Strongly Regular Spaces
A New Separation Axiom

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Abstract
In this paper we define the strongly regular spaces and we show that these spaces give the regular spaces but the converse is not true in general. Morever we prove that these spaces satisfy the topological property and the hereditary property . Also we study the relation between the strongly regular spaces and the connectednees.

الفضـــاءات المنتظمـــــة بقـــوه

الخلاصة
في هذا البحث عرفنا الفضاءات المنتظمة بقوه وبينا بأن هذه الفضاءات تعطي الفضاءات المنتظمة لكن العكس غير صحيح بصورة عامه .اضافة الى ذلك برهنا بأن هذه الفضاءات تحقق الخاصية التبولوجية و الخاصية الوراثيه. كذلك درسنا العلاقة بين الفضاءات المنتظمة بقوه والترابط.

1 Introduction
This paper gives a new separation axiom (strongly regular spaces) .

Section two includes the fundamental topological concepts such homeomorphism, topological property, hereditary property, connectedness, and regular spaces with lemma that we need them in this paper.

In section three we give the main results of this paper which are the definition of the strongly regular spaces with examples. Also we have proved the following theorems:

i) Every strongly regular space is regular but the converse is not true.
ii) Every not an indiscrete strongly regular space is disconnected.
iii) The strongly regular spaces satisfy the topological property and the hereditary property.

2 Fundamental Concepts
2.1 Definition:- Willard (1970) mentioned the following definition:
Let (X,T) and (Y,S) be topological spaces .A bijective function $f : X \to Y$ is called a homeomorphism when both f and f^{-1} (the inverse of f)are continuous .If such a function exists , then (X,T) and (Y,S) are called homeomorphic .

2.2 Examples : Let T_U denote the usual topology on the real numbers IR :

1-(a,b) as a subspace of (IR,T_U) is homeomorphic to (IR,T_U) .

2-(IR,T_U) is not homeomorphic to (IR,I) where I is the indiscrete topology on IR .

2.3 Definition: Sharma (1977) mentioned the following definition:
A property of a topological space (X,T) is said to be a topological property if any a topological space (Y,S) which is homeomorphic to (X,T) has that property.

2.4 Definition: Sharma (1977) mentioned the following definition:
A property of a topological space (X,T) is said to be a hereditary property if any subspace of (X,T) has that property.

2.5 Definition: Klaus Jänich (1984) mentioned the following definition:
A topological space (X,T) is called connected if it is not the union of two non-empty ,open ,disjoint subspaces of (X,T) :or in other words, X and the empty set ϕ are the only subsets which are clopen(closed and open) .Otherwise (X,T) is called disconnected.

2.6 Examples

1-(IR, T_U) is a connected space .

2-The subspace $X = [0,1] \cup (2,3) \subset IR$ is disconnected .

2.7 Definition:

Sharma (1977) mentioned the following definition:
A topological space (X,T) is called regular iff it satisfies the following axiom :
For any closed subset F of X and $x \notin F$,then there exists disjoint open sets G and H s.t. $F \subseteq G$ and x $\in$ H .

2.8 Definition:

Sharma (1977) mentioned the following definition:
Let (X,T) and (Y,S) be topological spaces and $f : X \to Y$ be a function. Then
i) f is said to be an open function iff f (G) is open in Y whenever G is open in X.
ii) f is said to be a closed function iff f (G) is closed in Y whenever G is closed in X.

2.9 Lemma:

Hu (1969) mentioned the following theorem :
Let (X,T) and (Y,S) be topological spaces . For any bijective function $f : X \to Y$ the following statements are equivalent :
i)The inverse function $f^{-1} : Y \to X$ is continuous . ii) f is open . iii) f is closed .

3. The Main Results

3.1 Definition :

A topological space (X,T) is said to be a strongly regular space iff it is satisfies the following axiom:
If F is a closed subset of X and $x \notin F$,then there exists a clopen (closed and open) subset G of X s.t. $x \in G$ and $G \cap F = \phi$.

3.2 Examples

1) Every discrete space is a strongly regular space.
2) Every indiscrete space is a strongly regular space.
3) Let X={a,b,c} and T={X, ϕ,{a},{b},{a,b}},then (X,T) is not a strongly regular space.

3.3 Theorem:- Every strongly regular space is a regular space.

proof : Let (X,T) be a strongly regular space , F be a closed subset of X and $x \notin F$.
Since (X,T) is a strongly regular space, then there exists a clopen subset G of X s.t.
$x \in G$ and $G \cap F = \phi$

$\Rightarrow$ F is contained in the open subset G^C of X .

Since $G \cap G^C = \phi$. then (X, T) is a regular space .

3.4 Remark : The converse of theorem (3.3) is not true :

Note that (IR, T_U) is a regular space but it is not a strongly regular space .

3.5 Theorem : Let (X,T) be not an indiscrete topological space . If (X,T) is a strongly regular space , then it is disconnected .

Proof:
Since (X,T) is not an indiscrete topological space, then there exists a non –empty closed subset F of X .
Now if $x \notin F$, then there exists a clopen subset G of X s.t. $x \in G$ and $G \cap F = \phi$.
$G \neq X$ since $G \cap F = \phi$ also $G \neq \phi$
Thus (X,T) is disconnected .

3.6 Examples :
1- Since (IR, T_U) is connected ,then it is not a strongly regular space (theorem 3.5).
2-Let $X = \{a,b,c,d\}$ and $T = \{X, \phi, \{a,b\}, \{c,d\}\}$, since (X,T) is a strongly regular space , then it is disconnected (theorem 3.5).

3.7 Theorem : A strongly regular space is a topological property .
Proof: Let (X,T) be a strongly regular space and (Y,S) be a topological space which is homeomorphic to (X,T) . Then there exists a homeomorphsim $f : X \to Y$.
Now, let F be a closed subset of Y and $y \notin F$. $\Rightarrow f^{-1}(F)$ is a closed subset of X and $f^{-1}(y) = x \notin f^{-1}(F)$.
Since (X,T) is a strongly regular space , then there exists a clopen subset G of X s.t.

$$x \in G \text{ and } G \cap f^{-1}(F) = \phi \Rightarrow y = f(x) \in f(G) \text{ also } f(G \cap f^{-1}(F)) = f(\phi) = \phi$$

But $f(G \cap f^{-1}(F)) = f(G) \cap f(f^{-1}(F))$ for f is 1-1.
$= f(G) \cap F$ for f is onto.
Since f(G) is a clopen subset of Y (Iemma 2.9) , $y \in f(G)$ and $f(G) \cap F = \phi$. ,then (Y,S) is a strongly regular space .

3.8 Theorem : A strongly regular space is a hereditary property .
Proof : Let (X,T) be a strongly regular space and (A, T_A) be a subspace of (X,T) .
Now let F be a closed subset of A and $x \notin F (x \in A)$.
$\Rightarrow F = A \cap G$, G is a closed subset of X and $x \notin G$. $x \in A \Rightarrow x \in X$.
Since (X,T) is a strongly regular space, then there exists a clopen subset W of X s.t.
$x \in W$ and $W \cap G = \phi \Rightarrow A \cap W$ is a clopen subset of A , $x \in A \cap W$ and

$$(A \cap W) \cap F = A \cap (W \cap G)$$
$$= A \cap \phi$$
$$= \phi$$

Thus (A, T_A) is a strongly regular subspace of (X,T) .

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