Analysis of the Effect of Randomness on the Performance of Lorenz Model using Lyapunov Exponent

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Abstract:

The effect of the choice of a cloud of initial conditions on the behavior of Lorenz model is studied. Attractors are generated by the usual method and by the snapshot, results of which prove the invariance of Lorenz model under the effect of randomness.

Keywords:

Randomness; Lorenz Model;Lyapunov Exponent

Introduction:

The model proposed by Lorenz [1] gained wide intreset and continues to do so since it was first derived in 1963, this intrest is due to the high similarity discovered by Haken [2] in 1975 between the Lorenz and Maxwell-Bloch models, as it was notised that lasers can behave in a manner similar to fluids and this type of instability was given the term"Chaos". In 1988 and the following years N.B.Abraham [3] proposed a series of research papers which linked the two models and the results obtained led to the possibility of different dynamical behaviours, which can be used in the study of the dynamics of laser equipment in general [4] .

In this study an investigation was carried out of the randomness of attractor dynamics resulting from solution of Lorenze's equations under the influence of a cloud of initial conditions based on the concepts by Romeiras et.al [5]. Since the system attractor may not provide sufficient information on the state of the dynamic system, hence the result concerning the state of the system is inconclusive as to whether the system is stable, periodic, or chaotic. Therefore Lyapunov exponents [6] are used to analyse the results of this influence since they give the magnitude of the divergence in convergent paths and are essential in the study of chaotic dynamics of dynamic systems [7].

Lorenz Model:

The Lorenz model is considered a simplification of the description of turbulence in fluid dynamics, especially in how the states of initially heated systems transform, which are characterized by complete irregularity transforming into very clear sudden regularity when the system departs from thermal equilibrium . The system of Lorenz equations is a simplification of Salzman's equations [1] for the study of convection with small capacities, as it is possible to write two equations for the stream function in two-dimensional flow and deviation from temperature compared to the case of no convection.

Lorenz performed a truncation of the resulting series to three terms only and obtained the following system of equations which are known as Lorenz's equations [9]:

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= (r - z)x - y \\ \dot{z} &= xy - bz \end{aligned} \qquad \textbf{(1)}$$

In the above equations x,y,z represent the proportion with the convective motion and the temperature gradient between the upward and downward currents and the distortion that affects the vertical temperature distribution for a regular strip of the fluid. The dot in the given equations refers to the first derivative with respect to time, $\sigma = k^{-1}v$ is prandtl's number where v and k are the kinematic viscosity and thermal conductivity, respectivly $r = \frac{R_a}{R_c}$, where R_c represents the critical value for Rayleigh's number. Finally $b = 4(1 + a^2)^{-1}$ is a constant and when $a^2 = \frac{1}{2}$ then $R_c = \frac{27}{4}\pi^4$ is a minimum.

Solution of the Lorenz model:

It is known that the fourth-order Runge-Kutta msthod is used for the solution of Lorenz's equations in order to determine the behavior with time of the three variables x,y,z in the usual method where the relation is drown between x and y, x and z, y and z to produce attractors which describe the dynamics of the system .

In this study an attempt was made to apply the problem of random effects and their influence on the resulting attractors by selecting a cloud of initial conditions and obtaining the results for each of x,y and z for a specified period of the time of integration and then selecting one instant (snapshot) to record the values of x,y and z for one initial condition (an initial value for x,y and z) .

In addition another initial condition is selected randomly withen the domain

$$x_0 = (-1,1)$$
$$y_0 = (-1,1)$$
$$z_0 = (-1,1)$$

Where $x_0, y_0 and z_0$ represent the initial conditions. This process is repeated for 10000 initial condition(x_0, y_0, z_0). The process is similar to taking a snapshot of the behavour of $x, y \, and \, z$ with time and studying its influence on the resulting attractor in the known classical form. The process is repeated for the values of $r, \sigma \, and \, b$ shown in table (1).

r	σ	b
50	**4**	**0.6**
50	**4.5**	**0.4**
50	**4.5**	**0.6**
50	**5**	**0.4**
100	**4.5**	**0.6**
100	**5**	**0.2**

Table (1) values of r, σ, b used in study

Methods used to Determine the State of the Dynamical System:

There are many methods used to determine the behavior of a dynamical system, of these methods:

1- Plotting the time behavior with its variables such as x,y and z .

2- Plotting the attractor as a relationship x and y, x and z, y and z.

3- Finding out the power spectrum for the time behavior for each of x,y and z .

4- Finding out Poincare's sections.

5- Determination of Lyapunov's exponent which is considered to be one of the useful methods to distinguish between chaotic and non-chaotic dynamics, as it points to the exponential separation of adjacent paths and measures the rate of growth of linear distances and places amaximum for the time of expectation for the system as the positive value signifies that the system has entered to a state of chaos, whereas a negative or zero value means that the system is periodic or stable, it is possible to calculate the Lyapunov exponent from the following expression [10]:

$$d(t) = d_0 e^{\lambda t} \qquad (2)$$

Which means that when two adjacent paths on a chaotic attractor start to separate a distance d_0 at time t=0, they diverge such that their separation is $d(t)$ at time t. From equation (2) we obtain:

$$\lambda = \frac{1}{t} \ln \frac{d(t)}{d_0} \qquad (3)$$

Equation (3) is considered as the definition of Lyapunov exponent for this model by selecting a cloud of initial conditions in a program for calculating Lyapunov exponent to obtain a Lyapunov spectrum which is considered a useful index for determination of the state of the system espescially when the time of integration is short and it is not possible to depend on the shape of the attractor only, but it is preferable to consult the results of this exponents. The attractors and Lyapunov exponent with time have been drown for all the cases.

Results and Discussion:

In order to obtain results the following is done:

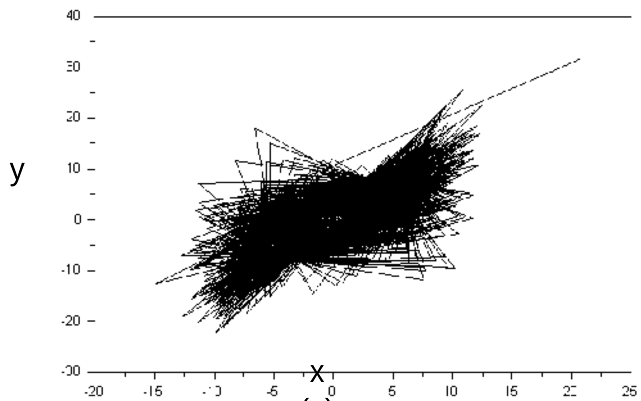
Selection of the initial conditions refered to in the solution of Lorenz's model, then selecting values for each of x,y,z at a specific time during the integration process and plotting the results to obtain the attractor, the prediction informs as that there will be no attractor due to the randomness of the process, however what actually was obtained were attractors with high similarity to the attractors obtained by ordinary integration for all the selected values shown in table(1) as depicted in fig.s (1-6) .

Each of the six figures is divided into two parts: a-the classical attractor and b-the snapshot attractor in c**-** the Lyapunov exponent where the positive values indicate that the system has started to enter the state of non-periodicity or chaos or is actually there. Inspection of the Lyapunov values show that the system for the cases considerd is within the states of non-periodicity and chaos for positive non-zero Lyapunov values.

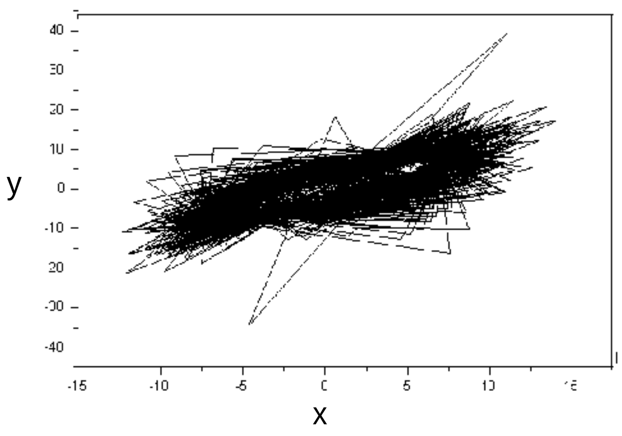
It is evident from comparing figures (a) and (b) for each case that Lorenz's model is relatively stable for random effects.The proposed technique is equivalent to producing a snapshot attractor[5],selection of a large number of initial conditions and continuation of the integration process for a specific value of time t then plotting the resulting points which is equivalent to taking a snapshot of the evolution with time for a set of points in the mode space which results in a new attractor however the general shape is not different than Lorenz's known attractor,also the result of Lyapunov's spectrum indicate different behaviours according to the values given in table(1).

Conclusion:

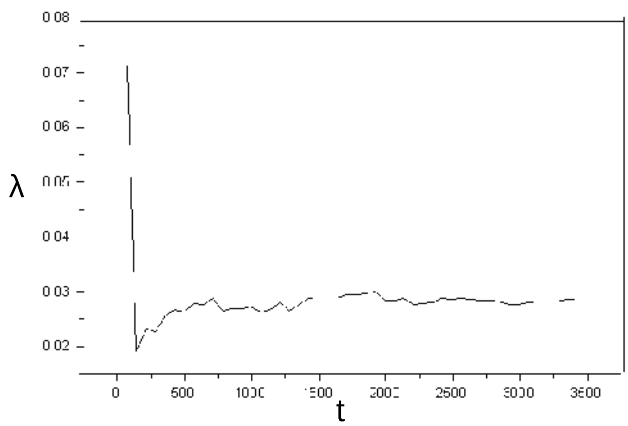
It is concluded that the Lorenz model is highly stable when exposed to an external randomness in the form of the random selection of a large number of initial conditions, this randomness led to results similar to the usual method used for producing attractors, and here attractors are meant to be those which result from plotting the relation between two or more of the variables for any system. This conclusion is clearly shown upon the calculation of the Lyapunov index which is the subject of this research.



(a)

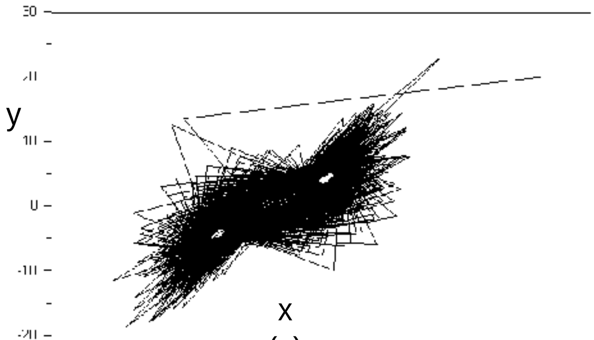


(b)

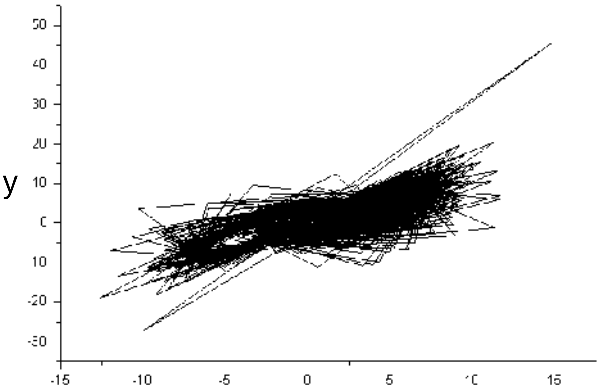


(c)

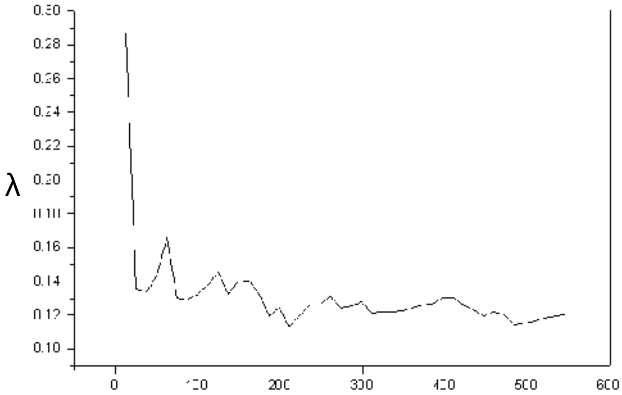
Fig.(1) Solution results for values σ=4, r=50, b=0.6



(a)



(b)



(c)

Fig.(2) Solution results for values σ=4.5, r=50, b=0.4

a-Classical attractor.

b-Snapshot attractor.

c-Lyapunov exponent (λ).

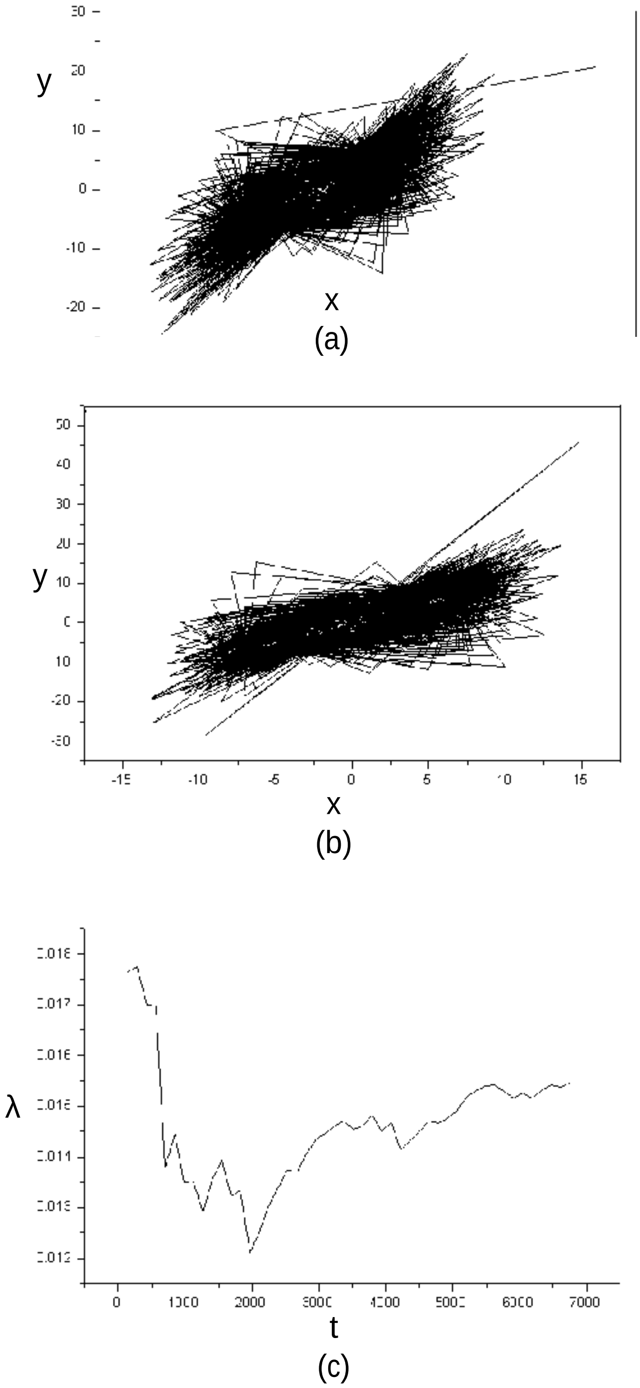
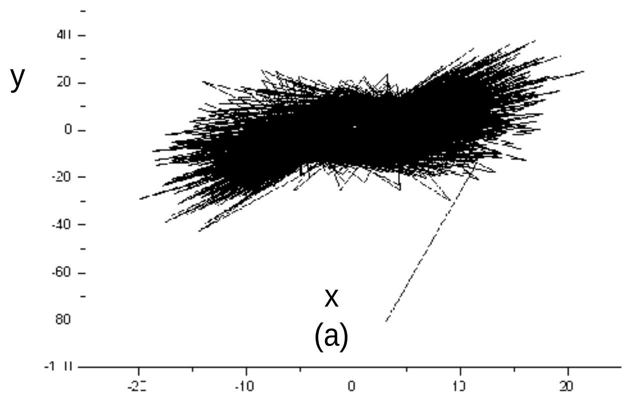
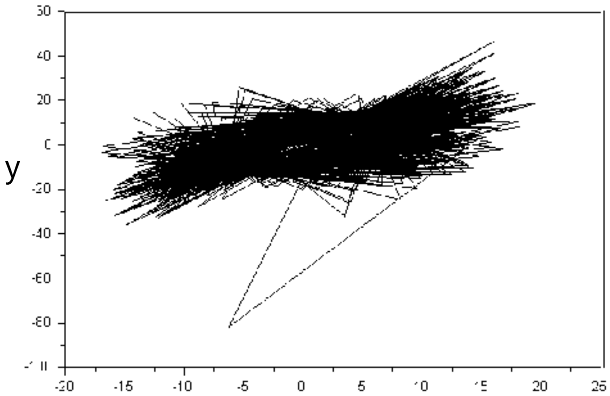


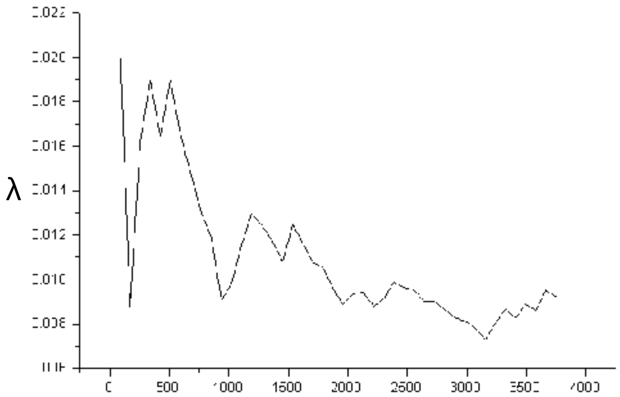
Fig.(3) Solution results for values σ=4.5, r=50, b=0.6
a-Classical attractor.
b-Snapshot attractor.
c-Lyapunov exponent(λ).



(a)



(a)



(c)

Fig.(4) Solution results for values σ=4.5, r=100, b=0.6

a-Classical attractor.

b-Snapshot attractor.

c-Lyapunov exponent(λ).

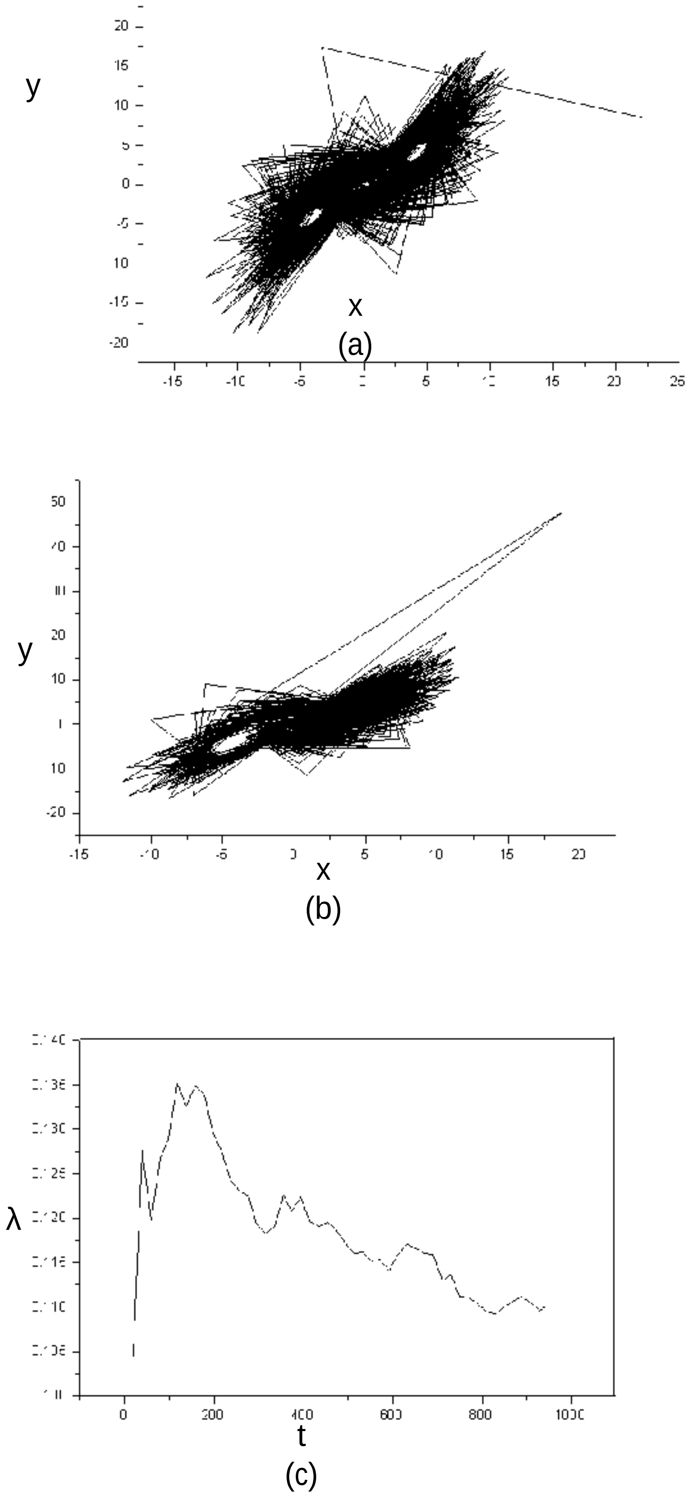
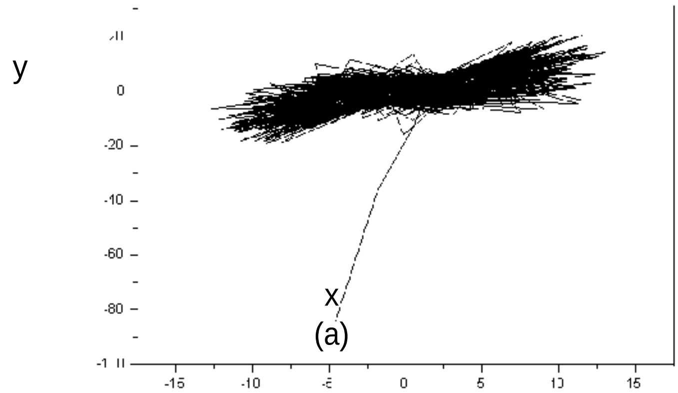
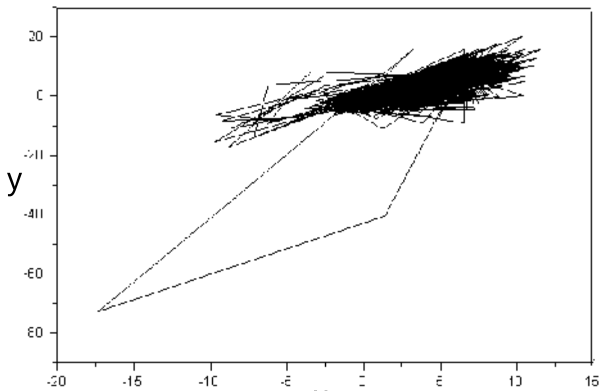


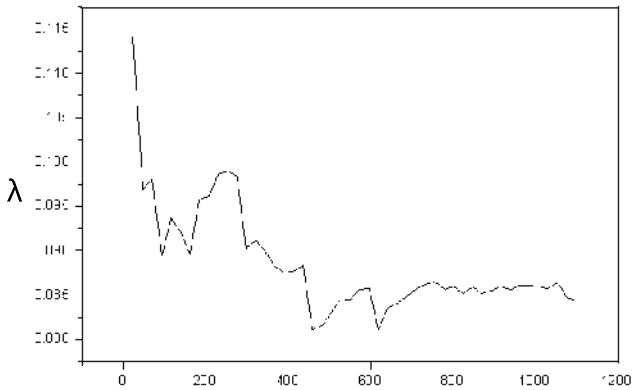
Fig.(5) Solution results for values σ=5, r=50, b=0.4
a-Classical attractor.
b-Snapshot attractor.
c-Lyapunov exponent(λ).





x

(a)



t

(c)

Fig.(6) Solution results for values σ=5, r=100, b=0.2

a-Classical attractor.

b-Snapshot attractor.

c-Lyapunov exponent(λ).

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أداء أنموذج لورنز تأثيرعشوائية تحليل نتائج

بأستعمال دليل ليابونوف

رنا داود الحلفي

قسم الرياضيات/ كلية العلوم /**جامعة البصرة**

البصرة /العراق

الخلاصة :

دراسة تأثير اختيار غيمة من الشروط الابتدائية على تصرف انموذج لورنز , اذ رسمت الجاذبات بالطريقة التقليدية وبطريقة اللقطة , وأثبتت النتائج صمود هذا الأنموذج تحت تأثير العشوائية وتبين ذلك ايضا من خلال حساب ورسم دليل ليابونوف.

الكلمات المفتاحية : العشوائية ; أنموذج لورنز; دليل ليابونوف