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Especial Case of Connectedness in Bitopological spaces

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الخلاصة

بالاستفادة من مفهوم (المجموعات – δ –المفتوحة) عرّف المؤلفان مفهوم الاتصال في الفضاءات الثنائية التبولوجية وبحثا المفاهيم والمبرهنات المتعلقة بالاتصال من هذا المنظور وحصلا على نتائج مشابهة لمثيلاتها في الفضاءات التبولوجية واخرى تخالف مثيلاتها في الفضاءات التبولوجية، وكذلك تم بحث مفهومي الاتصال المساري والاتصال الموضعي في الفضاءات ثنائية التبولوجيا.

1 – Introduction

In 1963 Kelly defined: a set equipped with two topologies is called a bitopological space, denoted by $(X.T,\Omega)$ where (X,Ω) and (X,T) are two topological spaces. Jaleel in 2003 definedopen sets in bitopological spaces and -δ generalized a part of toplogical notions in bitopological spaces : A subset A of X(in a bitopological space (X,T,Ω)) is said to be δ-open set if A$\subset$T-int(Ω-Cl(T-intA))). The complement of δ -open sets (in X) are called δ - closed sets. δ-int(A) is defined to be the union of all δ -open sets contained in A. δ-Cl(A) is defined to be the intersection of all δ -closed sets containing A. A function f from (X,T,Ω) to (Y,$T'.\Omega'$) is said to be δ -continuous if for every δ-open subset B of Y, f^{-1} (B) is δ-open in X.

Similarly δ-open -open, δ -closed and δ -homeomorphism between two bitopological spaces are defined. The following results (and others) were given in (Jaleel , 2003):

i) The union of any family of δ -open sets in X is δ-open.
ii) The intersection of any family of δ -closed sets inX is δ -closed.
iii) The intersection of any two δ -open sets is not necessary δ -open.
iv) Every T-open set in(X,T) is δ-open set of (X,T, Ω) .
v) A necessary condition for a subset A of X to be δ -open set is T-int(A) $\neq \phi$.

In (Jaleel , 2003) the collection of all δ -open sets of a bitopological space (X,T,Ω) is denoted by δ -O(X). It is clear that δ -O(X) is not a topology.

In this work we .First (in section 1),add some other results about the notions discussed in (Kelly ,1963) and generalized the notion of continuity and we get some depending results .Then (in section 2), we discuss in detail the notion of connectivity in bitopological spaces.

1.1 Remarks

Let (X,T, Ω) be a bitopological space :

i) The union of any two δ -closed sets is not necessarily δ-closed.
ii)T-int(A) $\subset \delta$-int(A).
iii)δ-int(A$\cap$B) $\subset \delta$ -int(A)$\cap$ δ-int(B). (The equality in general is not true). iv)δ-Cl(A) $\cup$ δ - Cl(B)$\subset \delta$-Cl(A$\cup$B) . (The equality in general is not true).

In (Jaleel , 2003) the relations (iii) and (iv) with "=" are wrongly ,proved, see the following example.

1.2 Example

Let X={ a,b,c,d }, T={X,ϕ , {a}, {d} ,{a,d}, {b,c}, {a,b,c}, {b,c,d}}. Ω = { X,ϕ, { a }, { d } , { a,d } } .Then

δ -O(X)= { X,ϕ, { a },{ d },{ a,d },{ b,c },{ a,b,c } { b,c.d }, { a,b },{ a,c }, { b,d },{ a,b,d }, { a,c,d } } .

Let A={ a,c } , B= { b,c } then A,B$\in\delta$ -O(X) but A$\cap$B = { c }$\notin\delta$-O(X). F={ b,d } , H= { a,d }are δ -closed sets in (X,T, Ω) but F$\cup$H = { a, b,c } is not a δ -closed set.

δ -int(A) =A , δ -int(B)=B soδ -int(A) $\cap$ δ-int(B) = A$\cap$B = { c },on the other handδ-int (A$\cap$ B)= δ-int({ c })= ϕ

i.e δ-int(A$\cap$ B)$\neq\delta$-int(A) $\cap$ δ-int(B) .Also we see that

δ -Cl(F)$\cup$ δ -Cl(H)=F$\cup$H = { a,b,d } where

δ-Cl(F$\cup$H)=δ -Cl({ a,b,d })=X $\neq\delta$ -Cl(F)$\cup$ δ -Cl(H).

1.3 Definition.

Let$(X.T,\Omega)$ and(Y,$T'.\Omega'$) be two bitoplogical spaces ,f a function from X into Y.

i)) f:(X,T) $\rightarrow$ (Y,$T'.\Omega'$) is said to be T-δ continuous if for each B $\in\delta$-O(Y) , f^{-1} (B) $\in$ T.

ii) f: $(X.T,\Omega)\rightarrow$(Y,T') is said to beδ-T continuous if for each B$\in T'$, f^{-1} (B) $\in\delta$-O(X).

iii)f:(X,T) $\rightarrow$ (Y,$T'.\Omega'$) is said to be T-δ open (T-δ closed)if for each A$\in$T(A a closed set in(X,T)), f(A)$\in\delta$ -O(Y)(f(A)is a δ-closed in (Y, $T'.\Omega'$))..

iv)f:(X,$T'.\Omega'$)$\rightarrow$(Y,T') is said to beδ-T-open(δ-T closed) if for each A$\in\delta$-O(X)(A a δ-closed set in$(X.T,\Omega)$), f(A)$\in T'$(f(A) is a closed set in (Y, T')).

1.4 Remarks.

Let$(X.T,\Omega)$ and (Y,$T'.\Omega'$) be two bitopological spaces ,f :X $\rightarrow$Y be a function

i)The notions of continuity andδ-continuity are independent.

ii)If f:(X,T)$\rightarrow$(Y, T') is continuous or f:$(X.T,\Omega)\rightarrow$(Y, $T'.\Omega'$) is δ-continuous then :$(X.T,\Omega)\rightarrow$(Y,T') isδ -T continuous.

iii)If f:(X,T)$\rightarrow$(Y,$T'.\Omega'$)is T-δ continuous then f:$(X.T,\Omega)\rightarrow$(Y,$T'.\Omega'$) is δ-continuous.

iv)If f:$(X.T,\Omega)\rightarrow$(Y,$T'.\Omega'$)isδ-open(δ-closed)then f :(X,T)$\rightarrow$(Y, $T'.\Omega'$) is T-δ open (T-δ closed).

v)If f :$(X.T,\Omega)\rightarrow$(Y, T') is T-δ open (δ -T closed) then f:(X,T)$\rightarrow$(Y,T') is open (closed).

vi)The notions of homeomorphism andδ-homeomorphism are independent.

The proofs of (ii) to (v) are obvious ,we only prepare examples for the other remarks .

1.5 Example.

Let T be the topology induced by the absolute value in R , where R be the of real number and Ω be the cofinite topology on R. Then (R,T,Ω) is a bitopological space and any subset of R which contains some open interval is a δ-open set in(R,T,Ω) ,for example if A=(0,1)$\cup${2 },then T-int(A)=(0,1) ,

Ω-Cl((0,1))=R,(Ω-closed sets are finite sets and R) and so T-intR=R.i.e $A\subset$T-int(Ω-Cl(T-intA))). Define f: R$\rightarrow$R , f(x)=1 if x$\in[0,1]$ and f(x)=0 if x$\notin[0,1]$. We can remake the following about f :

i) f:(R,T) $\rightarrow$(R,T) is not continuous.

ii) f: (R,T,Ω)$\rightarrow$(R,T,Ω) is δ-continuous ,because for any subset B of R , so f^{-1} (B) equals one of the sets ϕ, $[0,1]$, R-$[0,1]$or R which are all δ-open sets in (R,T,Ω), so f^{-1} (B) is δ-open set.

iii) f:(R,T,Ω)$\rightarrow$(R,T) is δ-T continuous for the same reason above ,but f:(R,T) $\rightarrow$(R,T,Ω) is not T-δ continuous.

iv) f is not open ,not δ-open ,not δ -T open and not T-δ open.

v) f is closed, δ-closed ,T-δ closed and δ-T closed.

Now define g: R$\rightarrow$R ,g(x)=sin x ,and note the following about g:

i) g:(R,T) $\rightarrow$(R,T) is continuous.

ii) g: (R,T,Ω)$\rightarrow$(R,T,Ω) is not δ-continuous ,because if B=$\{0\}\cup$(2,3),which is a δ-open set in(R,T,Ω),where f^{-1}(B)=$\{n\pi\}$,n$\in$Z, which is not δ-open in (R,T,Ω).

iii)g:(R,T)$\rightarrow$(R,T) is not open (f ((0,π))=(0,1], but g: (R,T,Ω)$\rightarrow$(R,T,Ω) is δ-open.

iv) g is closed and δ-closed.

2- Connectness in bitopological spaces

In this section we define δ-connected , δ-path connected and locally δ-connected spaces and list all the theorems (related to those notions) .The proofs are mostly omitted because they are precisely similar to the proofs of corresponding theorems in topological spaces ,see (Gemignani , 1971)

2.1 Definition.

A bitopological space (X,T, Ω) is said to be δ-disconnected if X can be expressed as the union of two disjoint , δ-open, non empty subsets of X. Otherwise ,X is called δ-connected, i.e X is δ-connected if there does not exist disjoint ,non empty, δ-open subsets U and V of X such that X=U$\cup$V.

A subset A of X is said to be δ-connected if the subspace A of (X,T,Ω) is δ-connected.

2.2 Example.

Let X=R ,T the topology induced by the absolute value , Ω the cofinite topology on R. Then (X,T) is connected ,but (X,T,Ω) is δ-disconnected (note that $[a,b]$and R-$[a,b]$ are both δ-open sets of (X,T,Ω),(see 1.5).

2.3 Theorem.

Suppose f is a δ-continuous mapping from (X,T,Ω) onto (Y,T'.Ω'),if X is δ-connected then so is (Y,T'.Ω').

2.4 Theorem.

Let (X,T,Ω) be a bitopological space, then the following statements are epuivalent:

i) X is δ-connected.

ii) X cannot be exepressed as the union of two disjoint non empty δ-closed sobsets.

iii)The only subsets of X which are both δ-open and δ-closed are X and ϕ.

iv) Let Y=$\{0,1\}$have the disecrete topology,D. Then there is no δ-T continuous function from (X,T,Ω) onto (Y,D).

2.5 Theorem.

Suppose (X,T,Ω) is a bitopological space such that X= U$\cup$V, where U and V are disjoint , δ-open, non empty subsets of X, let A be any δ-connected subspace of X ,then A $\subset$U or A $\subset$V.

2.6 Theorem.

If (X,T,Ω) is a bitopological space and X=$\bigcup_{I\in\Lambda} A_I$,where $\{A_i\}$, i$\in\Lambda$ is a collection of δ-connected subspaces of X .Then if $\bigcap_{I\in\Lambda} A_I \neq \phi$, X itself is δ-connected.

2.7 Theorem.

Let (X,T,Ω) be a bitopological space such that any two elements x and y of X are contained in some δ-connected subspace of X. Then X is δ-connected.

2.8 Theorem.

If A is a subspace of a bitopological space (X,T,Ω),then A is δ-connected if and only if A cannot be expressed as S $\cup$T, where Sand T are non empty subsets of X and S$\cap$(Cl(T))=(δ-Cl (S)) $\cap$T=ϕ.

2.9 Theorem.

Suppose A is δ-connected subspace of S $\cup$T, where S and T are two subsets of a bitopological space(X,T,Ω)such that

S$\cap$(δ-Cl T)=(δ-Cl S) $\cap$T=ϕ,then either A $\subset$S or A $\subset$T.

2.10 Theorem.

Suppose A is a δ-connected subspace of a bitopological space (X,T,Ω) and A$\subset$Y$\subset$ δ-Cl(A).Then Y is also a δ-connected subspace of X.

2.11 Definition.

Let (X,T,Ω) be a bitopological space. A subspace Y of X is said to be δ-path of X if there is a T-δ continuous function from $[0,1]$ (with the absolute value topology) onto (Y,T_Y,Ω_Y).

X is said to be δ-path connected if given any two points x and y in X ,there is a path in X containing x and y .

2.12. Remark.

If (X,T) is δ- path connected then it is not necessary that (X,T,Ω) be δ- path connected (see example 2.2).

2.13 Theorem.

If (X,T,Ω) is δ- path connected then (X,T,Ω) is δ-connected.

The converse of theorem 2.13 is not true ,see the following example.

2.14Example.

LetY=$\{(x,y) \mid y=\sin(1/x), x>0\}\cup\{(0,0)\}$,T the Pythagorean topology,Ω the discrete topology .Then (Y,T,Ω) is a bitopology space and δ-O(Y)=T. So(Y,T,Ω) is δ-connected but not δ- path connected (Gemignani ,1971)

2.15 Definition.

Let (X,T,Ω) be a bitopological space, A maximal δ-connected subspace of X is said to be δ-component of X.

2.16 Example.

Let X=$\{a,b,c,d\}$,T=$\{\phi, X,\{a\},\{d\},\{a,d\}\}$, $\Omega=\{\phi, X\}$,then

δ-O(X)= $\{\phi, X\{a\},\{d\},\{a,d\},\{a,b\},\{a,c\},\{a,b,c\},\{a,b,d\},\{b,d\},\{c,d\},\{b,c,d\},\{a.c.d\}\}$

and the δ -component of X are $\{a\},\{b\},\{c\}, and \{d\}$.

2.17 Theorem.

A δ -component of a bitopological space is a δ -closed set.

2.18 Remark.

A δ -component of a bitopological space is not necessary δ -open set. In 2.16 $\{b\} and \{c\}$ are not δ -open sets.

2.19 Theorem.

Let $\{A_i\}$, i$\in$ Λ be the set of all δ -components of a bitopological space (X,T,Ω).Then $\bigcup_{I\in\Lambda} A_I$ =X and if i $\neq$ j then $A_i \cap A_j = \phi$.

2.20 Definition.

A bitopological space (X,T,Ω) is said to be locally δ -connected if for each x$\in$X and each δ -open set G containing x ,there exists a δ -open set G' which is δ -connected and x$\in$ G' $\subset$G.

2.21 Examples.

i) The space in Example 2.16 is not locally δ -connected .

ii) Let X=R, T the topology induced by the absolute value, Ω the discrete topology, then δ -O(X)=T .If A=(-3,0)$\cup$(3,8) then A is a locally δ -connected subspace of X but not δ -connected (Naoum , 1974).

iii) Let X= R^2 ,T the pythagorain topology, Ω the discrete topology,then δ -O(X)=T and if A=$\{$ segments(in R^2) joining (1,0) to the points (1,1/n) $\}\cup\{(1,0)\}\cup\{(-1,0)\}$,A is δ -connected but not locally δ -connected. (Naoum , 1974).

2.22 Theorem.

Each δ -component of a locally δ -connected bitopological space (X,T,Ω) is δ -open and δ -closed set.

Proof.

Suppose x$\in$X and C(x) is the δ -component containing x,if y$\in$C(x) then C(y)=C(x),(by theorem 2.19), but for every δ -open set G containing y there is a δ -open,δ -connected set G' such that y $\in G' \subset$G which implies that y $\in G' \subset$ C(y)=C(x),because C(x) is maximal ,that is ,y$\in$ δ -int(C(x)) ,therefore C(x) is δ -open.

2.23 Theorem.

If f is a δ -continuous, δ -open function from a locally δ -connected bitopological space (X,T,Ω) onto a bitopological space (Y,T'.Ω') then Y is locally δ - connected.

Proof.

Assume that Y is not locally δ - connected ,then there is some $y_0 \in$Y and G$\in$ δ -O(Y) such that $y_0 \in$G and any subset G' (where $y_0 \in G' \subset$G)is δ -disconnected.

Let $x_0 \in f^{-1}(y_0)$ and H= f^{-1}(G) then H is δ -open in X and $x_0 \in$ H and so by the assumption of the theorem there is a δ -open, δ - connected subset H' of X such that $x_0 \in H' \subset$H,but f(H') is δ -open subset of Y(f is δ -open) and it is δ - connected (by theorem 2.3),also $y_0 \in$ f(H')$\subset$G which is a contradiction.

References

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