\}$

$\geq \min \{ Y(x), f(A)(y) \}$

Similerty $f(A)(x \cdot y) \geq \min \{ Y(y), f(A)(x) \}$

Hence $f(A)(x \cdot y) \geq \max \{ f(A)(x), f(A)(y) \}$

Therefore f (A) is a fuzzy ideal of Y.

Since A is maximal, then A_* is maximal ideal, $A(0) = 1$ and $1 \in$ $Im(A) = \{t,1\}$ by proposition (2.1.2) and proposition (2.1.5).

Let $x \in \ker f$, then $f(x) = f(0)$, but A is a f – invariant and $A(0) = 1$, therefore $A(x) = A(0) = 1$ ($x \in A_*$) implies that $\ker f \subseteq A_*$.

Hence $x \in A_*$, A_* is maximal and $\ker f \subseteq A_*$

We have that $f(A_*)$ is maximal ideal of R' by (11) ...(1)

The chain of level ideals of A is $A_* \subseteq R$ by lemma (7)

The chain of level ideals of f (A) is $f(A_*) \subseteq f(R) = R'$

$f(A)(0) = \sup \{A(x) \mid x \in f^{-1}(0)\}$

$\geq \{ A(0) = 1\}$ implies that $1 \in Im(f(A))$ ...(2)

From (1) and (2), f (A) is a maximal fuzzy ideal of R' by proposition (2.1.3, (2)).

$$(A \cap B)(x) = \begin{cases} 1 & \text{if } x \in A_* \cap B_* \\ t & \text{otherwise} \end{cases}$$

And $(A \circ B)(x) = \sup\{\min\{A(y), B(z)\} \mid x = y \cdot z\}$ $x, y, z \in R$,

If $y \in A_*$, $z \in B_* \Rightarrow (A \circ B)(x) = 1$

If $y \in A_*$, $z \notin B_* \Rightarrow (A \circ B)(x) = t$

If $y \notin A_*$, $z \in B_* \Rightarrow (A \circ B)(x) = t$

If $y \notin A_*$, $z \notin B_* \Rightarrow (A \circ B)(x) = t$

$$\therefore\ (A \circ B)(x) = \begin{cases} 1 & \text{if } x \in A_* \cap B_* \\ t & \text{otherwise} \end{cases}$$

Hence $A \circ B \subseteq A \cap B$

PROPOSITION 2.1.10:

Let $X : R \to [0,1]$, $Y : R' \to [0,1]$ are fuzzy rings. If A is a maximal fuzzy ideal of X, f is a homomorphism from R onto R' and A be a f – invariant then f (A) is a maximal fuzzy ideal of Y.

PROOF: We must prove that f (A) is a fuzzy ideal of R'

Let $x, y \in R'$ such that $f(a) = x$, $f(b) = y$ where $a, b \in R$

$f(A)(x - y) = \sup\{\min\{A(a), A(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}$

$\geq \sup\{\min\{X(a), A(b)\} \mid a = f^{-1}(x), b = f^{-1}(y)\}$

$= \sup\{\min\{Y(x), f(A)(y)\} \mid x = f(a), y = f(b)\}$

$\geq \sup\{\min\{f(A)(x), f(A)(y)\} \mid x = f(a), y = f(b)\}$

$\geq \min\{f(A)(x), f(A)(y)\}$

PROPOSITION 2.1.7 (9):

Let R be a ring and F denote the family of all maximal fuzzy ideals of R, then

$$J = (\bigcap \{M \mid M \subseteq F\})$$

PROPOSITION 2.1.8:

A fuzzy ideal A of R is maximal if and only if the level ideal $\{x \in R \mid A(x) = 1\}$ is maximal

PROOF: →

If A is maximal fuzzy ideal of R, then A_* is a maximal ideal of R, A(0) =1, (9).

But $A_* = \{x : x \in R \mid A(x) = A(0)\}$, then $A_* = \{x : x \in R \mid A(x) = 1\}$ is maximal ideal of R.

←

If the level ideal $\{x \in R \mid A(x) = 1\}$ is maximal ideal of R, then A_* is a maximal ideal of R and A(0) =1 implies that A is maximal fuzzy ideal of R (9).

PROPOSITION 2.1.9:

If A and B are two different maximal fuzzy ideals of R and Im(A) = Im(B), then $A \circ B \subseteq A \bigcap B$.

PROOF:

$$A(x) = \begin{cases} 1 & \text{if } x \in A. \\ t & \text{otherwise} \end{cases} \qquad B(x) = \begin{cases} 1 & \text{if } x \in B. \\ t & \text{otherwise} \end{cases}$$

where $x \in R$, $1 \neq t \in Im(A) = Im(B)$, clearly $A_* \neq B_*$, then

PROPOSITION 2.1.3 (9):

Let A be a fuzzy ideal of a ring R. Then

1. If A_* is a maximal ideal of R, then A is two- valued
2. If A_* is a maximal ideal of R and A(0) =1, then A is a maximal fuzzy ideal of R.

PROPOSITION 2.1.4 (9):

1. Let R be a ring and A be a non constant fuzzy ideal of R. Then there exists a maximal

   fuzzy ideal B of R such that $A \subseteq B$.
2. Let $I \neq R$ be an ideal of R. Then I is a maximal ideal of R if and only if λ_I is a maximal fuzzy ideal of R.

PROPOSITION 2.1.5 (10):

Let A be a fuzzy ideal of a ring R. Then A is a maximal fuzzy ideal of R if and only if

$Im(A) = \{t,1\}$, for some $t \in [0,1)$ and A_t is a maximal ideal of R.

PROPOSITION 2.1.6 (7):

Let $X = \{A \mid A$ fuzzy ideal of R, $Im(A) = \{t,1\}, t \in [0,1)\}$. Then $A \subseteq X$ is a maximal

fuzzy ideal if and only if for each $B \subseteq X$, either $B \subset A$ or $(A + B)_{(y)} = 1$ for all $y \in R$.

PROPOSITION 1.2.4 (7),(8):

Let A and B are fuzzy ideals of R, then $A \circ B$ is a fuzzy ideal of R.

PROPOSITION 1.2.5 (7),(8):

Let A and B are fuzzy ideals of R, then $A \cap B$, $A + B$ are fuzzy ideals of R. Moreover, if $A(0) = B(0)$, then $A, B \subseteq A + B$. Also $A + A = A$.

CHAPTER TWO

2.1 MAXIMAL FUZZY IDEALS

We are ready to define a maximal fuzzy ideal of a ring R and we give some properties of it.

DEFINITION 2.1.1 (1),(9):

Let A be a fuzzy ideal of a ring R. Then A is called maximal fuzzy ideal of R if

1. A is not constant, and
2. For any fuzzy ideals B of R, if $A \subseteq B$, then either $A_* = B_*$ or $B = \lambda_R$.

PROPOSITION 2.1.2 (9):

Let A be a maximal fuzzy ideal of R. Then

1. $A(0) = 1$
2. $|\,Im(A)\,| = 2$
3. A_* is maximal ideal of R.

We denote λ_R if $\lambda_R(x) = 1$ if $x \in R$ and $\lambda_R(x) = 0$ if $x \notin R$, (4).

Let f : R → R′ function. A fuzzy subset A of R is called f – invariant if f(x) = f(y), then

A(x) = A(y) where $x, y \in R$, (7).

1.2 FUZZY IDEAL

We will give definition and some basic properties about fuzzy ideals.

DEFINITION 1.2.1 (3):

A fuzzy subset A of R is called a fuzzy ideal of R if and only if for all $x, y \in R$,

$A(x - y) \geq \min\{A(x), A(y)\}$ and $A(x \cdot y) \geq \max\{A(x), A(y)\}$.

It is clear every fuzzy ideal of R is a fuzzy ring of R but the converse is not true.

PROPOSITION 1.2.2 (1),(3):

Let R be a ring and A be a fuzzy ideal of R, then A and A_t are ideals of R.

PROPOSITION 1.2.3 (6):

Let $\{A_i \mid i \in \Lambda\}$ be a family of fuzzy ideals of R. Then $\bigcap_{i \in \Lambda} A_i$ is a fuzzy ideal of R and

$\bigcup_{i \in \Lambda} A_i$ is a fuzzy ideal of R if A_i is a chain.

$x_t \in A$ if and only if $x_t \subseteq A$, (5).

Let $I^R = \{A_i \mid i \in \Lambda\}$ be a collection of fuzzy subset of R. Define the fuzzy subset of R

(intersection) by $(\bigcap_{i \in \Lambda} A_i)(x) = \inf \{ A_i(x) \mid i \in \Lambda \}$ for all $x \in R$,

(unian) by $(\bigcup_{i \in \Lambda} A_i)(x) = \sup \{ A_i(x) \mid i \in \Lambda \}$ for all $x \in R$, ((4), (5)).

Let A and B be a fuzzy subsets of R, the product $A \circ B$ define by

$$A \circ B\,(x) = \begin{cases} \sup \{\min \{A(y), B(z)\}\} & x = y \cdot z \\ 0 & x \neq y \cdot z \end{cases}$$

$y, z \in R$, for all $x \in R$.

The addition $A + B$ define by $(A + B)(x) = \sup \{\min \{A(y), B(z) \mid x = y + z\}$ $y, z \in R$,
for all $x \in R$, (5).

Let A be a fuzzy subset of R, A is called a fuzzy subgroup of R if for all $x, y \in R$,
$A(x + y) \geq \min \{A(x), A(y)\}$ and $A(x) = A(-x)$, Note that $A(0) \geq A(x)$ for all $x \in R$. (5).

Let A be a fuzzy subset of R, A is called a fuzzy ring of R if for all $x, y \in R$, $A(x - y) \geq \min \{A(x), A(y)\}$ and $A(x \cdot y) \geq \min \{A(x), A(y)\}$ and a fuzzy subring B of a fuzzy ring A is a fuzzy ring of R satisfying $B(x) \leq A(x)$ for all $x \in R$, (6).

CHAPTER ONE

1.1 PRELIMINARY CONCEPTS

Let (R,+,·) be a commutative ring with identity. A fuzzy subset of R is a function from R into [0,1].

Let A and B be fuzzy subset of R. We write $A \subseteq B$ if $A(x) \leq B(x)$ for all $x \in R$. If $A \subseteq B$ and there exists $x \in R$ such that $A(x) < B(x)$, then we write $A \subset B$ and we say that A is a proper fuzzy subset of B, A = B if and only if $A(x) = B(x)$, for all $x \in R$.

We let ϕ denote $\phi(x) = 0$ for all $x \in R$, the empty fuzzy subset of R, ((1), (2), (3)).

When we say fuzzy subset we mean a nonempty fuzzy subset.

We let Im(A) denote the image of A. We say that A is a finite – valued if Im(A) is finite. | Im(A) | denote the cardinality of Im(A), ((3), (4)).

For each $t \in [0,1]$, the set $A_t = \{ x \in R \mid A(x) \geq t \}$ is called a level subset of **R**

$A_* = \{ x \in R \mid A(x) = A(0) \}$, ((1), (4)).

Let $x \in R$ and $t \in [0,1]$, we let x_t denote the fuzzy subset of **R** defined by $x_t(y) = 0$ if

$x \neq y$ and $x_t(y) = t$ if $x = y$ for all $y \in R$. x_t is called a fuzzy singleton. If x_t and y_s are fuzzy singletons, then $x_t + y_s = (x + y)_\lambda$ and $x_t \circ y_s = (x \cdot y)_\lambda$ where $\lambda = \min \{ t,s \}$, ((1), (4)).

المثالي الضبابي الأعظم

اريج توفيق حميد*

المستخلص

بعد أن قدم زاده مفهوم المجموعات الضبابية(fuzzy sets) وقدم ليو مفهوم الحلقات الضبابية (fuzzy ring) وكذلك المثالي الضبابي (fuzzy ideal) ومنذ ذلك الحين أُجريت العديد من البحوث في مختلف المجالات الرياضية النظرية والتطبيقية حول هذا الموضوع.

الهدف الرئيسي من هذا البحث هو دراسة بعض خواص المثالي الضبابي الأعظم.

ABSTRACT

After, (Zadeh L.A.) 1965 introduced the concept of fuzzy sets and (Liu W.J.) 1982 formulated the term of fuzzy ring

Also, (Liu W.J.) 1982 introduced the concept of fuzzy ideal of a ring. Since that time many papers were introduced in different mathematical scopes of theoretical and practical applications.

The main aim of this paper is to study some proposition about maximal fuzzy ideal of a ring.

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* مدرس مساعد / جامعة بغداد / كلية التربية ابن الهيثم / قسم الرياضيات

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