Covariates in One-Way Repeated Measures Model with Satterthwaite Type Approximation to Quadratic Form

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Abstract

In this article we consider a set of p repeated measurements on r variables on each of the n individuals. Thus, data on each individual is a r×p matrix. The n individuals themselves may be divided and randomly assigned to g groups. We take the one-way Multivariate Repeated Measures Model and study the Satterthwaite Type Approximation to the distribution of Quadratic form.

***Key words*:**

(One-Way MRMM): One-Way Multivariate Repeated Measures Model, Λ_r: Wilks distribution ; $U*$ is $P\times P$ orthogonal matrix ; (Wishart) the multivariate- Wishart distribution ; (ANCOVA) analysis of variance and covariance contain on the covariates ; (Z_1 ,Z_2) Concomitant Variate or Covariates.

Introduction :-

The MRM (multivariate repeated measures model) generalizes RM (repeated measures model) in the sense that it allows a vector of observations at each measurements , the response variables are measured on each t occasions which regarded as t levels of a with in – units factors , we consider the case of multivariate response.

Boik(1988)[5] considered the Satterthwaite type approximations (multivariate approximation) to the distributions of various sums of squares and cross-products (SS & CP) matrices of the MANOVA table when the covariance matrix is any general positive definite matrix . K huri and Nel, (1994)[9] also have considered the problems relating to multivariate approximation to a SS&CP matrix .their approximation is different from Boik's (1988)[5] in the sense that the use a generalized variance (determinant of the covariance matrix) instead of a trice, for their approximations. All work on multivariate repeated measures done thus far in the literature has the basic assumption that Cov (y_{ij})=Ω, wheer Ω is a positive definite magtrtix. In my work the one –way multivariate repeated measures model is consider and Istudy the Satterthwaite type approximation to the distribution of a quadratic form.

(1.1) Covariate in One-Way Multivariate Repeated Measurements Design:

For convenience ,we define the following linear model and parameterization for the one-way repeated measurements design with one between units factor incorporation two covariates(Z_1, Z_2)[1] . For this model the two covariate is time-independent, that is measured only once :

$$Y_{ijk} = \mu + \tau_j + \gamma_k + (\tau\gamma)_{jk} + (Z_{1ij} - \overline{Z}_{1..})\beta_1 + (Z_{2ij} - \overline{Z}_{2..})\beta_2 + e_{ijk}$$
…(1.1)

Where

$(i = 1,..., n_j)$ is an index for experimental unit of level (j)

$(j = 1,..., q)$ is an index for levels of the between-units factor (Group).

$(k = 1,..., p)$ is an index for levels of the within-units factor (Time).

$Y_{ijk} = (Y_{ijk1},..., Y_{ijkr})'$ is the response measurement of within-units factors (Time) for unit i within treatment factors (Group).

$\mu = (\mu_1,..., \mu_r)'$ is the overall mean.

$\tau_j = (\tau_{j1},..., \tau_{jr})'$ is the added effect of the j^{th} level of the treatment factor (Group).

$\gamma_k = (\gamma_{k1},..., \gamma_{kr})'$ is the added effect of the k^{th} level of Time.

$(\tau\gamma)_{jk} = \left((\tau\gamma)_{jk1},..., (\tau\gamma)_{jkr}\right)'$ is the added effect of the interaction between the units factor (Group) at level of (Time).

$Z_{1ij} = (Z_{1ij1},..., Z_{1ijr})'$ is the value of covariate Z_1 at time k for unit i within group j .

$\overline{Z}_{1..} = (\overline{Z}_{1..1},..., \overline{Z}_{1..r})'$ is the mean of covariate Z_1 over all experimental units.

$\beta_1 = (\beta_{11},...,\beta_{1r})'$ is the slope corresponding to covariate Z_1.

$Z_{2ij} = (Z_{2ij1},...,Z_{2ijr})'$ is the value of covariate Z_2 at time k for unit i within group j .

$\bar{Z}_{2..} = (\bar{Z}_{2..1},...,\bar{Z}_{2..r})'$ is the mean of covariate Z_2 over all experimental units.

$\beta_2 = (\beta_{21},...,\beta_{2r})'$ is the slope corresponding to covariate Z_2.

$e_{ijk} = (e_{ijk1},...,e_{ijkr})'$ is the random error at time k for unit i within group j.

For the parameterization to be of full rank, we impose the following set of conditions :

$$\sum_{k=1}^{p}\gamma_k = 0 \quad , \quad \sum_{j=1}^{q}\tau_j = 0$$

$$\sum_{j=1}^{q}(\tau\gamma)_{jk} = 0 \quad , \quad \sum_{k=1}^{p}(\tau\gamma)_{jk} = 0 \quad \text{for each} \quad j,k = 1,...,p$$

$$\sum_{i=1}^{n_j} Z_{1ij} = \sum_{i=1}^{n_j} Z_{2ij} = 0 \quad , \quad \sum_{j=1}^{q} Z_{1ij} = \sum_{j=1}^{q} Z_{2ij} = 0$$

We assume that $e_{ijk}{}^{,s}$ is independent with

$e_{ijk} = (e_{ijk1},...,e_{ijkr})' \sim \quad i.i.d \quad N_r(0,\Sigma_e)$

Where N_r is denoted to the multivariate normal distribution , and Σ_e is $r \times r$ positive definite matrix [1].

Let $\quad Y_{ij} = (Y_{ij1},...,Y_{ijp})'$

$$Y_{ij} = \begin{bmatrix} Y_{ij11} & Y_{ij21} & \cdots & Y_{ijp1} \\ Y_{ij12} & Y_{ij22} & \cdots & Y_{ijp2} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{ij1r} & Y_{ij2r} & \cdots & Y_{ijpr} \end{bmatrix}$$

Let the covariance matrix of $\vec{Y}_{ij}$ is denoted by Σ , where

$\vec{Y}_{ij} = vec(Y_{ij})$

The vec(.) operator creates a column vector from a matrix A by simply stacking the column vectors of Y_{ij} below one another (see Vonesh and Chinchilli (1997)[12]).

Where the variance matrix and covariance ∑ of the model (1.1) satisfy the assumption of compound symmetry. i.e.

$$\Sigma = I_p \otimes \Sigma_e = \begin{bmatrix} \Sigma_e & 0 & \cdots & 0 \\ 0 & \Sigma_e & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_e \end{bmatrix}$$

…(1.2)

Where

I_p denote the $P \times P$ identity matrix.

J_p denote the $P \times P$ matrix of one's. and $\otimes$ be the Kroneker product

operation of two matrices.

(1.2) Transforming the one-way Repeated measurements Analysis of Covariance (ANCOVA) model :

Let U^* be any $P \times P$ orthogonal matrix is partitioned as follows[2]:

$$U^* = (p^{-\frac{1}{2}}_{j_p} \quad U)$$

Where j_p denote the $P \times 1$ vector of one's ,U is $(p-1) \times p$ matrix .

Let $Y_{ij}^* = Y_{ij} U^*$

So $$Cov(\vec{Y}_{ij}^*) = \begin{bmatrix} \Sigma_e & 0 & \dots & 0 \\ 0 & \Sigma_e & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Sigma_e \end{bmatrix}$$

The ANCOVA based on the set of transformed observations above the $Y_{ij1}^{*,s}$ provides the ANCOVA for between-units effects. This leads to the following form for the sum square terms:

$$Q_1 = P\sum_{j=1}^{q} n_j(\overline{Y}_{j..}^{*} - \overline{Y}_{...}^{*})(\overline{Y}_{j..}^{*} - \overline{Y}_{...}^{*})' = Y'\ A_1\ Y$$

$$Q_2 = p\sum_{j=1}^{q}\sum_{i=1}^{n_j}\frac{1}{\beta_1}\left[(\overline{Y}_{ij.}^{*} - \overline{Y}_{j..}^{*} - \beta_2^{*}Z_{2ij}^{*})(\overline{Y}_{ij.}^{*} - \overline{Y}_{j..}^{*} - \beta_2^{*}Z_{2ij}^{*})'\right] = Y'\ A_2\ Y$$

$$Q_3 = P\sum_{j=1}^{q}\sum_{i=1}^{n_j}\frac{1}{\beta_2}\left[(\overline{Y}_{ij.}^{*} - \overline{Y}_{j..}^{*} - \beta_1^{*}Z_{1ij}^{*})(\overline{Y}_{ij.}^{*} - \overline{Y}_{j..}^{*} - \beta_1^{*}Z_{1ij}^{*})'\right] = Y'\ A_3\ Y$$

$$Q_4 = P\sum_{j=1}^{q}\sum_{i=1}^{n_j}\left((Y_{ij.}^{*} - \overline{Y}_{j..}^{*} + \beta_1^{*}Z_{1ij}^{*} + \beta_2^{*}Z_{2ij}^{*})(Y_{ij.}^{*} - \overline{Y}_{j..}^{*} + \beta_1^{*}Z_{1ij}^{*} + \beta_2^{*}Z_{2ij}^{*})'\right) = Y'\ A_4\ Y$$

$$Q_5 = n\sum_{k=1}^{p}((\overline{Y}_{i,k}^{*} - \overline{Y}_{i..}^{*} - \overline{Y}_{..k}^{*} + \overline{Y}_{...}^{*})(\overline{Y}_{i,k}^{*} - \overline{Y}_{i..}^{*} - \overline{Y}_{..k}^{*} + \overline{Y}_{...}^{*})') = Y'\ A_5\ Y$$

$$Q_6 = \sum_{k=1}^{p}\sum_{j=1}^{q}\sum_{i=1}^{n_j}\left((\overline{Y}_{ijk}^{*} - \overline{Y}_{ij,}^{*} - \overline{Y}_{i.k}^{*} + \overline{Y}_{i..}^{*})(\overline{Y}_{ijk}^{*} - \overline{Y}_{ij,}^{*} - \overline{Y}_{i.k}^{*} + \overline{Y}_{i..}^{*})'\right) = Y'\ A_6\ Y$$

$$Q_7 = \sum_{k=2}^{p}\sum_{j=1}^{q}\sum_{i=1}^{n_j}\left((Y_{ijk}^{*} - \overline{Y}_{.jk}^{*} + \beta_1^{*}Z_{1ij}^{*} + \beta_2^{*}Z_{2ij}^{*})(Y_{ijk}^{*} - \overline{Y}_{.jk}^{*} + \beta_1^{*}Z_{1ij}^{*} + \beta_2^{*}Z_{2ij}^{*})'\right) = Y'\ A_7\ Y$$

<u>**(1.3) Exact null distribution of Q_j:**</u>

In the following, we derive the distribution of Q_j=Y' A_j Y, where A_j is an appropriately defined symmetric matrix of order np × np [3] . Fist, it is easy to show that $A_1\ \vartheta\ A_4$=$A_2\ \vartheta\ A_4$=$A_3\ \vartheta\ A_4$=$A_5\ \vartheta\ A_7$=$A_6\ \vartheta\ A_7$=0 (see,Geisser,S.& Greenhouse,(1958)[8]),where $\vartheta = I_n \otimes I$.

We observe that each column of Y is a multivariate normal vector of order np×1 and has a covariate matrix proportional to $\vartheta = I_n \square I$.

Since, by assumption, ϑ is a positive definite matrix, there exist $\vartheta^{1/2}$ and $\vartheta^{-1/2}$ such that $\vartheta = \vartheta^{1/2}\vartheta^{1/2}$ and $\vartheta^{-1} = \vartheta^{-1/2}\vartheta^{-1/2}$ and that $\vartheta^{-1/2}\vartheta^{1/2} = I_{np}$ Consider $Q_{j=}Y'A_j\ Y =$ $Y'\vartheta^{1/2}\vartheta^{-1/2}A_j\vartheta^{1/2}\vartheta^{-1/2}Y = C'B_jC$ with $C = \vartheta^{-1/2}Y$ and $B_j = \vartheta^{1/2}A_j\vartheta^{1/2}$.Now the rows of C form random sample of size nt from $N_r(o, \sum)$ under certain null hypotheses.

There exist an orthogonal matrix Γ such that $B_j = \Gamma\Lambda_j\Gamma'$ if B_i is positive definite and it is a symmetric matrix with rank νi, where $\Gamma\Gamma' = \Gamma'\Gamma = I_{np}$ and $\Lambda_{j=}Diag(\lambda_1, \dots, \lambda_{\nu j}, 0, \dots, 0), \lambda_1 > \cdots > \lambda_{\nu j}$ being the eigenvalues of B_j. Thus $Q_j = CB_jC = C'\Gamma\Lambda_j\Gamma'C = U'\Lambda_jU = \sum_{i=1}^{\nu j} \lambda_i U_i U_i'$, $U_i \sim N_r(0, \Sigma)$ and $U_1, \dots, U_{\nu j}$ are all independent, where $U = \Gamma'C$ and U_j is the j th row U. It is well known that $U_iU_i' \sim W_r(1, \sum)$.

In summary, we have the following:

$Q_1 \sim W_r(q-1, \sum)$, $Q_2 \sim W_r(1, \sum)$

$Q_3 \sim W_r(1, \sum)$, $Q_4 \sim \sum_{i=1}^{p-1} \lambda_i W_r^{(i)}(n-q-2, \sum)$

$Q_5 \sim \sum_{i=1}^{p-1} \lambda_i W_r^{(i)}(p-1, \sum)$, $Q_6 \sim \sum_{i=1}^{p-1} \lambda_i W_r^{(i)}((p-1)(q-1), \sum)$

$$Q_7 \sim \sum_{i=1}^{p-1} \lambda_i W_r^{(i)}((p-1)(n-q), \sum)$$

Where $\lambda_1, \dots, \lambda_{p-1}$ are the eigenvalues of $(I - (1/p)J)$ Iand $W_r^{(i)}(V_i, \sum)$, for $j = 1, \dots, p-1$ are Wishart random matrices with $v_j$degrees of freedom, and mutually independent . Suppose we want to test of no group effect . then the Wilks' for testing this is:

$$\Lambda_1 = \frac{|Q_1|}{|Q_1 + Q_4|}$$

$$\Lambda_2 = \frac{|Q_2|}{|Q_2 + Q_4|}$$

$$\Lambda_3 = \frac{|Q_3|}{|Q_3 + Q_4|}$$

The test is based on the usual asymptotic distribution (see,Rao(1973)[10]) of Λ_1 , Λ_2 , Λ_3. Then

$$-\left[(n-q-2)-\frac{(r-q)}{2}\right] \ln \Lambda_1 \sim x^2_{r(q-1)} \quad \text{approximately}$$

$$-\left[(n-q-2)-\frac{(r-q)}{2}\right] \ln \Lambda_2 \sim x^2_{r} \quad \text{approximately}$$

$$-\left[(n-q-2)-\frac{(r-q)}{2}\right] \ln \Lambda_3 \sim x^2_{r} \quad \text{approximately}$$

The approximate distributions of the test statistics for testing that there is on time effect, and that there is on time and group interaction Will be derived next.

(1.4) Approxiate null distribution of Q_5, Q_6 and Q_7:

In this section , we approximate the distribution of each of SS&CP matrices, Q_5, Q_6 and Q_7, to a scale multiple of a Wishart matrix, gW(h,Σ), for some constants g and h .As in the univariate case, the approximation is derived by equating the first two central moment. For that we first find the first two central moment of Q_j. It is well known (for example ,see ,Vonesh ,and Chinchilli,(1997)[12]) that for any $S \sim W_r(v, \Sigma)$, E(S)=$v\Sigma$ and D(S)=2$v\Sigma \otimes \Sigma$. Here , D(S) denotes the variance covariance matrix of all the random quantities in S using these formulae we have the following:

$$E(Q_5) = \left|(p-1)tr(I-\tfrac{1}{p}JI)\right|\Sigma \quad \text{... (1.3)}$$

$$E(Q_6) = \left|(p-1)(q-1)tr(I-\tfrac{1}{p}JI)\right| \ \Sigma \quad \text{... (1.4)}$$

$$E(Q_7) = \left|(p-1)(n-q)tr(I-\tfrac{1}{p}JI)\right| \ \Sigma \quad \text{... (1.5)}$$

Also

$$D(Q_5) = \left|2(p-1)tr(I-\frac{1}{p}JI)^2\right|\Sigma \otimes \Sigma \quad \text{... (1.6)}$$

$$D(Q_4) = \left|2(p-1)(p-1)tr(I-\frac{1}{p}JI)^2\right|\Sigma \otimes \Sigma \quad \text{... (1.7)}$$

$$D(Q_5) = \left|2(p-1)(n-q)tr(I-\frac{1}{p}JI)^2\right|\Sigma \otimes \Sigma \quad \text{... (1.8)}$$

Suppose we want to approximate the distribution of $Q_5$by a random matrix having the distribution $g_1 W_r(h_{1,}\Sigma)$ so that the two central moments of $Q_5$and $g_1 W_r(h_1, \Sigma)$ are the same. Then,

$$(p-1)tr\left|I-\frac{1}{p}JI\right|\Sigma = g_1 h_1 \Sigma \qquad \dots (1.9)$$

$$2(p-1)\left|tr\left(I-\frac{1}{p}JI\right)\right| \Sigma \otimes \Sigma = 2g_1^2 h_1 \Sigma \square \Sigma \qquad \dots (1.10)$$

From (1.9) and (1.10) :

$$g_1 = \frac{tr(I-\frac{1}{p}JI)^2}{tr(I-\frac{1}{p}JI)} \qquad \textit{and} \qquad h_1 = \frac{\left[tr(I-\frac{1}{p}JI)\right]^2}{tr(I-\frac{1}{p}JI)^2}$$

Next, to approximate the distribution of Q_6 by a random matrix having the distribution $g_2 W_p(h_2, \Sigma)$ so that the first two central moments $Q_6$and $g_2 W_p(h_2, \Sigma)$ are the same we have,

$$(p-1)(q-1)tr\left[\left(I-\tfrac{1}{p}JI\right)\right]\Sigma = g_2 h_2 \Sigma \qquad \dots (1.11)$$

$$2(p-1)(q-1)tr\left[\left(I-\tfrac{1}{p}JI\right)\right]\Sigma \otimes \Sigma = 2g_2 h_2 \Sigma \square \Sigma \qquad \dots (1.12)$$

From (1.11) and (1.12) we have,

$$g_2 = \frac{tr\left(I-\frac{1}{p}JI\right)^2}{tr\left(I-\frac{1}{p}JI\right)} = g_1 \qquad \text{and}$$

$$h_2 = (p-1)(q-1)\frac{\left[tr\left(I-\frac{1}{p}JI\right)\right]^2}{tr\left(I-\frac{1}{p}JI\right)^2} = (g-1)h_1$$

Next, to approximate the distribution of Q_7 by a random matrix having the distribution $g_3 W_p(h_3, \Sigma)$ so that the first two central moments Q_7 and $g_3 W_p(h_3, \Sigma)$ are the same we have,

$$(p-1)(n-q)tr\left[\left(I-\frac{1}{p}JI\right)\right]\Sigma = g_3 h_3 \Sigma \quad \text{... (1.13)}$$

$$2(p-1)(n-q)tr\left[\left(I-\frac{1}{p}JI\right)\right]\Sigma \otimes \Sigma = 2g_3 h_3 \Sigma \,\square\, \Sigma \quad \text{... (1.14)}$$

From (1.13) and (1.14) we have,

$$g_3 = \frac{tr\left(I-\frac{1}{p}JI\right)^2}{tr\left(I-\frac{1}{p}JI\right)} \doteq g_1 \qquad \text{and}$$

$$h_3 = (p-1)(n-q)\frac{\left[tr\left(I-\frac{1}{p}JI\right)\right]^2}{tr\left(I-\frac{1}{p}JI\right)^2} = (g-1)h_1$$

Now, for testing that there is time effect one can Wilks' Λ which is,

$$\Lambda_4 = \frac{|Q_5|}{|Q_5 + Q_7|}$$

$$\Lambda_5 = \frac{|Q_6|}{|Q_6 + Q_7|}$$

and the fact that

$-\left[(p-1)(n-q)h_1 - \left(\frac{r+1-(p-1)h_1}{2}\right)\right] In\Lambda_4 \sim x^2_{r(p-1)h_1}$ approximately.

$-\left[(p-1)(n-q)h_1 - \left(\frac{r+1-(p-1)(q-1)h_1}{2}\right)\right] In\Lambda_4 \sim x^2_{r(p-1)(q-1)h_1}$ approximately

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