ON α - TRANSITIVITY FOR DYNAMICAL SYSTEMS

Mohammed Nokhas Murad **Rewayda Razaq Mohsin**
University of Kufa college of Education ,Math .Dept. Najaf

Abstract

In this paper,we have studied the properties of α - transitivity for discrete dynamical systems defined on α - open topological spaces.

1-Introduction

The concepts of transitivity of discrete dynamical systems are introduced .we shall study various aspects of topological α - transitivity starting from some basic and elementary properties .we discuss some definitions and basic ideas of operator associated to a topology , α -open, α -compact space to discuss some important recent developments pertaining to α -transitivity. We are including an outline of proof for a theorem.we have divided this paper in to three sections the first section is the introduction ,the second section we have studied the basic definition ,in the section three we have studied the topological α –transitivity .

2-Dynamical systems and topological conjucacy.

To study the dynamics of a self-map $f:X\rightarrow X$ means to study the qualitative behavior of the sequence $\{f^n(x)\}$ as n goes to infinity when x varies in X ,where f^n denotes the composition of f with itself n times .

Definition 2.1([3]) Adiscrete dynamicals system is apair *(X,f)*,where *X* is a tolopological space and $f:X\rightarrow X$ acoutinuous self-map of X **.**

Definition 2.2 Let (A,τ) and (B,σ) be two topological space let $f:A\rightarrow A$ and $g: B\rightarrow B$ two continuous maps.

The maps *f* and *g* are said to be topological conjugate if there exists a homeomorphism $T:A\rightarrow B$ s.t $Tof(x)=g\,oT(x)$ holds for all $x\in A$…

The homeomorphism is called a topological conjugacy ,and if $T:A\rightarrow B$ is acontinuous surjection map and $Tof(x)=g\,oT(x)$ then *T* is called asemi conjugacy .

Suppose *f* and *g* are conjucate functions, with conjugacy *H*,then $H(f(x))=g(H(x))$

Remark 2.3 If x_0 is afixed point of *f* ,then $H(x_0)$ is a fixed point of *g* .

Proof : x_0 is afixed point of f therefore $f(x_0)=x_0$ and so $H(f(x_0))=H(x_0)$ then $g(H(x_0)) =H(g(x_0))=H(x_0)$ i.e $H(x_0)$ is afixed point of *g* .

Let*(X,f)* be adynamical systems ,a point $x\in X$ " moves under the rule f^n ,following 'the path " x ; $f(x)$; $f^2(x)$; …, which is called the trajectory of x , where f^n is the *nth* iretation of *f* (i.e $f^n = fof^{n-1}$ for every *n* in the set *N*); the point $f^n(x)$ being the position reached by *x* after *n* units of time .The pre-images of *x* under f^n are $y\in X$ for which $f^n(y)=x$ .the set $O(x)=\{y\in X \setminus f^n(x)=y; n\in N\}$ is called the (forword) orbit of *x* .Similary , for a set $A\subset X$,and any $n\in N$, $f^{-n}(A)$ is its n^{th} pre-image and $f^n(A)$ is its n^{th} image .If $f(A)\subset A$, *A* is said to be (forward)invariant under *f* ; and *A* is said to be backward invariant under *f* ,if $f^{-1}(A)\subset A$.

Definition 2.4 ([2]) A point $p\in X$ is called a periodic point of f if for each $n\in N$ for which $f^n(p)=p$, the least such *n* is called its period .(For *n=1*, such a point is called a fixed point).

3-Topological α - transitivity

In the study of dynamicas on a topological space ,it is natural and convenient to break the space into its irreducible parts and investigate the dynamics on each part .the topological property that precludes such decomposition is called topological transitivity and denoted by *TT* .Now we define topological transitivity as follows :

Definition 3.1([1]) Let *(X,f)* be adynamical system .If for every pair of non empty open sets *U*and *V* in *X* ,there is an $n\in N$ s.t $p^n(U)\cap V\neq\varphi$,then the system *(X,f)* is said to be topologically transitivite and then the map is said to be topologically transitive .
We want to discuss some definitions and basic ideas before we study the topological α - transitivity . we recall the definitions of operator associated to atopology , α - open , α - compact space , to discuss the α - transitivity.
Definition 3.2([4]) Let (X,Γ) be a topological space , *B* a subset of *X* and
α an operator from Γ to $P(X)$ i.e $\alpha:\Gamma\to P(X)$ (the set $P(X)$ consisting of all subsets of *X*) we say that α is an operator associated with Γ if $U\subseteq\alpha(U)$ for all $U\in\Gamma$.
Definition 3.3([4]) Let (X,T) be a topological space and *X* an open associated with *T* , a subset *A* of *X* is said to be α - open if for each $x\in A$ there exists an open set *U* containing *x* such that $\alpha(U)\subset A$ .a subset *B* is said to be α -closed if it is the compliment of α - open.
Definition 3.4 α -closure of *Y* is the intersection of all α -closed sets which contain *Y* and denoted by $(\alpha-cl(Y))$.
Definition 3.5 (α -Dense subset).A subset *Y* of *X* is α -dense in *X* if $\alpha-cl(Y)=X$.
Definition 3.6 Let $f:X\to X$ be cont's map of α -Compact metric space we say that *f* is topologically α -transitive if for any pair of non empty α -open sets *U* and *V in X* there is an $n\in N$ such that $f^n(U)\cap V\neq\varphi$ then the map is said to be topologically α -transitive.
Theorem 3.7: Let *(X,f)* a discrete dynamicals system. Then the following are equivalent :
1)*f* is topologically α -transitive.
2)For every non empty α - open set *U* in *X* , $\bigcup_{n=0}^{\infty} f^n(U)$ is α - dense in *X*
3) For every non empty α - open set *U* in *X* , $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is α -dense in *X*
4)If $E\subset X$ *is* α -closed and $f(E)\subset E$ then *E=X or E* is nowhere α -dense in *X*.
5)If $U\subset X$ is α - open and $f^{-1}(U)\subset U$ then $U=\varphi$ or *U* is α -dense in *X*.
Proof:
$1\Rightarrow 2$
Assume $\bigcup_{n=0}^{\infty} f^n(U)$ is not α -dense ,there exists a non empty α - open *V* such that $\bigcup_{n=0}^{\infty} f^n(U)\cap V=\varphi$ .This implies $f^n(U)\cap V=\varphi$, for all $n\in N$
This a contradiction to the α -transitivity of f, hence $\bigcup_{n=0}^{\infty} f^n(U)$ is α -dense in *X*. $2\Rightarrow 1$
Let *U* and *V* be two non empty α - open sets in *X*. $\bigcup_{n=0}^{\infty} f^n(U)$ is α -dense in $X\Rightarrow\bigcup_{n=0}^{\infty} f^n(U)\cap V\neq\varphi.$ This implies that $f^k(U)\cap V\neq\varphi.$ for some k .Hence *f* is
α -transitive.
$1\Rightarrow 3$
$\bigcup_{n=0}^{\infty} f^{-n}(U)$ is α - open and since f is α -transitive. It has to meet every
α -open set in *X* and hence is α -dense.
$3\Rightarrow 1$

Let U and V be α -open and non empty in X .Then $\bigcup_{n=0}^{\infty} f^{-n}(V)$ is α -dense X. As a result $U \cap \bigcup_{n=0}^{\infty} f^{-n}(V) \neq \varphi,$ this implies there exise $m \in N$ such that $U \cap f^{-m}(V) \neq \phi$. We further have $f^{-m}(U) \cap f^{-m}(V) \neq \varphi,$
therefore f is α -transitive.

$1 \Rightarrow 4$

f is α-transitive , $E \subset X$ is α-closed and $f(E) \subset E$ .Assume that $E \neq X$ and has a non empty interior . Define $U=X\backslash E$.Clearly U is α-open since E is α-closed . Let $V \subset E$ be α-open since E has a non empty interior .We have $f^{n}(V) \cap E$ is invariant , then $f^{n}(V) \cap V = \phi$ for all $n \in N$ .This is a contradiction to α- transitivity, hence $E=X$ or E is nowhere α -dense.

$4 \Rightarrow 1$

Let U be a non empty α -open set in X .Suppose f is not α -transitive, then from (3) of this theorem, $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is not α -dense , but α -open . Define $E = X | \bigcup_{n=0}^{\infty} f^{-n}(U)$. Clearly E is α -closed and $E \neq X$ .We need to prove that $f(E) \subset E$ .Suppose $f(E)$ is not subset of E .This implies $f(E) \cap \bigcup_{n=0}^{\infty} f^{-n}(U) \neq \varphi$. This further implies .

$f^{-1}[f(E) \cap \bigcup_{n=0}^{\infty} f^{-n}(U)] = E \cap \bigcup_{n=o}^{\infty} f^{-n}(U) \neq \phi$ This is a contradication to the definition of E .Thus $f(E) \subset E$ .Since $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is not α -dense,there exists a non empty α -open U in X such that $\bigcup_{n=0}^{\infty} f^{-n}(U) \cap V \neq \varphi.$This implies $V \subset E.$ This is a contradiction to the fact that E is now here α -dense , hence f is α -transitive.

$1 \Rightarrow 5$

f is α -transitive., $U \subset X$ is α -open and $f^{-1}(U) \subset U$ .Assume that $U \neq \varphi$ and U is not α -dense in X .Then $\exists$ anon empty α -open. V in X such that $U \cap V = \varphi$ .Further $f^{-n}(U) \cap V = \varphi.$ for all $n \in N$ .This implies $U \cap f^{-n}(V) = \varphi.$ for all $n \in N$,a contradiction to transitivity of f , hence $U = \varphi$ or U is α -dense in X .

$5 \Rightarrow 1$

Suppose f is not α -transitive.. For a non empty α -open. U in X ,let $W = \bigcup_{n=0}^{\infty} f^{-n}(U)$ is non empty α -open and not α -dense .Clearly $f^{-1}(W) \subset w.$. This a contradiction since $W \neq \varphi$ is α -dense this proves that f is α -transitive.

References

[1] Akin ,E. The general topology of dynamical systems, Graduate stud .Math.1,American Mathematical. Society. Providence(1993).
[2] Alse Da, L. and Kolyadad, S ,.periodic points for transitive maps ,transactions of the American mathematical society volume 351, number 4,Aprill(1999),Pages 1551-1573 .
[3]Block, L.S. and coppel, W.A. Dynamics in one dimension ,Lecture Notes in Mathematics .1513,Springer (1992).
[4] Rosas. E.vielma, J. Operator compactification of topological spaces .Divulgaciones Mathematics Vol.8 No.2(2000).pp.163-167.