Existence Results for Nonlinear Quasi-hemivariational Inequality Systems

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ABSTRACT

This paper aims at establishing the existence solutions of a non-standard quasi-hemivariational inequality, whose solutions are discussed in a subset K_n of a reflexive Banach space X_n for every $n = \overline{1,\iota}$. In addition, we introduce two applications. The first one shows the relationship between our inequality problems and equilibrium problems. The second one applies the extend some results concerning nonlinear quasi-hemivariational inequalities.

الملخص

هذا البحث يهدف لنشر وجود حلول لنظم المتراجحات شبه نصف التغايرية الغير قياسية. حيث ان الحلول نوقشت في المجموعة الجزئية K_n من فضاء بناخ X_n لكل $n = \overline{1,\iota}$.

بالإضافة الى ذلك قدمنا تطبيقين. الاول بين مشاكل متراجحتنا ومشاكل التوازن . اما الثاني يطبق نتيجتنا الرئيسية التي حصلنا عليها لتوسع بعض النتائج المتعلقة بألمتراجحات الشبه نصف تغايريه والغير خطية.

Keywords

Quasi-hemivariational inequality; Clarkes generalized gradient; locally Lipschitz functional; equilibrium problems.

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1. Introduction

The theory of hemivariational inequalities was introduced P.D. Panayiotopoulos at the beginning of the 1980s (see [18] and [19]). Within a very short period of time, this theory witnessed a remarkable development in both pure and applied mathematics. It has been proved very efficient to describe a variety of mechanical problems and engineering sciences, economics, differential inclusion and optimal control (see [5-8,12, 15, 16, 18, 21, 22] and [25]). In these papers, based on Clarkes
generalized directional derivative and Clarke's generalized gradient for locally Lipchitz functions, the researchers study the existence and uniqueness of solutions by using such as fixed point Theorems, KKM Theorems, critical point Theory, surjectivity Theorems for pseudomonotone and coercive operators (see [1, 3, 4, 11, 13, 14, 20, 23, 24] and [26]).

Generally, in the last few years, there were many authors were who interested in the study of various kinds of hemivariational inequalities and systems of hemivariational inequalities, these inequalities are a generalization of the variational inequalities, and related problems such as equilibrium problems. In order to, it is very useful to understand several problems of mechanics and engineering for non- convex, non-smooth energy functionals.

Quasi- hemivariational inequalities arise from hemivariational inequalities if in addition some constraints have to be taken into account. The solution of these inequality gives the position of the state equilibrium of the structure.

The main purpose of this work is to contribute in establishing the existence solutions for systems of generalized quasi- hemivariational inequalities. In order to achieve the aim, the study is divided into the following sections.

In section 2, we refer to some definitions and results that assist us in the study. In section 3 we formulate the system of generalized quasi-hemivariational inequality and prove the main results. In the last section of the paper, two applications are given. The first one present the relationship between our inequality problems and equilibrium problems. The second one apply of our inequality to a system of generalized quasi- hemivariational inequalities involving integrals of Clarke's generalized directional derivatives.

2. Preliminaries with basic assumptions

Throughout this paper, we assume that E_n are Banach space and that $E_n{}^*$ are the topological dual space of the Banach space E_n, white $\langle \cdot , \cdot \rangle_n$ and $\|\cdot\|_n$ denote the duality pairing between E_n and $E_n{}^*$, respectively for every $n = \overline{1,\imath}$.

In what follows, we are going to recall some definitions and notions from non-smooth analysis which will be used in this paper.

Assume that K is a nonempty, closed and convex subset of a Banach space E_n and assume that $F: K \times K \to R$ us a given bifunction satisfying the property $F(u, u) = 0$, for every $u \in K$. An equilibrium problem (for short, (EP)). In the sense of Blum, Mu and Zoettli (see [17]) is a problem of the form:

Find $x \in K$ such that $F(x, y) \geq 0, \forall y \in K$.

Definition 2.1: [8] A functional $J: E \to R$ is said to be locally Lipchitz if every point $u \in E$ possesses a neighborhooh W such that

$$|J(a) - J(b)| \leq M_u \|a - b\|_E \; \forall a, b \in W.$$

For a constant $M_u \geq 0$ which depends on W.

Definition 2.2: [8] Assume that $J: E \to R$ is a locally Lipchitz. The generalized derivative of J at the point $u \in E$ in the direction $z \in E$ is denoted by $J^0(u, z)$,

i.e.,

$$J^0(u, z) = \limsup_{\substack{a \to u \\ \lambda \searrow 0}} \frac{J(a + \lambda z) - J(a)}{\lambda}.$$

Similarly, one can define the partial generalized derivative and partial generalized gradient of locally Lipchitz functional in the r^{th} variable.

Definition 2.3: [8] Assume that $\emptyset: E_1 \times \ldots \times E_r \times \ldots E_n \to R$ be a locally lipschitz function in the r^{th} variable. The partial generalized derivative of $\emptyset(u_1, \ldots, u_r, \ldots, u_i)$ at the point $u_r \in E_r$ in the direction $z_r \in E_r$, denoted by $\emptyset^0(u_1, \ldots, u_r, \ldots, u_i; z_r)$, is

$$\emptyset^0{}_{,r}(u_1, \ldots, u_r, \ldots, u_i; z_r) = \limsup_{\substack{a_r \to u_r \\ \lambda \searrow 0}} \frac{\emptyset(u_1, \ldots, a_r + \lambda z_r, \ldots, u_i) - \emptyset(u_1, \ldots, a_r, \ldots, u_i)}{\lambda}.$$

While the partial generalized gradient of the mapping

$u_r \mapsto f(u_1, \ldots, u_r, \ldots, u_i)$ denoted by $\partial_r \emptyset(u)(u_1, \ldots, u_r, \ldots, u_i), \forall z_r \in E_r$ that is

$\partial_r \emptyset(u)(u_1, \ldots, u_r, \ldots, u_i) = \left\{ \varsigma_r \in E^*{}_r : \emptyset^0{}_{,r}(u_1, \ldots, u_r, \ldots, u_i; z_r) \geq \langle \varsigma_r, z_r \rangle E_r \right\}$.

Definition2.4: [8] Assume that E is a Banach space and $J: E \to R$ is locally lipschitz functional. We say that J is a regular (in the sense of Clarkes) at $u \in E$ if for each $z \in E$ the one sided directional derivative $J'(u, z)$ exists and $J^0(u, z) = J'(u, z)$. We say that J is regular at every point $u \in E$.

Proposition 2.5: [8] Let $J: E \to R$ be a function on a Banach space E, which is locally Lipchitz of rank M_u near the point $z \in E$, then

i) The $z \mapsto J^0(u, z)$ is subadditive, finite, positively homogeneous and satisfies
ii) $J^0(u, z) \leq M_u \|z\|$;
iii) $J^0(u, z)$ is upper semicontinuous as a function of (u,z) one can found a proof it in [19].

Definition 2.6: The generalized gradient of J at $u \in E$, which is a subset of a dual space E^*, is define by

$$\partial J(u) = \{\xi \in E^* : \langle \xi, z \rangle \leq J^0(u; z), \forall z \in E\}.$$

Lemma 2.7. [8] Assume that $\emptyset: E_1 \times \ldots \times E_i \to R$ is a regular, locally Lipchitz functional. Then the following are satisfied:

(1) $\partial \emptyset(u_1, \ldots, u_r, \ldots, u_i) \subset$
$\partial_1 \emptyset(u_1 \ldots, u_r, \ldots, u_i) \ldots \times \partial_i \emptyset(u)(u_1, \ldots, u_r, \ldots, u_i)$;

(2) $\emptyset(u_1, \ldots, u_r, \ldots u_i, z_1 \ldots, z_r, \ldots, z_i) \le \sum_{i=1}^{n} \emptyset^0_{,r}(u_1, \ldots, u_r, \ldots, u_i)$;
(3) $\emptyset(u_1, \ldots, u_r, \ldots, u_i, 0, \ldots, z_r, \ldots, 0) \le \emptyset^0_{,r}(u_1, \ldots, u_r, \ldots, u_i; z_r)$.

Definition2.8. Assume that $T: E \to E^*$ and $\eta: E \times E \to E$ are two single valued. T is said to be η- monotone, if for each $x, y \in \text{dom } T$ such that

$$\langle Tx - Ty, \eta(x,y)\rangle \ge 0. \tag{2.1}$$

At the end of this section, we recall Tarafdar fixed point theorem for set – valued mapping [25] which we shall use to prove the main results of the study.

Theorem 2.9. Assume that $K \subset E$ is a nonempty and convex of Hausdorff topology vector space E, and that $\alpha: K \multimap K$ be a set valued map such the following are satisfied

(i) For each $u \in K$, $\alpha(u)$ is a nonempty convex subset of K.
(ii) For each $z \in K$, $\alpha^{-1}(z) = \{u \in K: z \in \alpha(u)\}$ contain an open set O_z which may be empty.
(iii) $\bigcup_{z\in K} O_z = K$.
(iv) There exists a nonempty set U_0 contained in a compact subset U_1 of K such that $S = \bigcap_{z\in K} {O_z}^c$ is either empty or compact, where ${O_z}^c$ is the complement of $O_z \in K$.

Then there exists a point $u^0 \in K$ such that $u^0 \in \alpha(u^0)$.

3. Formulation of the problem and main results:

Assume that K_n is a nonempty bounded, closed and convex subset of a real reflexive Banach space X_n where n a positive integer. Our aim is to study the following system of nonlinear quasi- hemivariational inequality: (NQHIS)

Find $(u_1, \ldots, u_i) \in K_1 \times \ldots \times K_i$.

$$\begin{cases} \langle T_1(u_1,\dots,u_i),\eta_1(u_1,v_1)\rangle_{X_1} + H_1(u_1,\dots,u_i)j_1{}^0\big(A_1u_1,\dots,A_1u_i;A_1\eta_1(u_1,v_1)\big) \\ \geq \langle F_1(u_1,\dots,u_i),v_1-u_1\rangle_{X_1} \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ \langle T_i(u_1,\dots,u_i),\eta_i(u_i,v_i)\rangle_{X_i} + H_i(u_1,\dots,u_i)j_i{}^0\big(A_iu_1,\dots,A_iu_i;A_i\eta_i(u_i,v_i)\big) \\ \geq \langle F_i(u_1,\dots,u_i),v_i-u_i\rangle_{X_i} \end{cases}$$

For all $(v_1,\dots,v_i) \in K_1 \times \dots \times K_i$. Assume that $X_1,\dots,X_i$ are reflexive Banach space and that $E_1,\dots,E_i$ are Banach space. We assume that there exist compact linear operators $A_n: X_n \to E_n$ for every $n = \overline{1,\imath}$. In addition, let us assume that $\eta_n: X_n \times X_n \to X_n$ are single- valued functions, and $J: E_1 \times \dots \times E_n \to R$ is a regular locally Lipchitz functional, for every $n = \overline{1,\imath}$.

Notations: For every $n = \overline{1,\imath}$. Then

° $X = X_1 \times \dots \times X_i$; $E = E_1 \times \dots \times E_i$ and $E = K_1 \times \dots \times K_i$

° $u = (u_1,\dots,u_i)$ and $A_u = (A_1u_1,\dots,A_iu_i)$

° $\eta(u,v) = \big(\eta_1(u_1,v_1),\dots,\eta_i(u_i,v_i)\big)$; $A_\eta(u.v)$
$$= \big(A_1\eta_1(u_1,v_1),\dots,A_i\eta_i(u_i,v_i)\big)$$

° T_n and $F_n: X_1 \times \dots \times X_i \to X_n{}^*$ are nonlinear operators such that:

$$\langle F_u, v-u\rangle_X = \sum_{n=1}^{i} \langle F_n(u_1,\dots,u_i), v_n - u_n\rangle_{X_n}$$

$$\langle T_u, \eta(u,v)\rangle_X = \sum_{n=1}^{i} \langle T_n(u_1,\dots,u_i), \eta_n\,(u_n,v_{n)}\rangle_{X_n}$$

° $H_n: X_1 \times \dots \times X_i \to R \cup \{+\infty\}$ is a non negative and continuous on K_n such that $H(u) = \sum_{n=1}^{i}(u_1,\dots,u_i)$.

Hypotheses: For every $n = \overline{1,\imath}$. Then the following assertions are fulfilled

H_F: $\langle F_n(u_1,\dots,u_i), v_n - u_n\rangle_{X_n} \leq \liminf_m \langle F_n(u_1{}^m,\dots,u_i{}^m), v_n - u_n{}^m\rangle_{X_n}$,

Where $(u_1{}^m,\dots,u_i{}^m) \rightharpoonup (u_1,\dots,u_i)$ as $m \to \infty$ and $v_n \in X_n$ is fixed.

H_η: The mapping $\eta_i(\cdot\ ,\cdot): X_i \times X_i \to X_i$ satisfied the following conditions

(1) $\eta_n(u_n, u_n) = 0 \ \forall u_n \in X_n$;
(2) $\eta_n(u_n, 0)$ is a linear operator $\forall u_n \in X_n$;
(3) $\eta_n(u_n{}^m, v_n) \rightharpoonup \eta(u_n, v_n)$, whenever $u_n{}^m \rightharpoonup u_n$.

H_T:

(1) $\limsup_m \langle T_n(u_1{}^m, \dots, u_i{}^m), \ \eta_n(u_n{}^m, v_n)\rangle_{X_n} \leq$
$\langle T_n(u_1, \dots, u_i), \ \eta_n(u_n, v_n)\rangle_{X_n}$ where
$(u_1{}^m, \dots, u_i{}^m) \rightharpoonup (u_1, \dots, u_i)$ as $m \to \infty$ and $v_n \in X_n$ is fixed;
(2) $v_n \mapsto \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \ \eta_n(u_n, v_n)\rangle_{X_n}$ is a convex $\forall u_n \in X_n$

H_H: $\limsup_m H_i(u_1{}^m, \dots, u_i{}^m) = H_i(u_1, \dots, u_i)$ whenever $u_n{}^m \rightharpoonup u_n$ as $m \to \infty$.

Theorem 3.1. Assume that the nonempty, bounded, closed and convex set $K_n \subset X_n$ for every $n = \overline{1,\imath}$ .If the conditions H_F, H_η, H_T and H_H satisfy, then the system of nonlinear quasi- hemivariational inequalities (NQHZIS) admits at least one solution.

In what follows, we are going to present the following vector quasi-hemivariational inequality:

(VQHI) find $u \in K$,such that

$$\langle T_n\eta(u, v)\rangle + H(u)J^0\big(Au; A\eta(u, v)\big) \geq \langle Fu, v - u\rangle_X \qquad (3.1)$$

For all $v \in K$

Proposition 3.2. Assume that the assumptions H_F,H_η, H_T and H_H satisfy, and that $(u_1{}^0, \dots, u_i{}^0) \in K_1 \times \dots \times K_i$ is a solution of the inequality (VQHI), then u^0 is a solution of the system (NQHI).
Proof. Indeed, if we fix a point $u_n \in K_n$ for every $n = \overline{1,\imath}$ and for $n \neq h$. We assume that $u_h = u_h{}^0$. Therefore,from lemma 1.8 and $H_\eta(1)$,

$$0 \leq \langle Tu^0, \eta(u^0, v)\rangle_X + H(u^0)J^0(Au^0; A\eta(u^0, v)) - \langle Fu^0, v - u^0\rangle_X$$

$$\leq \sum_{h=1}^{i}\left[\begin{matrix}\langle \mathbf{T_h}(\mathbf{u_1}^0, \dots, \mathbf{u_i}^0), \boldsymbol{\eta}_\mathbf{h}(\mathbf{u_h}^0, \mathbf{v_h})\rangle_{X_h} + \mathbf{H_h}(\mathbf{u_1}^0, \dots, \mathbf{u_i}^0)\, \mathbf{J_h}^0\big(\mathbf{Au_1}^0, \dots, \mathbf{u_i}^0; \mathbf{A}\boldsymbol{\eta}_\mathbf{h}(\mathbf{u_h}^0, \mathbf{v_h})\big) \\ -\langle \mathbf{F_h}(\mathbf{u_1}^0, \dots, \mathbf{u_i}^0), \mathbf{v_h} - \mathbf{u_h}^0\rangle_{X_h}\end{matrix}\right]$$

$=$
$\langle \mathbf{T_n}(\mathbf{u_1}^{\mathbf{0}}, \dots, \mathbf{u_i}^{\mathbf{0}}), \boldsymbol{\eta}_{\mathbf{n}}(\mathbf{u_n}^{\mathbf{0}}, \mathbf{v_n})\rangle_{X_n} +$
$\mathbf{H_n}(\mathbf{u_1}^{\mathbf{0}}, \dots, \mathbf{u_i}^{\mathbf{0}})\mathbf{J_n}^{\mathbf{0}}\left(\mathbf{Au_1}^{\mathbf{0}}, \dots, \mathbf{u_i}^{\mathbf{0}}; \mathbf{A}\boldsymbol{\eta}_{\mathbf{n}}(\mathbf{u_n}^{\mathbf{0}}, \mathbf{v_n})\right) - \langle F_n(u_1^0, \dots, u_i^0), v_n - u_n^0\rangle_{X_n}, \quad \forall n = \overline{1.\imath}.$

This means that $(u_1^0, \dots, u_i^0) \in K_1 \times \dots \times K_i$ is a solution of the inequality (NQHIS). ■

Remark 3.3. The mapping $v_n \mapsto J^0{}_{,n}\big((u_1, \dots, u_i), \eta_n(u_n, v_n)\big)$ is a convex for every $(u_1, \dots, u_i) \in E_1 \times \dots \times E_i$. It follows from the convexity of $J^0{}_{,n}\big((u_1, \dots, u_i), v_n\big)$ and linearity of $\eta_n(u_n, \cdot)$ For every $n = \overline{1,\imath}$.

Remark 3.4. Since A is a linear compact operator we obtain that Av_n converges strongly to some $A_v \in K$ and $u \in K$, $A\eta(v_n, u)$, converges strongly to $A\eta(v_n, u)$.

Applying this fact, together with Proposition 2.4(2) , we get that

$$\limsup_{n} J^0\big(Av_n; A\eta(v_n, u)\big) \le J^0\big(Av; A\eta(u, v)\big). \qquad (3.2)$$

In what follows, we are going to prove of Theorem 3.1.

Proof. According to Proposition 3.2, it is enough to prove that problem (VQHI) has at least one solution. We argue by contradiction, let us assume that (VQHI) has no solution. Then for each $u \in K$ there exists $v \in K$ such that

$$\langle Tu, \eta(u, v)\rangle_X + H(u)J^0\big(Au; A\eta(u, v)\big) < \langle Fu, v - u\rangle_X \qquad (3.3)$$

Let as consider the set $\psi \subset K \times K$ as follows

$$\psi = \big\{(u, v) \in K \times K: \langle Tu, \eta(u, v)\rangle_X + H(u)J^0\big(Au; A\eta(u, v)\big) \ge \langle Fu, v - u\rangle_X\big\}. \quad (3.4)$$

We shall prove ψ satisfies the conditions of Tarafdar Lemma for weak topology of the space X. Let $\alpha: K \multimap K$ defined by

$$\alpha(u) = \{v \in K: (v, u) \notin \psi\} \cdot \qquad (3.5)$$

Claim 1: $\alpha(u)$ is nonempty and convex $\forall u \in K$.

Obviously, $\alpha(u)$ is a nonempty for each $u \in K$. Let us choose $v', v'' \in \alpha(u), t \in (0,1)$ and $v^t = tv' + (1-t)v''$. Using $H_T(2)$,one can have

$$\langle T(u), \eta(u, v^t) \rangle_X = \sum_{n=1}^{i} \langle T_n(u_1, \ldots, u_i), \eta_n(u_n, tv_n' + (1-t)v_n'') \rangle_{X_n}$$

$$\leq t \sum_{n=1}^{i} \langle T_n(u_1, \ldots, u_i), \eta_n(u_n, v_n \quad) \rangle_{X_n}$$

$$+ (1-t) \sum_{n=1}^{i} \langle T_n(u_1, \ldots, u_i), \eta_n(u_n, v_n) \rangle_{X_n}$$

$$= t\langle T(u), \eta(u, v') \rangle_{X_n} + (1-t)\langle T(u), \eta(u, v'') \rangle_{X_n}$$

$\forall\, t \in (0,1)$, so $v \mapsto \langle T(u), \eta(u, v^t) \rangle_X$ is convex. On the other hand, Remark (1,8) and continuity of H, we deduce that the mapping $v \mapsto H(u)J^0(Au; A\eta(u,v))$ is convex. Applying the fact that the mapping $v \mapsto \langle F(u), v-u \rangle_X$ is an affine, then $\alpha(u)$ is a convex for a fixed $u \in K$.

Claim 2. $[\alpha^{-1}(v)]^c = \{u \in K: (v,u) \in \psi\}$ is weakly set in K.

It is enough to prove that for any fixed point $v \in K$, the function $G: K \to R$,

$$G(u) = \langle T(u), \eta(u, v) \rangle_X + H(u)J^0(Au; A\eta(u, v)) - \langle F(u), v-u \rangle_X \qquad (3.6)$$

is weakly upper semi continuous. This is equivalent to say that mappings:

$\langle T(u), \eta(u, v) \rangle$,$H(u)J^0(Au; A\eta(u, v))$ is weakly upper semi continuous while the mapping $\langle F(u), v-u \rangle_X$ is weakly lower semi continuous for the fixed $v \in K$. Assume that $\{u^m\} \subset K$ be a sequence such that $u^m \rightharpoonup u \in K$ as $m \to \infty$. From $H_\eta(3)$ and H_T for every $n = \overline{1, \imath}$, then

$$\limsup_{m}\langle T(u^m)\eta(u^m,v)\rangle_X=\limsup_{m}\sum_{n=1}^{i}\langle T_n(u_1{}^m,\dots,u_i{}^m),\eta_n(u_n{}^m,v_n)\rangle_{X_n}$$

$$\le \textstyle\sum_{n=1}^{i}\limsup_m\langle T_n(u_1{}^m,\ \dots,u_i{}^m),\eta_n(u_n{}^m,v_n)\rangle_{X_n}$$

$$\le \sum_{n=1}^{i}\langle T_n(u_1,\dots,u_i),\eta_n(u_n,v_n)\rangle_{X_n}$$

$$=\langle T(u),\eta(u,v)\rangle_X$$

This means that the mapping $u\longmapsto\langle Tu,\eta(u,v)\rangle$ is weakly u.s.c.

One the other hand, from Remark 3.4, H_H and $J^0(u,v)$ is u.s.c. Then

$$\limsup_{m} H(u^m)J^0\big(Au^m;A\eta(u^m,v)\big)\le\limsup_{m}\left[\sum_{n=1}^{i}H_n\,(u_1{}^m,\dots,u_i{}^m)\right.$$

$$\left.-\sum_{n=1}^{i}H_n\,(u_1,\dots,u_i)\right]J^0\big(Au^m;A\eta(u_i{}^m,v_i)\big)$$

$$+\limsup_{m}\sum_{n=1}^{i}H_n\,(u_1,\dots,u_i)J^0\big(Au^m;A\eta(u_i{}^m,v_i)\big)$$

$$\le 0+\limsup_{m}\textstyle\sum_{n=1}^{i}H_n\,(u_1,\dots,u_i)J^0\big(Au^m;A\eta(u_i{}^m,v_i)\big)$$

$$=H(u)J^0\big(Au;A\eta(u,v)\big).$$

Finally, by applying H_F, one can get

$$\langle Fu,v-u\rangle_X=\sum_{n=1}^{i}\langle Fn(u_1,\dots,u_i),v_n-u_n\rangle_{X_n}$$

$$\le\liminf_{m}\sum_{n=1}^{i}\langle Fn(u_1{}^m,\ \dots,u_i{}^m),v_n-u_n{}^m\rangle_{X_n}$$

$$= \liminf_m \langle F(u^m), v - u^m \rangle_{X_n}.$$

Therefore, G is weakly u.s.c. Hence, the set $\{u \in K: G(u) \geq \beta\}$ is weakly closed for every $\beta \in R$. Taking $\beta = 0$,we get that $[\alpha^{-1}(v)]^c$ is weakly closed.

Claim 3: $K = \bigcup \alpha^{-1}(v)$.

It is sufficing to prove that $K \subset \bigcup \alpha^{-1}(v)$. Let $u \in K$,so by 3.3, there exists $v \in K$ such that $v \in \alpha(u)$, this means that $u \in \alpha^{-1}(v)$. Therefore, $K \subset \bigcup\left(\alpha^{-1}(v)\right)$.

Claim 4: $S = \bigcap_{v \in K} [\alpha^{-1}(v)]^c$ is empty or weakly compact.

Since $[\alpha^{-1}(v)]^c$ is weakly closed on a nonempty, bounded, closed and convex set K of reflexive Banach space X, then K is weakly compact set. Hence,S is weakly compact set as it an intersection of weakly closed subsets of weakly compact. Therefore.by conditions $(T_1 - T_2)$ of Lemma Tarafrar, there exists $u^0 \in \alpha(u^0)$ which implies

$$0 = \langle Tu^0, \eta(u^0, v)\rangle + H(u^0)J^0\left(Au^0; A\eta(u^0, v)\right) - \langle Fu^0, v - u^0\rangle_X < 0.$$

So, we have reached a contradiction. Therefore, (NQHIS) admits at least one solution.

H_S: let $\left(u^1, \dots, u^i\right)$,$\left(v^1, \dots, v^i\right) \in X_1 \times \dots \times X_i$ for every $n = \overline{1,\imath}$.

(i) $\eta_n(u_n, v_n) + \eta_n(v_n, u_n) = 0 \ \forall u_n \in X_n$;

(ii) $\langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n)\rangle_{X_n} + \langle T_n(v_1 \dots, v_i), \eta_n(v_n, u_n)\rangle_{X_n} \geq 0$;

(iii) $v_K \longmapsto \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n)\rangle_{X_n}$ is weakly l.s.c;

(iv) For each $v_i \in X_i$, the mapping $v \longmapsto \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n)\rangle_{X_n}$ is concave.

lemma 3.5. suppose that H_S holds. Then the following hold

(i) T is $\eta -$ monotone operator.

(ii) The mapping $u \longmapsto \langle Tu, \eta(u, v)\rangle_X$ is weakly u.s.c $\forall v \in X$.

(iii) $v_k \longmapsto \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n)\rangle_{X_n}$ is weakly l.s.c;

(iv) For each $v_i \in X_i$ the mapping
$v \longmapsto \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n) \rangle_{X_n}$ is concave.

Lemma 3.5. suppose that H_S holds. Then the following hold

(i) T is η –monotone operator.
(ii) The mapping $u \longmapsto \langle Tu, \eta(u, v) \rangle_X$ is weakly u.s.c $\forall v \in X$.
(iii) The mapping $v \longmapsto \langle Tu; \eta(u,v) \rangle_X$ is convex $\forall u \in X$.

Proof. From $H_S(i - ii)$, we have

$$\langle Tu - Tv, \eta(u, v) \rangle_X = \langle Tu, \eta(u, v) \rangle_X - \langle Tv, \eta(u, v) \rangle_X$$

$$= \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n) \rangle_{X_n}$$

$$- \sum_{n=1}^{i} \langle T_n(v_1, \dots, v_i), \eta_n(u_n, v_n) \rangle_{X_n}$$

$$= \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n) \rangle_{X_n}$$

$$+ \sum_{n=1}^{i} \langle T_n(v_1, \dots, v_i), \eta_n(v_n, u_n) \rangle_{X_n}$$

$$\geq 0.$$

Prove second assertion, let $\{u^m\} \subset X$ be a sequence which converges weakly to some $u \in X$. According to $H_S(i - ii)$ and the verity that $u^m \to u$, one can obtain

$$\limsup_m \langle T(u^m), \eta(u^m, v) \rangle_X = \limsup_m \sum_{n=1}^{i} \langle T_n({u_1}^m, \dots, {u_i}^m), \eta_n({u_n}^m, v_n) \rangle_{X_n}$$

$$- \liminf_m \sum_{n=1}^{i} \langle T_n({u_1}^m, \dots, {u_i}^m), \eta_n(v_n, {u_n}^m) \rangle_{X_n}$$

$$= -\sum_{n=1}^{i} \liminf_{m} \langle T_n(u_1{}^m, \dots, u_i{}^m), \eta_n(v_n, u_n{}^m)\rangle_{X_n}$$

$$\leq -\sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(v_n, u_n)\rangle_{X_n}$$

$$= \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n)\rangle_{X_n}$$

$$= \langle Tu, \eta(u, v)\rangle_X$$

Prove third assertion, let $u, v', v'' \in X$ then, by $H_S(i - iv)$ such that $v^t = tv' + (1-t)v''$ $\forall t \in (0,1)$, then

$$\langle Tu, \eta(u, v)\rangle_X = \sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(u_n, tv_n{}'(1-t)v_n{}'')\rangle_{X_n}$$

$$-\sum_{n=1}^{i} \langle T_n(u_1, \dots, u_i), \eta_n(tv_n{}'(1-t)v_n{}'', u_n)\rangle_{X_n}$$

$$\leq -\sum_{n=1}^{i} t\langle T_n(u_1, \dots, u_i), \eta_n(v_n{}', u_n\rangle_{X_n}$$

$$-\sum_{n=1}^{i} (1-t)\langle T_n(u_1, \dots, u_i), \eta_n(v_n{}'', u_n\rangle_{X_n}$$

$$= \sum_{n=1}^{i} t\langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n{}')\rangle_{X_n}$$

$$+ \sum_{n=1}^{i} (1-t)\langle T_n(u_1, \dots, u_i), \eta_n(u_n, v_n{}'')\rangle_{X_n}$$

$$= t\langle Tu, \eta(u, v')\rangle_X + (1 - t)\langle Tu, \eta(u, v'')\rangle_X.$$

Theorem 3.6. Assume that the nonempty, bounded, closed and convex set $K_n \subset X_n$ for every $n = \overline{1, \imath}$. If the condition H_F,H_η, H_S and H_H satisfy, then the system of nonlinear quasi- hemivariational inequalities (NQHIS) admits at least one solution.

Proof. Suppose that the set $\psi \subset K \times K$ defined as follows:

$$\psi = \{(u,v) \in K \times K: \langle Tv,\eta(u, v)\rangle_X + H(u)J^0(Au; A\eta(u, v)) - \langle Fu, v - u\rangle_X \geq 0\}.$$

One can follow the same steps in the Theorem 3.1 to conclude that the conditions required in Tarafdar fixed point Theorem are hold. Therefore, there exists $u_0 \in K$ such that

$$\langle Tv,\eta(u_0, v)\rangle_X + H(u_0)J^0(Au_0; A\eta(u_0, v)) \geq \langle Fu_0,v - u_0\rangle_X. \quad (3.7)$$

Since T is η-monotone operator, so

$$\langle Tu - Tv, \eta(u, v)\rangle_X \geq 0. \quad (3.8)$$

Adding (3.7) and (3.8) we have u_0 solves (VQHVI), which applying Proposition.

In order to highlight the advantage of our inequality problem dedicate the last section to two applications, one in equilibrium problem and the other in an abstract nonlinear quasi-hemivariational inequality.

4. APPLICATIONS

4.1. First application

In this subsection, we present a result one the connection between equilibrium problems and our inequalities in the case T is set-valued. It is clear any solution of the quasi-hemivariational inequality is a solution of the equilibrium problem, where the equilibrium bifunction

$\Lambda(u, v): K \times K \to R$ is defined by

$$\Lambda(u,v) = \sup_{z\in T(u)} \langle z, \eta(u,v)\rangle_X + H(u)J^0\big(Au; A\eta(u,v)\big) - \langle Fu, v-u\rangle_X. \qquad (4.1)$$

$u,v \in K$.

In what following, let rewrite (3.3) where T is set- valued:

Find $u \in K$ and $z \in T(u)$ such that

$$\langle z, \eta(u,v)\rangle_X + H(u)J^0\big(Au; A\eta(u,v)\big) \geq \langle Fu, v-u\rangle_X \qquad (4.2)$$

$\forall v \in K$.

The next result was introduced by B. Allechea, V. Radulescu and M. Sebaouia in [2]. We extend this result.

Corollary 4.1. If a set-valued $T: X \multimap X^*$ has a nonempty,convex and weak*compact values,then any solution of the equilibrium problem (EP) is a solution of the problem (4.2).

Proof. Let $u^* \in K \subset X$ such that $\Lambda(u^*, v) \geq 0$ for every $v \in K$. Arguing by contradiction. Let us assume that problem (4.2) has no solutions. So, there exists $v_z \in K$ for any $z \in T(u^*)$ in which

$\langle z, \eta(u^*, v_z)\rangle_X + H(u^*)J^0\big(Au^*; A\eta(u^*, v_z)\big) - \langle Fu^*, v_z - u^*\rangle_X < 0.$

Define P: $X^* \times K \times K \to R$ such that

$$P(z, v, u^*) := \langle z, \eta(u^*, v)\rangle_X + H(u^*)J^0\big(Au^*; A\eta(v, u^*)\big) - \langle Fu^*, v - u^*\rangle_X$$

One can easily show that for any $v \in K$, the mapping defined on X^* by

$$z \longmapsto P(z, v, u^*)$$

Is weak*-continuous. Let $S := -P$ so, S is weak*-continuous with respect to first variable where $S(z, v, u^*) > 0$. Define $\Upsilon_{v,u^*}: X \to R$ such that $\Upsilon_{v,u^*}(z) := S(z, v, u^*)$ is weak*-continuous too, So,

$\forall z \in T(u^*)$ there exists $v_z \in K$ such that

$x \in \bigcup_{v\in K, \rho>0}\{w : S(z, v, u^*) > \rho\} = \bigcup_{v\in K, \rho>0} \Upsilon_{v,u^*}{}^{-1}(\rho, +\infty)$ is weak*-open. Which shows

$$T(u^*) \subset \bigcup_{v\in K} \Upsilon_{v,u^*}{}^{-1}(\rho, +\infty).$$

Since $T(u^*)$ is weak*-compact,there is a finite sub cover

$\Upsilon_{v_1,u^*}{}^{-1}(\rho_1, +\infty), \ldots, \Upsilon_{v_{n1},u^*}{}^{-1}(\rho_n, +\infty)$, i.e.,

$$T(u^*) \subset \bigcup_{i=1}^{n} \Upsilon_{v_i,u^*}{}^{-1}(\rho_i, +\infty).$$

Letting $\rho = \min\{\rho_i, \dots, \rho_n\}$. So,

$$T(u^*) \subset \bigcup_{i=1}^{n} \Upsilon_{v_i, u^*}{}^{-1}(\rho, +\infty).$$

For every $z \in T(u^*)$ then

$$\underset{n=\overline{1,\imath}}{\mathrm{Max}}\, S(z, v, u^*) > \rho$$

It means that

$$\underset{n=\overline{1,\imath}}{\mathrm{Min}}\, S(z, v, u^*) < -\rho.$$

For every $z \in T(u^*)$. One can easily show that

$$z \longmapsto P(z, v_i, u^*)$$

is convex and proper with domain containing $T(u^*)$. Then by Theorem 21.1 in [23] there exists $\lambda \geq 0$, such that $\sum_{i=1}^{n} \lambda_i = 1$ for every $n = \overline{1,\imath}$. So, $\sum_{i=1}^{n} \lambda_i P(z, v_i, u^*) < -\rho$. It means that

$$\sum_{i=1}^{n} \lambda_i \left[\langle z, \eta(u^*, v_i)\rangle_X + H(u^*)J^0\big(Au^*; A\eta(u^*, v_i)\big) - \langle Fu^*, v_i - u^*\rangle_X\right] < -\rho.$$

Set $v^* = \sum_{i=1}^{n} \lambda_i v_i$. Therefore, by use remark (3.3) and remark (3.4) we have

$$\langle z, \eta(u^*, v^*\rangle_X + H(u^*)J^0\big(Au^*; A\eta(u^*, v^*)\big) - \langle Fu^*, v^* - u^*\rangle_X < -\rho$$

for every $z \in T(u^*)$, which implies that $\Lambda(u^*, v^*) < 0$ is a contradiction.

4.2. Second application.

In this subsection, we are going to apply our main result, expressed in the previous section for systems of an abstract nonlinear quasi- hemivariational inequality. We assume that Ω is a bounded, open subset of $\mathbb{R}^{\mathbb{N}}$. $j: \Omega \times R^N \times \dots \times R^N \to \mathbb{R}$ is a Carathe'odory function and that $j(z, \dots, \cdot)$ is a locally Lipchitz for every $z \in \Omega$ and the following conditions hold:

H_j: there exists $a_n \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}_+)$ in which

$$|w_n| \leq a_n(x) + b_n(x)|y|^{p-1} \quad (1 < p < \infty) \tag{4.3}$$

is almost $z \in \Omega$ and every $w_n \in \partial_n j(z, y_1, \dots, y_i)$ and every $y = (y_1, \dots, y_i) \in \mathbb{R}^N \times \dots \times \mathbb{R}^N$. Let us assume that $A = (A_1, \dots, A_i): X_1 \times \dots \times X_n \to L^p(\Omega, \mathbb{R}^N) \times \dots \times L^p(\Omega, \mathbb{R}^N)$ and that $J \circ A\colon K_1 \times \dots \times K_i \to \mathbb{R}$ is define by

$$J(Au) = \int_{\Omega} j\big(z, A_1 u_1(z), \dots, A_i u_i(z)\big)dz.$$

One can apply the Aubin-Clarke theorem [5] to get that

$$J_n^{\ 0}(Au;A_n v_n)) \le \int_\Omega J_n^{\ 0}\big(z, A_1u_1(z), \dots, A_iu_i(z); A_n v_n(z)\big)\, dz.$$

For every $n = \overline{1,\imath}$ and $v_n \in X_n$. Since $H_n: X_1 \times \dots \times X_i \in \mathbb{R} \cup \{+\infty\}$ is a non negative and continuous on K_n. Then

$$H_n(u_1, \dots, u_i) J_n^{\ 0}(Au; A_n v_n)$$

$$\le H_n(u_1, \dots, u_i) \int_\Omega J_n^{\ 0}\big(z, A_1u_1(z), \dots, A_iu_i(z); A_n v_n(z)\big)\, dz.$$

Corollary 4.2. Assume that the nonempty, bounded, closed and convex set $K_n \subset X_n$ for every $n = \overline{1,\imath}$. If the conditions H_F, H_T, H_i, H_H and $\eta(u_n, v_n) = v_n - u_n$ are satisfies. Then the following system of nonlinear quasi-hemivariational inequalities admits at least one solution. Finding $(u_1, \dots, u_i) \in K_1 \times \dots \times K_i$, in which

$$\langle T_1(u_1, \dots, u_i), v_1 - u_1\rangle + H_1(u_n) \int_\Omega j_1^{\ 0}(A_1u_1(x), \dots, A_i u_i(x); A_n v_n(x)$$

$$-A_n u_n(x))dx \ \ge \langle F_1(u_1, \dots, u_i), v_1 - u_1\rangle_{X_1}$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \qquad (4.4)$$

$$\langle T_i(u_1, \dots, u_i), v_i - u_i\rangle + H_i(u_n) \int_\Omega j_i^{\ 0}(A_1u_1(x), \dots, A_i u_i(x); A_n v_n(x)$$

$$- A_n u_n(x))dx \ \ge \langle F_i(u_1, \dots, u_i), v_i - u_i\rangle_{X_i}.$$

For all $(v_1, \dots, v_i) \in K_1 \times \dots \times K_i$.

It is worth mentioning that this kind of inequalities was studied by several authors (see [10,22]).

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