The Elliptic Property and Local Truncation Error for the Full Finite Volume Method of the Convection-Diffusion Problem

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Abstract

In this paper, a full finite volume method is studied for the two-dimensional linear convection- diffusion problem. A linear convection term is approximated by the upwind finite element method considered over a mesh to the triangular grid, whereas the linear diffusion term is approximated by using divergence theorem and approximate the direction derivative by difference quotient. The elliptic property, the discrete conservation law and local truncation error of this method are proved under some assumption on the numerical fluxes.

Key words. Finite volume method, convection-diffusion problem, elliptic property, local truncation error.

1. Introduction

In this paper we consider the following two-dimensional linear convection-diffusion initial boundary problem:

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u + b \cdot \nabla u + cu = f \qquad (x, y, t) \in \Omega \times (0, T] = D; \qquad (1.1)$$

$$u(x, y, t) = 0 \qquad (x, y, t) \in \Gamma \times (0, T]; \qquad (1.2)$$

$$u(x, y, 0) = u^o(x, y) \qquad (x, y) \in \overline{\Omega}. \qquad (1.3)$$

Where $\Omega \subset \mathbb{R}^2$is a bounded domain with polygonal boundary Γ. The positive parameter ε is called diffusion coefficient, the vector $b: D \longrightarrow \mathbb{R}^2$ is called convection coefficient and $c, f: D \longrightarrow \mathbb{R}$ are given functions. Whereas $u^o: \Omega \longrightarrow \mathbb{R}$ is given function.

Convection-diffusion processes appear in many areas of science and technology. For example, fluid dynamics, heat and mass transfer hydrology and so on. This is the reason that the numerical solution of convection-diffusion problem attracts a number of specialty. From an extensive literature devoted to linear problems, let us mentioned some papers [1], [10] and the reference therein, few approaches to the solution of nonlinear problems mentioned in the papers [5], [6], and [11].

It is a well-known fact that the use of classical Galerkin method with continuous piecewise linear finite elements leads to spurious oscillations when the local Peclet number is large. To obtain an effective scheme in the case of that convective term is dominate or the Peclet number is large, it is required to consider a suitable approximation for the convective term $b \cdot \nabla u$. The partial upwind finite element scheme is known as the method solve linear convection-diffusion problem when the convection term is dominated [3], [9] and [10]. In [7], [11] and [13] the partial upwind finite element scheme for two dimensional nonlinear convection-diffusion problem is studied. In [5] and [6] investigates a combined finite volume-finite element method for two dimensional nonlinear convection-diffusion problem which the convection term only is nonlinear.

The purpose of this paper is to investigate a full finite volume method working on unstructured meshes and preserving elliptic (coercivity) property. The paper starts with a detailed derivation of the full finite volume scheme. Firstly; we used the divergence theorem to the diffusion term on the control domain and approximate direction derivative by difference quotient. Secondary, we used the upwind finite volume scheme (see [12],[15]). The elliptic property and local truncation error of the method are proved. This paper consists sex section. In section 2, formulation of the problem and some notations. The finite volume space is defined, and the full finite volume scheme in section 3. The discrete elliptic property is proven in section 4. In section 5, the discrete mass conservation law is proven. Finally, in section 6, the local truncation error of the scheme is shown.

2. Formulation of the Problem and Some Notations

Throughout this paper, we will use C (with or without subscript or superscript) to denote generic constant independent of discrete parameter. $w_p^m(\Omega)$ denotes usual Sobolev spaces, where m, p are nonnegative integer. The corresponding norm and semi-norm are $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ [14]. Particular, for $p = 2$, $H^m(\Omega) = w_2^m(\Omega)$, the corresponding norm and semi-norm are $\|\cdot\|_{m,2,\Omega}$ and $|\cdot|_{m,2,\Omega}$ respectively. Let $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$, then

$$(u, v) = \int_\Omega uv\, dx.$$

As usual $H_0^1(\Omega) = \{v \in H^1(\Omega);\ v|_{\partial\Omega} = 0\}$ denote the subspaces of $H^1(\Omega)$.

We assume the coefficient of problem (1.1)-(1.3) satisfied the following conditions:

(A1) $b = (b_1, b_2) \in [W_\infty^1(\Omega)]^2$, $c \in W_\infty^1(\Omega)$, $f \in W_q^1(\Omega)$ with some $q > 2$,

(A2) $c - \frac{1}{2}\nabla \cdot b \geq a_o > 0$ on Ω, where a_o does not depend on ε and $x \in \Omega$.

The weak form of problem (1.1)-(1.3) is, find $u: [0, T] \longrightarrow H_0^1(\Omega)$ such that

$(u_t^n, v) + a_\varepsilon(u^n, v) = (f^n, v)$, for all $v \in H_0^1(\Omega)$,
(2.1)

$u(0) = u^o$,
(2.2)

where

$$(u_t^n, v) = \int_\Omega u_t^n \, v dx, \qquad a_\varepsilon(u^n, v) = \int_\Omega [\varepsilon \nabla u \cdot \nabla v + (b \cdot \nabla u + cu)v] dx,$$

$$(f^n, v) = \int_\Omega f^n v \, dx.$$

We assume that the weak solution u of problem (1.1)-(1.3) satisfied the following conditions:

(A3) $u \in L^\infty(0, T; H^2(\Omega)) \cap L^\infty(0, T; W_\infty^1(\Omega)),\ u_t, u_{tt}, u_{ttt} \in L^\infty(0, T; L^\infty(\Omega))$.

3. The Finite Volume Space and Full Finite Volume Scheme

Let us consider a family of regular triangulation $\{\mathcal{T}_h\}$ in $\bar{\Omega}$ (see [4]). For a fixed triangulation $\mathcal{T}_h$ we defined the mesh parameter h by $h = \max_{\mathbb{T} \in \mathcal{T}_h} h_{\mathbb{T}}$, where $h_{\mathbb{T}}$ is the diameter of the triangle $\mathbb{T}$.

We assume that the triangulation family $\{\mathcal{T}_h\}$ is regular and weakly acute type, i.e.

(A4) There exists $\alpha_o \in (0, \frac{\pi}{2})$ independent of h, such that all interior angles α of the triangles are bounded as follows:

$$\alpha \in [\alpha_o, \frac{\pi}{2}],$$

For a given triangulation $\mathcal{T}_h$ with nodes $\{x_i\} \subset \bar{\Omega}$ $(1 \le i \le K)$, where K is positive integer dependent on the triangulation, we construct a secondary partition. Namely, we introduce regions

$$\Omega_i^{\mathbb{T}} = \{x: x \in \mathbb{T}, |x - x_i| \le |x - x_j| \text{ for all } x_j \in \mathbb{T}\},$$

where $|x - x_i|$ is the distance of node x and node x_i. We consider the dual decomposition $\tilde{\mathcal{T}}_h = \{\Omega_i\}$, where Ω_i is circumcentric domain associated with nodal point x_i [9]

$$\Omega_i = \bigcup \Omega_i^{\mathbb{T}}, \quad \mathbb{T} \in \mathcal{T}_h$$

We say that two nodes x_i, x_j are adjacent if and only if $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j \neq \emptyset$. The set of indices of all interior nodes $x_i \in \Omega$ is denoted by Λ whereas the set Λ_i contain the indices of all nodal points in $\bar{\Omega}$ adjacent to $x_i \in \bar{\Omega}$. Moreover, we defined $d_{ij} = |x_i - x_j|$ and $x_{ij} = \frac{1}{2}(x_i + x_j)$. The area of Ω_i is denoted by $m_i = meas_1(\Omega_i)$ and for the length of the straight-line segment $\Gamma_{ij}(i \in \Lambda_i)$ we use the notation $m_{ij} = meas_2(\Gamma_{ij})$. If $m_{ij} > 0$, then Γ_{ij} has a uniquely defined unit outward normal ν_{ij} with respect to Ω_i.

The elements of this partition are defined as follows. Obviously, the straight-line segment Γ_{ij} and the node x_i can be regarded as an edge and the corresponding opposite vertex, respectively, of some triangle $\mathbb{T}_{ij}$. Now, an element Q_{ij} of the partition is defined by $Q_{ij} = \mathbb{T}_{ij} \square \mathbb{T}_{ji}$.
Furthermore, the triangle $\mathbb{T}_{ij}$ can be represented as the union of two triangles $\mathbb{T}_{ij}^{(k)} (k = 1,2)$, the common boundary of which is part of the edge connecting the node x_i with the node x_j (see figure(1)). We set

$$\Gamma_{ij}^{(k)} = \Gamma_{ij} \cap \mathbb{T}_{ij}^{(k)} \quad \text{and} \quad m_{ij}^{(k)} = meas_2(\Gamma_{ij}^{(k)}).$$

It is not difficult to see that Q_{ij} can also be decomposed as $Q_{ij} = Q_{ij}^{(1)} \square Q_{ij}^{(2)}$,
where $Q_{ij}^{(k)} = \mathbb{T}_{ij}^{(k)} \square \mathbb{T}_{ji}^{(3-k)} (k = 1,2)$.

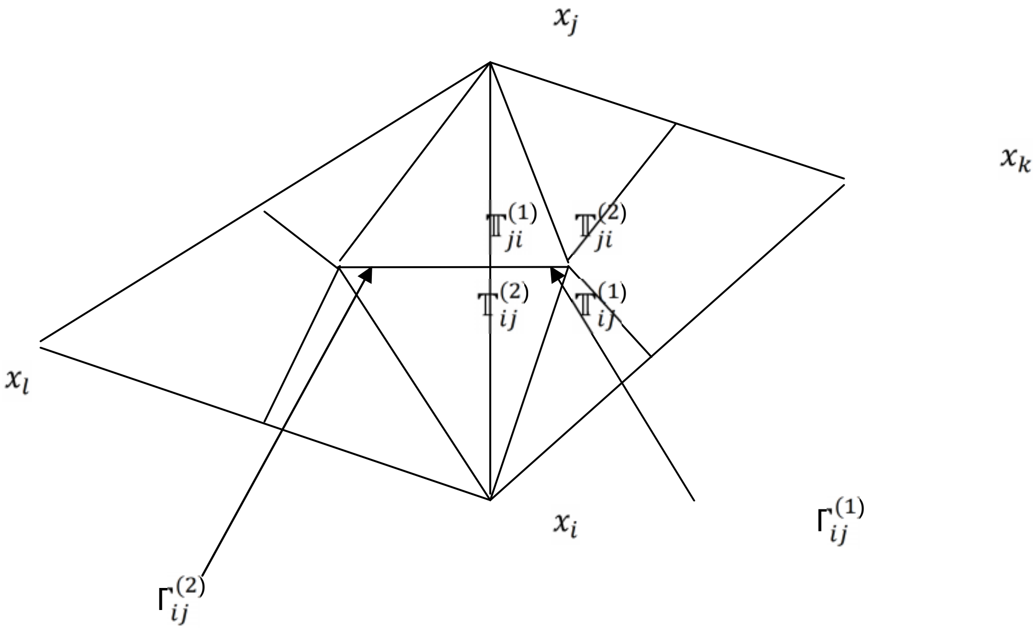

Figure (1) The auxiliary triangles $\mathbb{T}_{ij}^{(k)}$ and $\mathbb{T}_{ji}^{(k)}$

We mention the following relation:

$$meas_1(Q_{ij}^{(k)}) = 2\, meas_1(\mathbb{T}_{ij}^{(k)}) = \tfrac{1}{2} m_{ij}^{(k)} d_{ij}.$$

(3.1)

For $\ell = \{0,1,2,\dots\}$, $\mathbb{T} \square \mathcal{T}_h$ we denote by $P_\ell(\mathbb{T})$ the space of all polynomials on $\mathbb{T}$ of degree $\le \ell$. In what follows the following finite element spaces

$X_h = \{v_h | v_h \square C(\overline{\Omega}); v_h|_{\mathbb{T}} \square P_1(\mathbb{T}), \square\mathbb{T} \square \mathcal{T}_h\} \square H^1(\Omega)$,

$V_h = \{v_h | v_h \square X_h ; v_h = 0 \text{ on } \partial\Omega\} \square H_0^1(\Omega)$,

and the finite volume space

$Y_h = \{v_h | v_h \square L^2(\Omega) ; v_h|_{\Omega_i} \square P_0(\Omega_i), \square\Omega_i \square \tilde{\mathcal{T}}_h\}$.

By making use of the characteristic function $\hat{\mu}_i$ of circumcentric domain Ω_i, the mass lumping operator $\wedge$ is now defined by $\wedge : w_h \in C(\overline{\Omega}) \longrightarrow \widehat{w}_h \in Y_h$, such that

$$\widehat{w}_h(x) = \sum_{i=1}^{K} w_h(x_i)\hat{\mu}_i(x),$$

where $\widehat{w}_h(x) = w_h(x_i) = w_{hi}$.

Then we have some important lemmas:

Lemma (1) [9]. If $\mathcal{T}_h$ is regular triangulation of weakly acute type we have

$$|w_h|^2_{1,2,\Omega} \leq \frac{6}{\hat{k}^2} \| \widehat{w}_h \|^2_{0,2,\Omega} \quad ,\forall\, w_h \in X_h,$$

where $\hat{k} = \min\{\hat{k}_{\mathbb{T}};\ \mathbb{T} \in \mathcal{T}_h\}$, $\hat{k}_{\mathbb{T}}$ = minimum perpendicular length of $\mathbb{T}$.

Lemma (2) [9]. For all $w \in X_h$ with $p \geq 1$and all $w \in W^1_p(\Omega)$ with $p > 2$

$$\| \widehat{w} - w \|_{0,p,\Omega} \leq Ch|w|_{1,p,\Omega}.$$

Lemma (3) [9]. For all $w \in X_h$

$$(\nabla w, \nabla\varphi_i) = -\sum_{j \in \Lambda_i} (w_j - w_i)\frac{m_{ij}}{d_{ij}}, \qquad 1 \leq i \leq K$$

where φ_i is base function of finite element space X_h.

We turn to the derivation of the discrete scheme. We start by integrating the equation (1.1) over Ω_i and using the relation

$$b \cdot \nabla u^n = \nabla \cdot (bu^n) - (\nabla \cdot b)u^n$$

$$\int_{\Omega_i} \frac{\partial u}{\partial t}\, dx - \int_{\Omega_i} \nabla \cdot (\varepsilon \nabla u^n) dx + \int_{\Omega_i} \nabla \cdot (bu^n) dx - \int_{\Omega_i} (\nabla \cdot b)u^n\, dx + \int_{\Omega_i} cu^n\, dx = \int_{\Omega_i} f^n dx. \tag{3.2}$$

We approximate $\frac{\partial u}{\partial t}$ by the forward difference

$$\int_{\Omega_i} \frac{\partial u}{\partial t}\, dx \approx \frac{1}{\tau}(u_i^{n+1} - u_i^n)m_i, \tag{3.3}$$

where $\tau > 0$ is time step, for n =0,1,…,N_τ-1 with $N_\tau = T/\tau$.

Applying Gauss's theorem in second term of equation (3.2) we obtain

$$\int_{\Omega_i} \nabla \cdot (\varepsilon \nabla u^n) dx = \int_{\partial\Omega_i} \nu \cdot (\varepsilon \nabla u^n) ds,$$

where ν is the unit outer normal on $\partial\Omega_i$. Then, using the concrete structure of the boundary of Ω_i, we can write

$$\int_{\Omega_i} \nabla \cdot (\varepsilon \nabla u^n) dx = \sum_{j \in \Lambda_i} \int_{\Gamma_{ij}} \nu_{ij} \cdot (\varepsilon \nabla u^n) ds.$$

we approximate the direction derivatives by difference quotients

$$\nu_{ij} \cdot \nabla u^n \approx \frac{u_j^n - u_i^n}{d_{ij}}, \qquad (3.4)$$

then,

$$\int_{\Omega_i} \nabla \cdot (\varepsilon \nabla u^n) dx \approx \sum_{j \in \Lambda_i} \frac{\varepsilon}{d_{ij}} (u_j^n - u_i^n) m_{ij}. \qquad (3.5)$$

To approximate the third term

$$\int_{\Omega_i} \nabla \cdot (b u^n) dx = \sum_{j \in \Lambda_i} \int_{\Gamma_{ij}} \nu_{ij} \cdot b u^n ds$$

$$\approx \sum_{j \in \Lambda_i} \gamma_{ij} \left(r_{ij} u_i^n + (1 - r_{ij}) u_j^n \right) m_{ij}, \qquad (3.6)$$

where $\nu_{ij} \cdot b|_{\Gamma_{ij}} \approx \gamma_{ij}$ is constant, and $u^n|_{\Gamma_{ij}} \approx r_{ij} u^n(x_i) + (1 - r_{ij}) u^n(x_j)$ $r_{ij} \in [0,1]$ is a parameter and depends on ε, γ_{ij} and d_{ij}.

It remains in the left-hand side of equation (3.2), we approximate as follows:

$$\int_{\Omega_i} (\nabla \cdot b) u^n \, dx = \sum_{j \in \Lambda_i} u_i^n \int_{\Omega_i} \nabla \cdot b \, dx$$

$$= \sum_{j \in \Lambda_i} u_i^n \int_{\Gamma_{ij}} \nu_{ij} \cdot b \, dx$$

$$\approx \sum_{j \in \Lambda_i} u_i^n \gamma_{ij} m_{ij}, \qquad (3.7)$$

and

$$\int_{\Omega_i} c u^n \, dx \approx c_i u_i^n m_i. \qquad (3.8)$$

The approximation of the right-hand side in equation (3.2) is as follows

$$\int_{\Omega_i} f^n dx \approx f_i^n m_i. \qquad (3.9)$$

Thus, we obtain the following discrete version of equation (3.2) :

$$\frac{1}{\tau}(u_{hi}^{n+1} - u_{hi}^n) m_i + \sum_{j \in \Lambda_i} \frac{\varepsilon}{d_{ij}} \left(1 - (1 - r_{ij}) \frac{\gamma_{ij} d_{ij}}{\varepsilon} \right) (u_{hi}^n - u_{hj}^n) m_{ij} + c_i u_{hi}^n m_i = f_i^n m_i. \qquad (3.10)$$

Taking an arbitrary function $\upsilon_h \in V_h \subset H_0^1(\Omega)$ multiplying equation (3.10) by υ_{hi} and summing all these expression over $i \in \Lambda$, the resulting discrete problem can be written in the form

$$(D_\tau u_h^n, \upsilon_h) + a_h(u_h^n, \upsilon_h) = (\hat{f}^n, \upsilon_h) \quad \text{for all } \upsilon_h \in V_h, \tag{3.11}$$

where

$$(D_\tau u_h^n, \upsilon_h) = \sum_{i \in \Lambda} \upsilon_{hi} \left(\frac{u_{hi}^{n+1} - u_{hi}^n}{\tau}\right) m_i,$$

$$a_h(u_h^n, \upsilon_h) = \sum_{i \in \Lambda} \upsilon_{hi} \left\{ \sum_{j \in \Lambda_i} \frac{\varepsilon}{d_{ij}} \left(1 - (1 - r_{ij}) \frac{\gamma_{ij} d_{ij}}{\varepsilon}\right)(u_{hi}^n - u_{hj}^n) m_{ij} + c_i u_{hi}^n m_i \right\},$$

$$(\hat{f}^n, \upsilon_h) = \sum_{i \in \Lambda} \upsilon_{hi} \, f_i^n \, m_i.$$

Moreover, we introduce the following norm [2].

$$\| \upsilon_h \|_h = \sqrt{(\upsilon_h, \upsilon_h)_h} = \| \hat{\upsilon}_h \|_{0,2,\Omega}, \tag{3.12}$$

$$\| \upsilon_h \|_\varepsilon = \sqrt{\varepsilon |\upsilon_h|_{1,2,\Omega}^2 + \| \upsilon_h \|_{0,2,\Omega}^2}. \tag{3.13}$$

The scheme (3.11) can also be defined for control functions $r: \mathbb{R} \longrightarrow [0,1]$, where the control function r is defined as [1]

$$r(\gamma_{ij} d_{ij}/\varepsilon) = r(z) = 1 - \frac{1}{z} + \frac{1}{e^z - 1}.$$

However, we have that these functions satisfy the following properties:

(P1) $\lim_{z \to -\infty} r(z) = 0$, $\lim_{z \to \infty} r(z) = 1$,

(P2) $1 + z r(z) \geq 0$ for all z,

(P3) $[1 - r(z) - r(-z)]z = 0$ for all z,

(P4) $[\frac{1}{2} - r(z)]z \leq 0$ for all z.

For example, we can take the function $r(z) = \frac{1}{2}[\text{sign}\, z + 1]$, (see [3])

4. Discrete Elliptic Property

Lemma (4)

Let condition (A1),(A2) and (A4) be fulfilled. Then, for sufficiently small $h_o > 0$ there exists a constant $\lambda > 0$ such that for all $h \in (0, h_o]$ and $\upsilon_h \in V_h$ the relation.

$$a_h(\upsilon_h, \upsilon_h) \geq \lambda \| \upsilon_h \|_\varepsilon^2,$$

hold, where h_o can be chosen independently of ε, and λ does not depend on ε and h.

Proof

We decompose $a_h(\upsilon_h, \upsilon_h)$ into three parts as follows :

$$a_h(v_h, v_h) = \sum_{k=1}^{3} a_h^{(k)}(v_h, v_h),$$

where

$$a_h^{(1)}(v_h, v_h) = \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} \frac{\varepsilon}{d_{ij}}(v_{hi} - v_{hj})v_{hi}m_{ij},$$

$$a_h^{(2)}(v_h, v_h) = \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} [(1 - r_{ij})v_{hj} - (\tfrac{1}{2} - r_{ij})v_{hi}]\,\gamma_{ij}v_{hi}m_{ij},$$

$$a_h^{(3)}(v_h, v_h) = \sum_{i\in\Lambda} c_i\, v_{hi}^2\, m_i - \frac{1}{2}\sum_{i\in\Lambda} \sum_{j\in\Lambda_i} v_{hi}^2\, \gamma_{ij}\, m_{ij}.$$

The first term, apply lemma(3) we get

$$a_h^{(1)}(v_h, v_h) \geq \varepsilon\, |v_h|_{1,2,\Omega}^2. \tag{4.1}$$

The second term, we use a symmetry argument. Namely, changing the succession of summation and taking into consideration the boundary values of v_h, $a_h^{(2)}(v_h, v_h)$ that can be written in the following manner:

$$a_h^{(2)}(v_h, v_h) = \frac{1}{2} \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} \{[(1 - r_{ij})v_{hj} - (\tfrac{1}{2} - r_{ij})v_{hi}]\gamma_{ij}v_{hi}m_{ij}$$
$$+ [(1 - r_{ji})v_{hi} \quad - (\tfrac{1}{2} - r_{ji})v_{hj}]\gamma_{ji}v_{hj}m_{ij}\}$$
$$= \frac{1}{2} \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} \{[(1 - r_{ij})\gamma_{ij} + (1 - r_{ji})\gamma_{ji}]v_{hi}v_{hj} - [(\tfrac{1}{2} - r_{ij})\gamma_{ij}v_{hi}^2$$
$$+ (\tfrac{1}{2} - r_{ji})\gamma_{ji}v_{hj}^2\,]\}\ m_{ij}.$$

Since $r_{ij} + r_{ji} = 1$ and $\gamma_{ij} = -\gamma_{ji}$, in view of the relations

$$(1 - r_{ji})\gamma_{ji} = -r_{ij}\gamma_{ij}, \tag{4.2}$$

and

$$(\tfrac{1}{2} - r_{ji})\gamma_{ji} = (\tfrac{1}{2} - r_{ij})\gamma_{ij},$$

we get

$$a_h^{(2)}(v_h, v_h) = \frac{1}{2} \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} \{(1 - 2r_{ij})\gamma_{ij}v_{hi}v_{hj} - (\tfrac{1}{2} - r_{ij})(v_{hi}^2 + v_{hj}^2)\gamma_{ij}\}m_{ij}$$
$$= -\frac{1}{2} \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} \{(\tfrac{1}{2} - r_{ij})\gamma_{ij}(v_{hi}^2 - 2v_{hi}v_{hj} + v_{hj}^2)\}m_{ij}$$
$$= -\frac{1}{2} \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} \{(\tfrac{1}{2} - r_{ij})\gamma_{ij}(v_{hi}^2 - v_{hj}^2)^2\}m_{ij}.$$

Now, using the property (P4) then

$$a_h^{(2)}(v_h, v_h) \geq 0. \tag{4.3}$$

To estimate the remaining expression,

$$a_h^{(3)}(v_h, v_h) = \sum_{i\in\Lambda} c_i\, v_{hi}^2\, m_i - \frac{1}{2}\sum_{i\in\Lambda} \sum_{j\in\Lambda_i} v_{hi}^2\, \gamma_{ij}\, m_{ij} + \sum_{i\in\Lambda} c\ v_{hi}^2\, m_i - \sum_{i\in\Lambda} c\ v_{hi}^2\, m_i$$

$$= \int_\Omega \hat{c}\,\hat{v}_h^2\,dx - \frac{1}{2}\int_\Omega \nabla\cdot b\,\hat{v}_h^2\,dx + \int_\Omega c\,\hat{v}_h^2\,dx - \int_\Omega c\,\hat{v}_h^2\,dx$$

$$= \int_\Omega (\hat{c}-c)\,\hat{v}_h^2\,dx + \int_\Omega (c - \tfrac{1}{2}\nabla\cdot b)\,\hat{v}_h^2\,dx.$$

Thus, we can write.

$$a_h^{(3)}(v_h, v_h) = \big((\hat{c}-c)\hat{v}_h, \hat{v}_h\big) + ((c - \tfrac{1}{2}\nabla\cdot b)\hat{v}_h, \hat{v}_h)$$
$$= a_h^{(31)}(v_h, v_h) + a_h^{(32)}(v_h, v_h).$$

To estimate $a_h^{(31)}(v_h, v_h)$, we use Lemma (2)

$$|a_h^{(31)}(v_h, v_h)| \le \| \hat{c} - c \|_{0,\infty,\Omega} \| \hat{v}_h \|_{0,2,\Omega}^2$$
$$\le Ch|c|_{1,\infty,\Omega} \| \hat{v}_h \|_{0,2,\Omega}^2$$

(4.4) To estimate $a_h^{(32)}(v_h, v_h)$, we have in view of (A2)

$$a_h^{(32)}(v_h, v_h) = ((c - \tfrac{1}{2}\nabla\cdot b)\hat{v}_h, \hat{v}_h) \ge a_o \| \hat{v}_h \|_{0,2,\Omega}^2.$$

(4.5)

It follows, from equation (4.1), (4.3), (4.4) and (4.5) we obtain

$$a_h(v_h, v_h) \ge \varepsilon\,|v_h|_{1,2,\Omega}^2 + \{a_o - Ch|c|_{1,\infty,\Omega}\} \| \hat{v}_h \|_{0,2,\Omega}^2.$$

Now, it remains to choose h_o such that for all $h \in (0, h_o]$, the term $a_o - Ch_o|c|_{1,\infty,\Omega}$ becomes positive. Then,

$$a_h(v_h, v_h) \ge \lambda \| v_h \|_\varepsilon^2.$$

□

5. The Discrete Conservation Law

Theorem (1)

The numerical solution of equation (3.11) satisfies the discrete conservation law

$$\int_\Omega D_\tau u_h^n\, d\Omega = \int_\Omega \hat{f}^n\, d\Omega.$$

Proof:

Let the test function $v_h = \varphi_i$, where φ_i is a basis function of V_h and let $c = 0$ then, we can write

$$a_h(u_h^n, \varphi_i) = a_h^{(1)}(u_h^n, \varphi_i) + a_h^{(2)}(u_h^n, \varphi_i),$$

(5.1)

where

$$a_h^{(1)}(u_h^n, \varphi_i) = \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} \frac{\varepsilon}{d_{ij}}(u_{hi}^n - u_{hj}^n) m_{ij},$$

$$a_h^{(2)}(u_h^n, \varphi_i) = \sum_{i\in\Lambda} \sum_{j\in\Lambda_i} (r_{ij}\,u_{hi}^n + (1 - r_{ij})u_{hj}^n)\gamma_{ij} m_{ij}.$$

We have shown in the proof that both $a_h^{(1)}(u_h^n, \varphi_i)$ and $a_h^{(2)}(u_h^n, \varphi_i)$ have vanished. In the first term of equation (5.1), we apply a symmetry argument in the following manner:

$$a_h^{(1)}(u_h^n, \varphi_i) = \sum_{i \in \bar{\Lambda}} \sum_{j \in \Lambda_i} \frac{\varepsilon}{d_{ji}}(u_{hj}^n - u_{hi}^n) m_{ji}$$
$$= - \sum_{i \in \bar{\Lambda}} \sum_{j \in \Lambda_i} \frac{\varepsilon}{d_{ij}}(u_{hi}^n - u_{hj}^n) m_{ij}.$$

That is

$$a_h^{(1)}(u_h^n, \varphi_i) = 0, \tag{5.2}$$

applying again a symmetry argument to $a_h^{(2)}(u_h^n, \varphi_i)$ and use equation (4.2) we have

$$a_h^{(2)}(u_h^n, \varphi_i) = \sum_{i \in \Lambda} \sum_{j \in \Lambda_i} (r_{ji} u_{hj}^n + (1 - r_{ji}) u_{hi}^n) \gamma_{ji} m_{ji}$$
$$= - \sum_{i \in \Lambda} \sum_{j \in \Lambda_i} ((1 - r_{ij}) u_{hj}^n + r_{ij} u_{hi}^n) \gamma_{ij} m_{ij}.$$

Consequently, this term vanishes, too. Therefore,

$$a_h(u_h^n, \varphi_i) = 0, \tag{5.3}$$

it follows that

$$\int_\Omega D_\tau u_h^n \, d\Omega = \int_\Omega \hat{f}^n \, d\Omega.$$

□

6. Local truncation error

Let us suppose that the exact solution $u: (0, T) \longrightarrow V$ of problem (2.1)-(2.2) satisfies the conditions (A3), where u_t and u_{tt} denote the first and second derivative of the mapping $u: (0, T) \longrightarrow V$. In what follows we will denote $u^n = u(t_n) = u(\cdot, t_n)$.

Theorem (2)

Under condition (A3), for $t_n \in [0, T)$, if $\tau = O(h)$ then,

$$\text{L.T.E} \leq O(\tau^2) \qquad , \tau > 0 \tag{6.1}$$

Proof:

In equation (2.1), the exact solution u satisfies at $t = t_{n+1}$, we have

$$(u_t^{n+1}, \upsilon) + a_\varepsilon(u^{n+1}, \upsilon) = (f^{n+1}, \upsilon), \quad \text{for all } \upsilon \in V = H_0^1(\Omega) \tag{6.2}$$

where

$$a_\varepsilon(u^{n+1}, \upsilon) = (\varepsilon \nabla u^{n+1}, \upsilon) + (b \cdot \nabla u^{n+1}, \upsilon) + (c u^{n+1}, \upsilon).$$

Adding and subtracting $(\frac{u^{n+1} - u^n}{\tau}, \upsilon)$ to the above equation we get :

$$\left(\frac{u^{n+1} - u^n}{\tau}, \upsilon\right) + (u_t^{n+1}, \upsilon) + a_\varepsilon(u^{n+1}, \upsilon) = (f^{n+1}, \upsilon) + \left(\frac{u^{n+1} - u^n}{\tau}, \upsilon\right), \forall \upsilon \in V$$

setting now $\upsilon = \upsilon_h \in V_h$ and multiplied by τ, we find that

$(u^{n+1} - u^n, v_h) + \tau a_\varepsilon(u^{n+1}, v_h) = \tau(f^{n+1}, v_h) - \tau(u_t^{n+1}, v_h) + (u^{n+1} - u^n, v_h).$
(6.3)

Now, the discrete formula from equation (3.11), we can be written at $t = t_{n+1}$ and multiplied by τ:

$(u_h^{n+1} - u_h^n, v_h) + \tau a_h(u_h^{n+1}, v_h) = \tau(\hat{f}^{n+1}, v_h),$ for all $v_h \in V_h$
(6.4)

where

$$a_h(u_h^{n+1}, v_h) = \sum_{i\in\Lambda}\sum_{j\in\Lambda_i} v_{hi}\,(u_{hi}^{n+1} - u_{hj}^{n+1})\frac{\varepsilon}{d_{ij}}m_{ij} + \sum_{i\in\Lambda}\sum_{j\in\Lambda_i} v_{hi}\,(1 - r_{ij})(u_{hj}^{n+1} - u_{hi}^{n+1})\gamma_{ij}m_{ij} + \sum_{i\in\Lambda} v_{hi}c_i u_{hi}^{n+1}m_i.$$

By subtracting equation (6.3) from (6.4) and using Lemma (3) we get

$$(u_h^{n+1} - u_h^n, v_h) - (u^{n+1} - u^n, v_h) + \tau[(\varepsilon\nabla u_h^{n+1}, \nabla v_h) + \sum_{i\in\Lambda}\sum_{j\in\Lambda_i} v_{hi}\,(1 - r_{ij})(u_{hj}^{n+1} - u_{hi}^{n+1})\gamma_{ij}m_{ij} + (cu_h^{n+1}, v_h) - (\varepsilon\nabla u^{n+1}, \nabla v_h)$$
$$-(b\cdot\nabla u^{n+1}, v_h) - (cu^{n+1}, v_h)] = \tau[(\hat{f}^{n+1}, v_h) - (f^{n+1}, v_h)] + \tau(u_t^{n+1}, v_h) - (u^{n+1} - u^n, v_h),$$
(6.5)

adding and subtracting $\tau(b\cdot\nabla u_h^{n+1}, v_h)$ to the equation (6.5) we have

$$(u_h^{n+1} - u^{n+1}, v_h) - (u_h^n - u^n, v_h) + \tau[(\varepsilon\nabla(u_h^{n+1} - u^{n+1}), v_h)] + \tau[(b\cdot\nabla(u_h^{n+1} - u^{n+1}), v_h)] + \tau[(c(u_h^{n+1} - u^{n+1}), v_h)] = \sum_{k=1}^{3} M^{(k)},$$

where

$M^{(1)} = \tau(\hat{f}^{n+1} - f^{n+1}, v_h),$

$M^{(2)} = \tau(u_t^{n+1}, v_h) - (u^{n+1} - u^n, v_h),$

$M^{(3)} = \tau[(b\cdot\nabla u_h^{n+1}, v_h) - \sum_{i\in\Lambda}\sum_{j\in\Lambda_i} v_{hi}\,(1 - r_{ij})(u_{hj}^{n+1} - u_{hi}^{n+1})\gamma_{ij}m_{ij}.$

To estimate $M^{(1)}$, we use Lemma (2)

$$|M^{(1)}| = |\tau(\hat{f}^{n+1} - f^{n+1}, v_h)| \le \tau\,\|\hat{f}^{n+1} - f^{n+1}\|_{0,q,\Omega}\|v_h\|_{0,2,\Omega} \le C\tau h|f^{n+1}|_{1,q,\Omega}\|v_h\|_{0,2,\Omega} \le C\,\tau h\,\|v_h\|_{0,2,\Omega}.$$
(6.6)

To estimate $M^{(2)}$,

$$|M^{(2)}| = |(u^{n+1} - u^n, v_h) - \tau(u_t^{n+1}, v_h)| = |(u(t_{n+1}) - u(t_n), v_h) - \tau(u_t(t_{n+1}), v_h)|.$$

Using Taylor's Theorem with integral remainder, such that

$$u(t_{n-1}) = u(t_n) + u_t(t_n)(t_{n-1} - t_n) + \int_{t_n}^{t_{n-1}} u_{tt}(t)(t_{n-1} - t)dt,$$

Put $n = n + 1$,

$$u(t_n) = u(t_{n+1}) + u_t(t_{n+1})(t_n - t_{n+1}) + \int_{t_{n+1}}^{t_n} u_{tt}(t)(t_n - t)dt,$$

$$u(t_{n+1}) - u(t_n) + u_t(t_{n+1})(t_n - t_{n+1}) = -\int_{t_{n+1}}^{t_n} u_{tt}(t)(t_n - t)dt,$$

since $\tau = t_{n+1} - t_n$

$$u(t_{n+1}) - u(t_n) - \tau u_t(t_{n+1}) = \tau \int_{t_n}^{t_{n+1}} u_{tt}(t)dt,$$

$$(u(t_{n+1}) - u(t_n), \upsilon_h) - \tau(u_t(t_{n+1}), \upsilon_h) = \tau(u_t(t_{n+1}) - u_t(t_n), \upsilon_h),$$

$$(u(t_{n+1}) - u(t_n), \upsilon_h) - \tau(u_t(t_{n+1}), \upsilon_h) = \tau^2(\frac{u_t(t_{n+1}) - u_t(t_n)}{\tau}, \upsilon_h).$$

Taking into account that

$$(u_t(t_{n+1}) - u_t(t_n), \upsilon_h) = \int_{t_n}^{t_{n+1}} (u_{tt}(t), \upsilon_h)dt,$$

we see that

$$(u(t_{n+1}) - u(t_n), \upsilon_h) - \tau(u_t(t_{n+1}), \upsilon_h) = \tau^2(u_{tt}(t), \upsilon_h).$$

Then

$$|M^{(2)}| \leq \tau^2 \| u_{tt}(t) \|_{0,2,\Omega} \| \upsilon_h \|_{0,2,\Omega}$$
$$\leq C\,\tau^2 \| \upsilon_h \|_{0,2,\Omega}.$$

(6.7)

To estimate $M^{(3)}$, we use the proof of $A^{(32)}$ and $A^{(33)}$ in (Theorem 2, [8]), we get

$$| M^{(3)}| \leq C\,\tau h |u_h^{n+1}|_{1,2,\Omega} \| \upsilon_h \|_{0,2,\Omega}$$
$$\leq C\,\tau h \| \upsilon_h \|_{0,2,\Omega}.$$

(6.8)

Now, from equation (6.6), (6.7) and (6.8) we have

$$\text{L.T.E} \leq C\,\tau h \| \upsilon_h \|_{0,2,\Omega} + C\,\tau^2 \| \upsilon_h \|_{0,2,\Omega}$$
$$\leq O(\tau^2) \quad , \quad \tau > 0.$$

□

7. conclusion

In this paper we saw that the bilinear form $a_h(u_h^n, \upsilon_h)$, represent the full finite volume scheme of the convection-diffusion problem satisfied the important property so-called elliptic property, and also satisfied the discrete conservation law of the scheme. Another issue the local truncation error coincide with error estimate(Theorem 2, [8]) of order (τ^2).

المستخلص

في هذا البحث قمنا بدراسة طريقة الحجم التام المحددة لمسألة الحمل والانتشار الخطية ذات البعدين0 حد الحمل الخطي تم تقريبه بواسطة طريقة أبوند (upwind) للعنصر المحدد على شبكة التثليث بينما حد الانتشار الخطي تم تقريبه بواسطة استخدام نظرية التباعد (divergence theorem) وتقريب المشتقة الاتجاهية بواسطة الفروقات المحددة0 ثم برهنا الخاصية الاهليجية لنظام التقطيع وخاصية الحفظ المتقطع وخطأ البتر المحلي تحت بعض الشروط للفيض العددي.

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