Fixed Point Theorems for Several Contractive Mappings and Expansive Mappings in G-Metric Spaces

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Abstract

The purpose of this paper is to define two of types mappings in G-metric spaces and find the fixed points of these mappings. The first type is called several contractive mappings, where we define the contraction condition on the closure of an orbit, where the orbit is bounded and orbitally complete. Also, we will discuss the uniqueness of a fixed point only in this orbit. Finally, we prove the existence of a fixed point for surjective expansive mapping.

1. Introduction

In 2005, a new structure of generalized metric spaces was introduced by Mustafa Z. and Brailey Sims [1] called G-metric spaces. Mustafa Z. and Brailey Sims [1], Mustafa Z. [2] studies the convergence concept and the continuity of G-metric function. Also, Mustafa Z. [2] introduced some theorems of fixed point theory.

Mustafa Z. [3] and Mustafa Z. et al. [4] gave certain type of contractive mapping in G-metric spaces. Here we will recall such mapping by several contractive mapping.

In section two, we introduce some fixed point theorems for several contractive mapping in the closure of an orbit of the space. Here, the orbit must be bounded and orbitally complete, and the uniqueness of a fixed point will be discussed in this orbit.

In section three, we introduce some fixed point theorems for expansive mappings, depending on the convergence of the iterative sequences in G-metric spaces.

2- Preliminaries

Now, we give the following definitions and propositions concerning the G-metric spaces.

Definition (2.1), [1], [2]:

Let X be a nonempty set, and $G : X \times X \times X \longrightarrow R^+$, ($R^+$ is a set of all non-negative real numbers), be a function satisfying the following:

(G1) $G(x, y, z) = 0$ if $x = y = z$.

(G2) $0 < G(x, x, y)$, for all $x, y \in X$, with $x \neq y$.

(G3) $G(x, x, y) \leq G(x, y, z)$, for all x, y. $z \in X$ with $z \neq y$.

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$; (symmetry in all three variables).

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality)

Then the function G is called a **generalization metric**, or a **G-metric** on X, and the pair (X, G) is a **G-metric space**.

We recall the following proposition without proof:

Proposition (2.2), [1]:

Let X be a G-metric space and $x_0 \in X$, $r > 0$. Then the G-ball with center x_0 and radius r, is:

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}$$

Proposition (2.3), [1]:

Let (X, G) be a G-metric space. Then for any $x_0 \in X$ and $r > 0$:

1. If $G(x_0, x, y) < r$, then $x, y \in B_G(x_0, r)$.
2. If $y \in B_G(x_0, r)$, then there exist a $\delta > 0$, such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.

Proof:

(1) Follows directly from (G3), while (2) follows from (G5) with $\delta = r - G(x_0, y, z)$. ■

It follows from (2) of the above proposition that the family of all G-balls:

$$B = \{B_G(x, r) : x \in X, r > 0\}$$

is the base of a topology $\tau(G)$ on X, the G-metric topology.

Definition (2.4), [1], [2]:

Let (X, G) be a G-metric space, and $\{x_n\}$ be a sequence of X. If there exist a point $x \in X$, such that $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, then the sequence $\{x_n\}$ is G-convergent to x, and x is said to be the limit point of the sequence.

Or, for any $\varepsilon > 0$, there exists $k \in N$ (through this paper, we mean by N to the set of natural numbers), such that $G(x, x_n, x_m) < \varepsilon$, for all n, $m \geq k$.

Definition (2.5), [1], [2]:

Let (X, G) be a G-metric space. The sequence $\{x_n\} \subseteq X$ is said to be G-Cauchy sequence if for every $\varepsilon > 0$, there exists $k \in N$, such that $G(x_n, x_m, x_\ell) < \varepsilon$, for all n, m, $\ell \geq k$.

Definition (2.6), [1], [2]:

A G-metric space (X, G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Definition (2.7), [1], [2]:

Let (X, G), (X′, G′) be two G-metric spaces and let $f : (X, G) \longrightarrow (X', G')$ be a mapping, then f is said to be G-continuous at a point $a \in X$, if given $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in X$; and $G(a, x, y) < \delta$, implies $G'(f(a), f(x), f(y)) < \varepsilon$.

A mapping f is G-continuous at x if and only if, it is G-continuous at all $a \in X$.

Definition (2.8), [2]:

Let (X, G) be a G-metric space, the function G is jointly continuous in all three of its variables, if for any convergent sequences

$\{u_n\}$, $\{v_n\}$, $\{w_n\}$ in G-metric space X, where $\{u_n\}$ converge to $u \in X$, $\{v_n\}$ converge to $v \in X$ and $\{w_n\}$ converge to $w \in X$. Then, $\{G(u_n, v_n, w_n)\}$ converges to G(u, v, w).

Mustafa, Z. and Sims, B. [1] introduced the following propositions, here we mention to their without proof.

Proposition (2.9):

Let (X, G), (X′, G′) be G-metric spaces. Then a mapping $f : X \longrightarrow X$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x one has $\{f(x_n)\}$ is G-convergent to f(x).

Proposition (2.10):

Let (X, G) be a G-metric space. Then the function G is jointly continuous in all three of its variables.

Analogous to [5] and [6], we define the following concepts in the G-metric space.

Definition (2.11):

Let (X, G) be a G-metric space. Let $f : X \longrightarrow X$ be a function:

1. The orbit of f at the point $x \in X$ is the set:

   $o(x) = \{x, fx, f^2x, \ldots\}$

2. An orbit o(x) of x in X is said to be **G-bounded (or bounded)** if there exists a constant $k > 0$, such that $G(u, v, w) \leq k$, for all $u, v, w \in o(x)$. The constant k is called **G-bound (or bound)**.

3. An orbit o(x) is called an **orbitally complete** if every Cauchy sequence in o(x) converges to a point in X.

Definition (2.12):

Let (X, G) be a G-metric space and S be a nonempty subset of X. Define the diameter of S, as:

$\delta_G(S) = \sup\{G(x, y, z) : x, y, z \in S\}$

3- Fixed Point Theorems for Several Contractive Mapping in G-Metric Spaces

In this section, we will prove some theorems of fixed point for several contractive mappings. Mustafa Z. et al. [3], introduced the following definition, here, we will recall that by "several contractive mapping:

Definition (3.1), [3]:

Let (X, G) be a G-metric space an $f : X \longrightarrow X$ be a mapping, then f is said to be **several contractive mapping** on G-metric space if,

$$G(fx, fy, fz) \leq aG(x, fx, fx) + bG(y, fy, fy) + cG(z, fz, fz) \qquad \ldots(3.1)$$

for all $x, y, z \in X$, where $0 < a + b + c < 1$.

Also, if the contraction condition (3.1) restricted on all x, y, z in $O(x_0)$ then we say that f is **several contractive on the orbit** $O(x_0)$.

Now, we need to prove the following proposition:

Proposition (3.2):

Let (X, G) be a G-metric space. Let $x \in X$, such that $x_n = f^n x$, $n \in \mathbb{N}$. If the orbit $\{x_n\}$ is bounded. Define:

$\gamma_i = \delta_G\{x_i, x_{i+1}, x_{i+2}, \ldots\}$, $i = 1, 2, \ldots$

Then,

1. γ_n is finite for all $n \in N$.
2. $\{\gamma_n\}$ is non-increasing sequence, for all $n \in N$.

   Moreover, $\gamma_n \longrightarrow \gamma \geq 0$, as $n \longrightarrow \infty$.

Proof:

For (1), since $\{x_n\}$ is bounded, then the diameter of $\{x_n\}$ is finite

Therefore, γ_n is finite for all n

For (2), let $\gamma_r, \gamma_{r+1} \in \{\gamma_n\}$

$\gamma_r = \delta_G\{x_r, x_{r+1}, x_{r+2}, \ldots\} \geq \delta_G\{x_{r+1}, x_{r+2}, x_{r+3}, \ldots\} = \gamma_{r+1}$, for all $n \in N$.

Therefore, from (1) and (2), we have $\gamma_n \longrightarrow \gamma \geq 0$, as $n \longrightarrow \infty$. ■

Theorem (3.3):

Let (X, G) be a G-metric space and $f : X \longrightarrow X$ be a mapping. If there exists $x_0 \in X$, such that $O(x_0)$ is G-bounded, and f is several contractive mapping on the orbit $O(x_0)$. Then, $\{f^n x_0\}$ is a G-Cauchy sequence in $O(x_0)$.

Proof:

Let $x_n = f^n x_0$, $n \in N$

Since f is several contractive mapping on the orbit $O(x_O)$, we have:

$$G(x_n, x_{n+p}, x_{n+p+t}) = G(fx_{n-1}, fx_{n+p-1}, fx_{n+p+t-1})$$

$\leq aG(x_{n-1}, fx_{n-1}, fx_{n-1}) + bG(x_{n+p-1}, fx_{n+p-1}, fx_{n+p-1}) + cG(x_{n+p+t-1}, fx_{n+p+t-1}, fx_{n+p+t-1})$

$= aG(x_{n-1}, x_n, x_n) + bG(x_{n+p-1}, x_{n+p}, x_{n+p}) + cG(x_{n+p+t-1}, x_{n+p+t}, x_{n+p+t})$

Taking the supremum over p and t, we get:

$\gamma_n \leq a\gamma_{n-1} + b\gamma_{n+p-1} + c\gamma_{n+p+t-1}$ (since the orbit $\{x_n\}$ is G-bounded)

Since $\{\gamma_n\}$ is nonincreasing sequence (by proposition (3.2)). Hence,

$\gamma_n \leq a\gamma_{n-1} + b\gamma_{n-1} + c\gamma_{n-1}$

$\leq (a + b + c)\gamma_{n-1}$

But, $0 < a + b + c < 1$, we have:

$(a + b + c)\gamma_{n-1} < \gamma_{n-1}$

Thus,

$\gamma_n < \gamma_{n-1}$

Taking the limit as $n \longrightarrow \infty$, we get $\gamma < \gamma$ and if $\gamma > 0$, which is a contradiction.

Hence, $\gamma = 0$, that is $\gamma_n \longrightarrow 0$, as $n \longrightarrow \infty$. Then,

$G(x_n, x_{n+p}, x_{n+p+t}) \leq (a + b + c)\gamma_{n-1} \longrightarrow 0$, as $n \longrightarrow \infty$

Hence $\{f^n x_0\}$ is a G-Cauchy sequence. ■

Theorem (3.4):

Let (X, G) be a G-metric space and $f : X \longrightarrow X$ be a several contractive mapping on the orbit $\overline{O(x_0)}$. Then f has a unique fixed point in $\overline{O(x_0)}$.

Proof:

For existence, since $O(x_0)$ is bounded and $\{x_n\}$ is a sequence in $O(x_n)$, then from Theorem (3.3), we have $\{x_n\}$ is a Cauchy sequence.

Since $O(x_0)$ is orbitally complete, there exists $p \in X$, such that $\{x_n\}$ converges to p

For $n \in N$, and since f is several contractive mapping on the orbit $\overline{O(x_0)}$, we have:

$$G(x_n, fp, fp) \leq aG(x_{n-1}, x_n, x_n) + bG(x_{n-1}, x_n, x_n) + cG(p, fp, fp) \qquad \ldots(3.2)$$

Since G jointly continuous in three variables

Taking the limit into both sides of the inequality (3.2) as $n \longrightarrow \infty$, we have:

$$G(p, fp, fp) \leq aG(p, p, p) + bG(p, p, p) + cG(p, fp\ fp)$$

Thus:

$$G(p, fp, fp) \leq cG(p, fp, fp) < G(p, fp, fp)$$

If $G(p, fp, fp) > 0$, which is not true. Thus: $G(p, fp, fp) = 0$, and then $fp = p$, and p is a fixed point of f in $\overline{O(x_0)}$

To prove the uniqueness, suppose that q is another fixed point of f in $\overline{O(x_0)}$, i.e., $fp = p$, $fq = q$

By using the property of several contraction mapping , we have:

$$G(p, q, q) \leq G(fp, fq, fq) \leq aG(p, fp, fp) + bG(q, fq, fq) + cG(q, fq, fq)$$

$$\leq aG(p, fp, fp) + (b + c)G(q, fq, fq)$$

Hence, $p = q$

Therefore, p is the unique fixed point of f in $\overline{O(x_0)}$. ■

Corollary (3.5):

Let (X, G) be a G-metric space and $f : X \longrightarrow X$ be a mapping, if there exists $x_0 \in X$, such that $O(x_0)$ is bounded and orbitally complete, where:

$$G(fx, fy, fy) \leq aG(x, fx, fx) + bG(y, fy, fy) \tag{3.3}$$

for all $x, y \in \overline{O(x_0)}$, where $0 < a + b < 1$. Then f has a unique fixed point in $\overline{O(x_0)}$.

Proof:

The proof follows directly from Theorem (3.4), by putting z = y in inequality (3.1), then we see that every mapping satisfies inequality (3.3) satisfies the inequality (3.1) on the orbit $\overline{O(x_0)}$. ■

Corollary (3.6):

Let (X, G) be a G-metric space and $f : X \longrightarrow X$ be a mapping, if there exists $x_0 \in X$, such that $O(x_0)$ is bounded and orbitally complete, where:

$$G(fx, fy, fy) \leq dG(x, y, y) \quad \ldots(3.4)$$

for all $x, y \in \overline{O(x_0)}$, where $0 < d < 1/4$. Then f has a unique fixed point in $\overline{O(x_0)}$.

Proof:

By using property (G5) of G-metric function, we have:

$$G(x, y, y) \leq G(x, fx, fx) + G(fx, y, y) \quad \ldots(3.5)$$

$$G(fx, y, y) \leq G(fx, fy, fy) + G(fy, y, y) \quad \ldots(3.6)$$

$$G(fy, y, y) \leq G(y, fy, fy) + G(fy, y, y) \quad \ldots(3.7)$$

Hence, from inequalities (3.5)-(3.7), we see that inequality (3.4) becomes:

$$\begin{aligned} G(fx, fy, fy) &\leq dG(x, y, y) \\ &\leq dG(x, fx, fx) + dG(fx, fy, fy) + 2dG(y, fy, fy) \end{aligned} \quad \ldots(3.8)$$

Then, f will satisfy the following inequality:

$$G(fx, fy, fy) \leq aG(x, fx, fx) + bG(y, fy, fy) \quad \ldots(3.9)$$

for all x, y $\in \overline{O(x_0)}$, where $a = \frac{d}{1-d}$ and $b = \frac{2d}{1-d}$.a + b < 1, since d ≤ 1/4.

Therefore, inequality (3.4) is satisfied and the proof follows from corollary (3.5). ■

Now, we prove the following theorem by supposing that the iterative sequence has a convergent subsequence:

Theorem (3.7):

Let (X, G) be a G-metric space, and f be a self map on X. If there exists a point $x_0 \in X$, such that the sequence $\{f^{n_i}x_0\}$ is a convergent sequence in X, where:

$$G(fx, fy, fz) \leq qG(x, y, z) \quad \ldots(3.10)$$

for all x, y, z ∈ X and for some 0 ≤ q < 1. Then f has a unique fixed point in X.

Proof:

For existence, suppose that $\{f^{n_i}x_0\}$ is a convergent sequence in X

Then, there exists a point t ∈ X, and $\lim_{i\to\infty} f^{n_i}x_0 = t$

To show $\lim_{i\to\infty} f^{n_i+1}x_0 = ft$, by using inequality (4.1)

$$G(f^{n_i+1}x_0, f^{n_i+1}x_0, ft) \le qG(f^{n_i}x_0, f^{n_i}x_0, t) \qquad \text{...(3.11)}$$

By taking the limit to the both sides of inequality (3.11) as i ⟶ ∞, we get $\lim_{i\to\infty} f^{n_i+1}x_0 = ft$

If ft ≠ t, there exist k ∈ N, such that if i > k, then there exist two G-open balls $B_1 = (t, \varepsilon)$ and $B_2 = B(ft, \varepsilon)$, where:

$$\varepsilon < \min\{G(t, ft, ft), G(ft, t, t)\}$$

and

$$G(f^{n_i}x_0, f^{n_i+1}x_0, f^{n_i+1}x_0) > \varepsilon, \text{ for all } i > k \qquad \text{...(3.12)}$$

From inequality (3.10), we have:

$$G(f^{n_i+1}x_0, f^{n_i+2}x_0, f^{n_i+2}x_0) \le qG(f^{n_i}x_0, f^{n_i+1}x_0, f^{n_i+1}x_0) \qquad \text{...(3.13)}$$

for all $\ell > j > k$, and by inequality (3.13), we get:

$$\begin{aligned} G(f^{n_\ell}x_0, f^{n_\ell+1}x_0, f^{n_\ell+1}x_0) &\le qG(f^{n_\ell-1}x_0, f^{n_\ell}x_0, f^{n_\ell}x_0) \\ &\le q^2G(f^{n_\ell-2}x_0, f^{n_\ell-1}x_0, f^{n_\ell-1}x_0) \\ &\;\vdots \\ &\le q^{n_\ell-n_j}G(f^{n_j}x_0, f^{n_j+1}x_0, f^{n_j+1}x_0) \end{aligned}$$

Taking $\ell \longrightarrow \infty$, we get:

$$\lim_{\ell\to\infty} G(f^{n_\ell}x_0, f^{n_\ell+1}x_0, f^{n_\ell+1}x_0) \leq 0$$

This is a contradiction with (3.12)

Therefore, ft = t and t is a fixed point of f in X.

For uniqueness, suppose that r be another fixed point of f in X

This means that ft = t and fr = r and $t \neq r$

By inequality (3.10), we have:

$$G(t, t, r) = G(ft, ft, fr) \leq qG(t, t, r), \text{ for some } 0 \leq q < 1$$

This means that:

$$G(t, t, r) \leq qG(t, t, r) < G(t, t, r)$$

Which is a contradiction if $G(t, t, r) > 0$

Hence, $G(t, t, r) = 0$ and therefore, t = r.

Then, t is a unique fixed point of f

Therefore, f has a unique fixed point in X. ■

Corollary (3.8):

Let (X, G) be a G-metric space, and f be a self-mapping on X. If there exists a point $x_0 \in X$, such that the sequence $\{f^{n_i}x_0\}$ is a convergent sequence in X, where:

$$G(fx, fz, fz) \leq qG(x, z, z) \qquad \ldots(3.14)$$

for all $x, z \in X$ and for some $0 \leq q < 1$. Then f has a unique fixed point in X.

Proof:

The proof follows from Theorem (3.7), by taking y = z in inequality (3.10), consequently f has a unique fixed point in X. ■

4. Fixed Point Theorems for Expansive Mapping in G-Metric Spaces

In this section, we will prove some theorems for fixed point theory of expansive mappings in G-metric spaces.

Analogous to [5], [6], we define the expansive mapping which defined on G-metric space:

Definition (4.1):

Let (X, G) be a G-metric space and f be a self-mapping on X. Then, f is called **expansive mapping** if there exists a constant $q > 1$, such that:

$$G(fx, fy, fz) \geq qG(x, y, z) \quad \ldots(4.1)$$

for all $x, y, z \in X$.

Theorem (4.2):

Let (X, G) be a G-metric space and f be an expansive and surjective self-mapping on X. If there exist $x_0 \in X$, such that $\{f^{n_i} x_0\}$ be a convergent sequence in X. Then f has a unique fixed point in X.

Proof:

Suppose that f is a surjective on X

To show that f is injective mapping on X

Let $x, y \in X$, such $fx = fy$

Then, $G(fx, fx, fy) = 0$

Since, f is an expansive mapping, we have:

$$G(fx, fx, fy) \geq qG(x, x, y)$$

Thus, $qG(x, x, y) \leq 0$

Hence, $G(x, x, y) = 0$ and $x = y$

Then, f is an injective mapping, but f is a surjective mapping

Thus, f is a bijective mapping

Therefore, f is an invertible mapping

Suppose that g is the inverse mapping of f

Hence:

$$G(x, y, z) = G(f(gx), f(gy), f(gz)) \geq qG(hx, hy, hz)$$

Then, we get:

$$G(hx, hy, hz) \leq pG(x, y, z)$$

for all $x, y, z \in X$, where $p = \frac{1}{q} < 1$.

Now, we have the inverse mapping g satisfies all these conditions in Theorem (3.7)

By Theorem (3.7), g has a unique fixed point u in X, $gu = u$

But, $u = f(gu) = fu$

Thus, u is also a fixed point of f in X.

Uniqueness. Suppose that v is another fixed point of f in X, $f(v) = v$, $v \neq u$

Then:

$$fv = v = f(g(v)) = g(f(v))$$

Thus, fv is another fixed point of g in X

By uniqueness of fixed point, we get:

$$v = fv = u$$

Thus, u is a unique fixed point of f in X

Therefore, f has a unique fixed point in X. ■

Corollary (4.3):

Let (X, G) be a G-metric space, and f be a surjective self-mapping on X. If there exists

$x_0 \in X$, such that $\{f^{n_i} x_0\}$ be a convergent sequence in X, where:

$$G(fx, fz, fz) \geq qG(x, z, z) \qquad \ldots(4.2)$$

for $x, z \in X$ and for some $q > 1$. Then f has a unique fixed point in X.

Proof:

Suppose that f is a surjective mapping on X

To show that f is injective mapping on X

Let $x, y \in X$, such that $fx = fy$

Then, $G(fx, fy, fy) = 0$

By using inequality (4.2), we have:

$$G(fx, fy, fy) \geq qG(x, y, y)$$

Thus, $qG(x, y, y) \leq 0$

Hence, $G(x, y, y) = 0$ and $x = y$

Then, f is an injective mapping, but f is a surjective mapping

Thus, f is a bijective mapping

Therefore, f is invertible mapping, i.e., f has inverse mapping, say, g is inverse mapping of f.

By using inequality (4.2), we see:

$$G(x, z, z) = G(f(gx), f(gz), f(gz)) \geq qG(gx, gz, gz)$$

Thus:

$$G(gx, gz, gz) \leq pG(x, z, z)$$

for all x, y, z $\in$ X, where $p = \frac{1}{q} < 1$.

By using Corollary (3.8) of Theorem (3.7), we have g has a unique fixed point, say w, in X, g(w)=w

But, w = f(g(w)) = fw

Thus, w is also a fixed point of f in X.

For the uniqueness, suppose that v another fixed point of f in X, such f(v) = v, v $\neq$ w

Then, fv = v = f(g(v)) = g(f(v))

Thus, fv is another fixed point of g in X

By uniqueness of a fixed, we get v = fv = w

Thus, w is a unique fixed point of f in X

Therefore, f has a unique fixed point in X. ■

<u>Corollary (4.4):</u>

Let (X, G) be a G-metric space, and f be a surjective self-mapping on X. If there exists

$x_0 \in X$, such that $\{f^{n_i} x_0\}$ be a convergent sequence in X, where:

$$G(fx, fy, fz) \geq q\{G(x, y, y) + G(z, y, y)\} \quad \ldots(4.3)$$

for $x, y, z \in X$ and for some $q > 1$. Then f has a unique fixed point in X.

Proof:

The proof follows from corollary (4.3), by taking y = z in inequality (4.3). ■

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