On Spectrum of unperturbed Second Derivative Operator

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Abstract:

Spectrum of the unperturbed second derivative operator is investigated. Spectral asymptotics for operator is calculated. It is proven that the gaps between the spectral bands disappear when $n \to \infty$.

Key words: Spectral asymptotics, differential operator, spectral theory.

المستخلص:

طيف المشتقة الثانية للمؤثر غير المضطرب قد تم توضيحه. كما حسبت المحاذيات الطيفية للمؤثر. أثبتنا بأن الفجوات بين الحزم الطيفية تختفي عندما $n \to \infty$.

1. Introduction, Definition of the operator

Differential and pseudo differential operators are widely used in applications to quantum and atomic physics to produce exactly solvable models of complicated physics phenomena[1, 2, 4]. The application of this method to solid state physics are of particular interest since these models reproduce the geometry of the problem extremely well [1].

Each operator H is a self-adjoint extension of the unperturbed second-derivative operator $H_o = -\frac{d^2}{dx}$ restricted to the set of functions from $W_2^2(R)$ vanishing in a neighborhood of the point $x = n$ is explained in [5] and they using Bloch's theorem in their work. In [2] they described periodic one dimensional Schrödinger operators, with the property that the widths of the forbidden gaps increase and the gap to band ratio is not Small. Such systems can be realized by periodic array of geometric scatterers. In [4] They show that the spectrum of the generalized kroing-Benny model has always infinitely many gaps.

Definition an operator: The unperturbed second derivative operator is;

$$H_o = -\Delta$$

Where Δ is the Lap lace operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

With $D(H_o) = L_2(\Re^n)$ satisfying the boundary condition

$$\begin{pmatrix} \Psi(0^+) \\ \Psi'(0^+) \end{pmatrix} = \Lambda \begin{pmatrix} \Psi(0^-) \\ \Psi'(0^-) \end{pmatrix} \text{ where } \Lambda = e^{i\theta} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \theta \in [0,2\pi).$$

The main aim of this work is to study the spectral asymptotics for the operator H_o. In section two we described the transfers matrix and dispersion relation. This relation is used to calculate the spectral bands. In particular it is proven that the gaps between the spectral bands disappear when $n \to \infty$.

2. Transfer Matrix and Dispersion relation

The transfer matrix for the operator H_o in the interval (0,1) is defined in [3] as following:

$$M(0,1) = \begin{pmatrix} \cos k & -k \sin k \\ \frac{1}{k} \sin k & \cos k \end{pmatrix}, \text{ where } k = \sqrt{\lambda}$$

The characteristic determinant of the transfer matrix is;

$$\det(M^{\lambda} - \lambda \mathrm{I}) = \lambda^2 - \lambda TrM^{\lambda} + \det M^{\lambda}$$

$$= \lambda^2 - \lambda TrM^{\lambda} + 1$$

since $\det M^{\lambda} = 1$.

The spectrum of the operator H_o coincides with the set of λ for which the zeroes of the characteristic determinant are non-real [8],

i.e. $\left| TrM^{\lambda} \right| \le 2$,

then $\left| 2\cos k \right| \le 2$;

Suppose that $f(k) = 2\cos k$;

The equality above is called the dispersion relation which determine the form of the spectrum for the operator.

Theorem[6]:

The unperturbed second derivative operator is self-adjoint and its spectrum is pure absolutely continuous covers the interval $[0,\infty)$.

3. Spectral Asymptotics for operator

In this section, the spectral asymptotics of the operators H_o will be studied in detail. To calculate the spectral asymptotics for the operator above we must prove the following lemma.

Lemma(3.1)

The spectrum of the operator H_o consists of an infinite number of bands $\Delta_n = [a_n^2, b_n^2]$ situated for sufficiently large n inside the intervals $[\pi^2 n^2, \pi^2 (n+1)^2]$. The asymptotics of the band edges are;

$$a_n = n\pi + O(\frac{1}{n}).$$

$$b_n = (n+1)\pi$$

The length $|\Delta_n|$ and the middle point m_n of the band Δ_n are asymptotically given by: $|\Delta_n| = 2\pi^2 n + \pi^2 + O(\frac{1}{n}),\ as\ n \to \infty$.

$$m_n = \pi^2 n^2 + \pi^2 n + \frac{\pi^2}{2} + O(\frac{1}{n}),\ as\ n \to \infty .$$

Proof:

We first prove that exactly one band Δ_n of the absolutely continuous spectrum is situated in each interval $I_n = [\pi^2 n^2, \pi^2 (n+1)^2]$ for large enough values of k ,

since $f(k) = 2\cos k$

then the values of the end points of each interval I_n,

$$\boldsymbol{f}(\boldsymbol{n}\pi) = 2\cos(\boldsymbol{n}\pi)$$
$$= (-1)^{\boldsymbol{n}} 2$$

implies that $k = n\pi$, where $\boldsymbol{n} = 0, \mp 1, \pm 2, \ldots$

therefore each interval I_n contains exactly one extreme point for the function f

when $n \to \infty$, since f is continuous and monotonous between the extreme points;

It follows that, there is precisely one interval where $|f(k)| \le 2$ in each I_n if n is sufficiently large;

Since the points a_n and b_n are close to πn and $\pi(n+1)$, respectively, the following representations can be used:

$a_n = n\pi + \alpha_n$, $b_n = (n+1)\pi + \beta_n$, where $\boldsymbol{\alpha_n}, \boldsymbol{\beta_n}$ are constant.

the equation for the left end point,

$$f(k) = 2\cos k$$
$$(-1)^n 2 = 2\cos(n\pi + \alpha_n)$$
$$(-1)^n 2 = 2[\cos(n\pi)\cos\alpha_n - \sin(n\pi)\sin\alpha_n]$$

keeping the first terms of the perturbation theory [7], we get:

$$\alpha_n = O(\frac{1}{n}), \ as\ n \to \infty,$$

it follows that $a_n = n\pi + O(\frac{1}{n})$, $as\ n \to \infty$.

In the same way, the equation for the right point,

$$(-1)^n 2 = 2\cos((n+1)\pi + \beta_n),$$

keeping the first terms of the perturbation theory, we get:

$$\beta_n = 1,$$

then $\quad b_n = (n+1)\pi$

since the length and the middle point of the band Δ_n are given by:

$$|\Delta_n| = b_n^2 - a_n^2 \quad , \quad m_n = \frac{a_n^2 + b_n^2}{2},$$

then

$$\left|\Delta_n\right| = 2n\pi^2 + \pi^2 + O(\frac{1}{n})$$

$$m_n = \frac{\pi^2 n^2 + \pi^2 n^2 + 2n\pi^2 + \pi^2}{2}$$

$$= \pi^2 n^2 + \pi^2 n + \frac{\pi^2}{2} + O(\frac{1}{n}) \quad \blacksquare$$

Theorem(3.2)

If the spectrum of the operator H_o with periodic point interactions consists of an infinite number of bands Δ_n of the absolutely continuous spectrum, then the gaps between the spectral bands disappear when $n \to \infty$.

Proof:

Since $a_n = n\pi + O(\frac{1}{n})$, $as\ n \to \infty$ and $b_n = (n+1)\pi$;

Then the length of the gap G_n between the bands with the numbers n and $n+1$ can be calculated as follows:

$$\left|G_n\right| = a_{n+1}^2 - b_n^2$$

$$= \pi^2(n+1)^2 + O(\frac{1}{n}) - \pi^2(n+1)^2$$

then $\left|G_n\right| = O(\frac{1}{n}) \quad as\ n \to \infty$;

Hence the gaps between the spectral bands disappear when $n \to \infty$. $\blacksquare$

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