Non-Polynomial Splines technic to Approach the Solution of Sixth-Order Boundary-Value Problems

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Abstract: In this paper, we promoted numerical methods to get smooth approximations for the sixth-order Boundary Value Problem by applied non-polynomial spline functions .

Key Words: Non-polynomial spline functions method, sixth-order BVP,

1- Introduction

Let the sixth-order BVP,

$$\boldsymbol{y^{(6)}(x) = f(x, y), \quad a < x < b,} \tag{1}$$

with the boundary conditions:

$$\begin{aligned} &y(a) = A_0, \quad y^{(2)}(a) = A_2, \qquad y^{(4)}(a) = A_4 \\ &y(b) = B_0, \quad y^{(2)}(b) = B_2, \qquad y^{(4)}(b) = B_4 \end{aligned} \tag{2}$$

where $y(x)$ and $f(x,y)$ are continuous functions defined in the interval $x \in [a,b]$. It's assumed that $f(x,y) \in C^6[a,b]$ is real function and that $A_i, B_i, i = 0,2,4$, are finite real numbers. The literature on the numerical solution of sixth-order BVPs is sparse. Such problems are known to arise in astrophysics; the narrow convicting layers bounded by stable layers, which are believed to surround A-type stars, may be modeled by sixth- order BVPs (Toomre et al. [11]). Also in (Glatzmaier [9]) it is given that dynamo action in some stars may be calculated by such equations . (Chandrasekhar [6]) determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is an ordinary convection, the ordinary differential equation is sixth order .

Theorems, which list the conditions for the existence and uniqueness of solutions of sixth-order BVPs, are thoroughly discussed in the book by Agarwal (Agarwal [1]). Non-numerical techniques for solving such problems are contained in papers (Baldwin [3], Baldwin [4]) .

Numerical methods of solution are contained implicitly in (Chawala and Katti [7]), although those authors concentrated on numerical methods for fourth-order problems. E. H. Twizell (Twizell [12]) developed a second-order method for solving special and general sixth-order problems and in his later work (Twizell and Boutayeb [13]) developed finite-difference methods of order two, four, six and eight for solving such problems. The authors in (Siddiqi and Twizell [10]) used sixth-degree splines, where spline values at the mid knots of the interpolation interval and the corresponding values of the even order derivatives are related through consistency relations. Sinc-Galerkin method for the solutions of sixth order BVPs was used in (Gamel et al. [8]) whereas decomposition and modified domain decomposition methods were used in (Wazwaz [14]) to investigate solution of the sixth-order BVPs .

The spline function suggested in this study has the form:

T_7=Span $\{1,\ x,\ x^2,\ x^3,\ x^4\ ,\ x^5,\ cos\, kx,\ sin\, kx\}$

Such that *k* is the hesitancy of the trigonometric part to the splines function. Therefor in each Sub-interval $x_i \le x \le x_{i+1}$, we have:

Span $\{1,\ x,\ x^2,\ x^3,\ x^4\ ,\ x^5,\ cos\, kx,\ sin\, kx\}$,

Span $\{1,\ x,\ x^2,\ x^3,\ x^4\ ,\ x^5,\ cosh\, kx,\ sinh\, kx\}$, or

Span $\{1,\ x,\ x^2,\ x^3,\ x^4\ ,\ x^5, x^6, x^7\}$, $\quad (\ k \to 0)$

The above correlation We can explained The above correlation by the following:

T_7 = Span $\{1,\ x,\ x^2,\ x^3,\ x^4\ ,\ x^5,\ cos\, kx,\ sin\, kx\}$

$$= \text{Span}\begin{pmatrix} 1,\ x,\ x^2,\ x^3,\ x^4\ ,\ x^5,\ \frac{7!}{k^7}\left(kx - sin\, kx - \frac{(kx)^3}{6} + \frac{(kx)^5}{120}\right), \\ \frac{6!}{k^6}\left(1 - con\, kx - \frac{(kx)^2}{2} + \frac{(kx)^4}{24}\right) \end{pmatrix} \qquad (3)$$

2- Numerical methods

We can develop the approximation to the problem (1) and (2), By divide the interval $[a, b]$ is into *n* equal subinterval and using the grid points $x_i = a + ih\ ,\ i = 0, 1, \dots, n$ where $h = \frac{b-a}{n}$.

For each segment $[x_{i-1},\ x_i]$, the polynomial $P_{i-1/2}(x)$ has the form :

$$P_{i-1/2}(x) = a_{i-1/2} sin\, k(x - x_{i-1/2}) + b_{i-1/2} cos\, k(x - x_{i-1/2}) + c_{i-1/2}(x - x_{i-1/2})^5$$
$$d_{i-1/2}(x - x_{i-1/2})^4 + e_{i-1/2}(x - x_{i-1/2})^3 + g_{i-1/2}(x - x_{i-1/2})^2 + l_{i-1/2}(x - x_{i-1/2}) \quad + r_{i-1/2} \qquad (4)$$

where $a_{i-1/2}$, $b_{i-1/2}$, $c_{i-1/2}$, $d_{i-1/2}$, $e_{i-1/2}$, $g_{i-1/2}$, $l_{i-1/2}$ and $r_{i-1/2}$ are constants and *k* is free parameter. Let $y_{i-1/2}$ be an approximation to $y(\ x_{i-1/2}\ )$, obtained by the segment $P_{i-1/2}(x)$ of the mixed splines function passing through the points

$(x_{i-1/2}, y_{i-1/2})$ and $(x_{i+1/2}, y_{i+1/2})$. To determine the coefficients in (4) at the common nodes $(x_{i-1/2}, y_{i-1/2})$, we first define :

$$P_{i-1/2}(x_{i\pm j}) = y_{i\pm j},\ P'_{i-1/2}(x_{i\pm j}) = Z_{i\pm j},\ P''_{i-1/2}(x_{i\pm j}) = M_{i\pm j},\ P^{(vi)}_{i-1/2}(x_{i\pm j}) = S_{i\pm j},$$
$$i = 1, 2, \dots, n, j = 1/2$$

By algebraic manipulation, get the following expressions for coefficients:

$$a_{i-1/2} = \frac{1}{k^6}(S_{i-1/2}\cos(\theta) - S_{i+1/2}\sin(\theta),$$

$$b_{i-1/2} = \frac{S_{I-1/2}}{k^6},$$

$$c_{i-1/2} = \frac{\cos(\theta)}{2h^5k^6}\begin{pmatrix} h^2k^2\sin(\theta)S_{i+1/2} - h^2k^2\sin(\theta)S_{i-1/2} + h^2k^6\sin(\theta)M_{i-1/2} \\ -h^2k^6\sin(\theta)M_{i+1/2} - 6hkS_{i-1/2} + 6hk\cos(\theta)S_{i+1/2} \\ -6hk\cos(\theta)S_{i-1/2} + 6hkS_{i+1/2} + 6hk^6\sin(\theta)Z_{i-1/2} - 12\sin(\theta)S_{i+1/2} \\ +12\sin(\theta)S_{i-1/2} + 12k^6\sin(\theta)y_{i-1/2} - 12k^6\sin(\theta)y_{i+1/2} \end{pmatrix}$$

$$d_{i-1/2} = \frac{\cos(\theta)}{2h^4k^6}\begin{pmatrix} 2h^2k^2\sin(\theta)S_{i+1/2} - 3h^2k^2\sin(\theta)S_{i-1/2} + 3h^2k^6\sin(\theta)M_{i-1/2} \\ -2h^2k^6\sin(\theta)M_{i+1/2} - 14hkS_{i-1/2} + 14hk\cos(\theta)S_{i+1/2} \\ -16hk\cos(\theta)S_{i-1/2} + 16hkS_{i+1/2} + 16hk^6\sin(\theta)Z_{i-1/2} - 30\sin(\theta)S_{i+1/2} \\ +30\sin(\theta)S_{i-1/2} + 30k^6\sin(\theta)y_{i-1/2} - 30k^6\sin(\theta)y_{i+1/2} \end{pmatrix}$$

$$e_{i-1/2} = \frac{\cos(\theta)}{2h^5k^6}\begin{pmatrix} 20k^6\sin(\theta)y_{i-1/2} - 20k^6\sin(\theta)y_{i+1/2} - h^2k^6\sin(\theta)M_{i+1/2} \\ +h^2k^2\sin(\theta)S_{i+1/2} + 12hkS_{i+1/2} + 8hk\cos(\theta)S_{i+1/2} + 8hk^6\sin(\theta)Z_{i+1/2} \\ -8hkS_{i-1/2} + 3h^2k^6\sin(\theta)M_{i-1/2} - 3h^2k^2\sin(\theta)S_{i-1/2} - 12hk\cos(\theta)S_{i-1/2} \\ +12hk^6\sin(\theta)Z_{i-1/2} - 20\sin(\theta)S_{i+1/2} + 20\sin(\theta)S_{i-1/2} \end{pmatrix}$$

$$g_{i-1/2} = \frac{1}{2k^4}(-S_{i-1/2} + M_{i-1/2}k^4),$$

$$l_{i-1/2} = \frac{\cos(\theta)}{k^5}(-S_{i-1/2}\cos(\theta) + S_{i+1/2} + Z_{i-1/2}\sin(\theta)),$$

$$r_{i-1/2} = \frac{S_{i-1/2} - y_{i-1/2}k^6}{k^6},\ (\theta = kh). \tag{5}$$

By using the continuity condition for third , fourth and the fifth derivatives at $(x_{i-1/2}, y_{i-1/2})$ i.e. $P_{i-1}^{(n)}(x_{i\pm j}) = P_i^{(n)}(x_{i\pm j})$, where $n = 3,4,5,$ we obtain,

$$\left.\begin{array}{l} -\frac{1}{k^3}\sin(\theta)\,S_{i-1/2} - \frac{72}{h^2k^5}\cot(\theta)\,S_{i+1/2} + \frac{24}{h^2k^5}\cot(\theta)\,S_{i-1/2} + \frac{24}{h^2k^5}\cot(\theta)\,S_{i+3/2} \\ -\frac{1}{k^3}\cos(\theta)\,\cot(\theta)\,S_{i-1/2} + \frac{2}{k^3}\cot(\theta)\,S_{i+1/2} - \frac{18}{hk^4}S_{i+1/2} + \frac{3}{hk^4}S_{i-1/2} - \frac{3}{h}M_{i-1/2} \\ +\frac{18}{h}M_{i+1/2} + \frac{36}{h^2k^5}\csc(\theta)\,S_{i-1/2} - \frac{48}{h^2k^5}\csc(\theta)\,S_{i+1/2} - \frac{24}{h^2}Z_{i-1/2} + \frac{120}{h^3k^6}S_{i+1/2} \\ -\frac{60}{h^3k^6}S_{i-1/2} - \frac{60}{h^3}y_{i-1/2} + \frac{120}{h^3}y_{i+1/2} + \frac{36}{h^2k^5}\csc(\theta)\,S_{i+3/2} - \frac{60}{h^3}y_{i+3/2} \\ -\frac{3}{h}M_{i+3/2} + \frac{3}{hk^4}S_{i+3/2} + \frac{24}{h^2}Z_{i+3/2} - \frac{60}{h^3k^6}S_{i+3/2} - \frac{1}{k^3}\csc(\theta)\,S_{i+3/2} \end{array}\right\} = 0, \qquad (6)$$

$$\left.\begin{array}{l} -\frac{168}{h^3k^5}cot(\theta)\,S_{i+3/2} + \frac{168}{h^3k^5}cot(\theta)\,S_{i-1/2} - \frac{24}{h^2k^4}S_{i+3/2} + \frac{24}{h^2}M_{i+3/2} \\ -\frac{192}{h^3k^5}csc(\theta)\,S_{i+3/2} - \frac{168}{h^3}Z_{i+3/2} + \frac{360}{h^4k^6}S_{i+3/2} + \frac{360}{h^4}y_{i+3/2} \\ +\frac{24}{h^2k^4}S_{i-1/2} - \frac{24}{h^2}M_{i-1/2} + \frac{192}{h^3k^5}csc(\theta)\,S_{i-1/2} - \frac{168}{h^3}Z_{i-1/2} \\ -\frac{384}{h^3}Z_{i+1/2} - \frac{360}{h^4k^6}S_{i-1/2} - \frac{360}{h^4}y_{i-1/2} \end{array}\right\} = 0, \qquad (7)$$

$$\left.\begin{array}{l} \frac{1}{k}csc(\theta)\,S_{i-1/2} + \frac{1}{k}cos(\theta)\,cot(\theta)\,S_{i-1/2} - \frac{2}{k}cot(\theta)\,S_{i+1/2} - \frac{120}{h^3k^4}S_{i+1/2} \\ +\frac{60}{h^3k^4}S_{i-1/2} - \frac{60}{h^3}M_{i-1/2} + \frac{120}{h^3}M_{i+1/2} + \frac{360}{h^4k^5}csc(\theta)\,S_{i-1/2} - \frac{720}{h^4k^5}csc(\theta)\,S_{i+1/2} \\ -\frac{360}{h^4}Z_{i-1/2} + \frac{1440}{h^5k^6}S_{i+1/2} - \frac{720}{h^5k^6}S_{i-1/2} - \frac{720}{h^5}y_{i-1/2} + \frac{1440}{h^5}y_{i+1/2} + \frac{60}{h^3k^4}S_{i+3/2} \\ -\frac{60}{h^3}M_{i+3/2} + \frac{360}{h^4k^5}csc(\theta)\,S_{i+3/2} + \frac{360}{h^4}Z_{i+3/2} - \frac{720}{h^5k^6}S_{i+3/2} - \frac{720}{h^5}y_{i+3/2} \\ -\frac{720}{h^4k^5}cot(\theta)\,S_{i+1/2} + \frac{360}{h^4k^5}cot(\theta)\,S_{i-1/2} + \frac{360}{h^4k^5}cot(\theta)\,S_{i-1/2} + \frac{1}{k}cscS_{i+3/2} \end{array}\right\} = 0, \qquad (8)$$

Replacing *i* by $i+1, i-1, i+2, i-2,$ in Eqs. (6), (7), (8) we get the following equations:

$$\left.\begin{array}{l}
-\frac{1}{k^3}\sin(\theta)\, S_{i+1/2} - \frac{72}{h^2k^5}\cot(\theta)\, S_{i+3/2} + \frac{24}{h^2k^5}\cot(\theta)\, S_{i+1/2} + \frac{24}{h^2k^5}\cot(\theta)\, S_{i+5/2} \\
-\frac{1}{k^3}\cos(\theta)\,\cot(\theta)\, S_{i+1/2} + \frac{2}{k^3}\cot(\theta)\, S_{i+3/2} - \frac{18}{hk^4}S_{i+3/2} + \frac{3}{hk^4}S_{i+1/2} - \frac{3}{h}M_{i+1/2} \\
+\frac{18}{h}M_{i+3/2} + \frac{36}{h^2k^5}\csc(\theta)\, S_{i+1/2} - \frac{48}{h^2k^5}\csc(\theta)\, S_{i+3/2} - \frac{24}{h^2}Z_{i+1/2} + \frac{120}{h^3k^6}S_{i+3/2} \\
-\frac{60}{h^3k^6}S_{i+1/2} - \frac{60}{h^3}y_{i+1/2} + \frac{120}{h^3}y_{i+3/2} + \frac{36}{h^2k^5}\csc(\theta)\, S_{i+5/2} - \frac{60}{h^3}y_{i+5/2} \\
-\frac{3}{h}M_{i+3/2} + \frac{3}{hk^4}S_{i+5/2} + \frac{24}{h^2}Z_{i+5/2} - \frac{60}{h^3k^6}S_{i+5/2} - \frac{1}{k^3}\csc(\theta)\, S_{i+5/2}
\end{array}\right\} = 0, \qquad (9)$$

$$\left.\begin{array}{l}
-\frac{1}{k^3}\sin(\theta)\, S_{i-3/2} - \frac{72}{h^2k^5}\cot(\theta)\, S_{i-1/2} + \frac{24}{h^2k^5}\cot(\theta)\, S_{i-3/2} + \frac{24}{h^2k^5}\cot(\theta)\, S_{i+1/2} \\
-\frac{1}{k^3}\cos(\theta)\,\cot(\theta)\, S_{i-3/2} + \frac{2}{k^3}\cot(\theta)\, S_{i-1/2} - \frac{18}{hk^4}S_{i-1/2} + \frac{3}{hk^4}S_{i-3/2} - \frac{3}{h}M_{i-3/2} \\
+\frac{18}{h}M_{i-1/2} + \frac{36}{h^2k^5}\csc(\theta)\, S_{i-3/2} - \frac{48}{h^2k^5}\csc(\theta)\, S_{i-1/2} - \frac{24}{h^2}Z_{i-3/2} + \frac{120}{h^3k^6}S_{i-1/2} \\
-\frac{60}{h^3k^6}S_{i-3/2} - \frac{60}{h^3}y_{i-3/2} + \frac{120}{h^3}y_{i-1/2} + \frac{36}{h^2k^5}\csc(\theta)\, S_{i+1/2} - \frac{60}{h^3}y_{i+1/2} \\
-\frac{3}{h}M_{i+1/2} + \frac{3}{hk^4}S_{i+1/2} + \frac{24}{h^2}Z_{i+1/2} - \frac{60}{h^3k^6}S_{i+1/2} - \frac{1}{k^3}\csc(\theta)\, S_{i+1/2}
\end{array}\right\} = 0, \qquad (10)$$

$$\left.\begin{array}{l}
-\frac{1}{k^3}\sin(\theta)\, S_{i+3/2} - \frac{72}{h^2k^5}\cot(\theta)\, S_{i+5/2} + \frac{24}{h^2k^5}\cot(\theta)\, S_{i+3/2} + \frac{24}{h^2k^5}\cot(\theta)\, S_{i+7/2} \\
-\frac{1}{k^3}\cos(\theta)\,\cot(\theta)\, S_{i+3/2} + \frac{2}{k^3}\cot(\theta)\, S_{i+5/2} - \frac{18}{hk^4}S_{i+5/2} + \frac{3}{hk^4}S_{i+3/2} - \frac{3}{h}M_{i+3/2} \\
+\frac{18}{h}M_{i+5/2} + \frac{36}{h^2k^5}\csc(\theta)\, S_{i+3/2} - \frac{48}{h^2k^5}\csc(\theta)\, S_{i+5/2} - \frac{24}{h^2}Z_{i+3/2} + \frac{120}{h^3k^6}S_{i+5/2} \\
-\frac{60}{h^3k^6}S_{i+3/2} - \frac{60}{h^3}y_{i+3/2} + \frac{120}{h^3}y_{i+5/2} + \frac{36}{h^2k^5}\csc(\theta)\, S_{i+7/2} - \frac{60}{h^3}y_{i+7/2} \\
-\frac{3}{h}M_{i+7/2} + \frac{3}{hk^4}S_{i+7/2} + \frac{24}{h^2}Z_{i+7/2} - \frac{60}{h^3k^6}S_{i+7/2} - \frac{1}{k^3}\csc(\theta)\, S_{i+7/2}
\end{array}\right\} = 0, \qquad (11)$$

$$\left.\begin{array}{l}
-\frac{1}{k^3}\sin(\theta)\, S_{i-5/2} - \frac{72}{h^2k^5}\cot(\theta)\, S_{i-3/2} + \frac{24}{h^2k^5}\cot(\theta)\, S_{i-5/2} + \frac{24}{h^2k^5}\cot(\theta)\, S_{i-1/2} \\
-\frac{1}{k^3}\cos(\theta)\,\cot(\theta)\, S_{i-5/2} + \frac{2}{k^3}\cot(\theta)\, S_{i-3/2} - \frac{18}{hk^4}S_{i-3/2} + \frac{3}{hk^4}S_{i-5/2} - \frac{3}{h}M_{i-5/2} \\
+\frac{18}{h}M_{i-3/2} + \frac{36}{h^2k^5}\csc(\theta)\, S_{i-5/2} - \frac{48}{h^2k^5}\csc(\theta)\, S_{i-3/2} - \frac{24}{h^2}Z_{i-5/2} + \frac{120}{h^3k^6}S_{i-3/2} \\
-\frac{60}{h^3k^6}S_{i-5/2} - \frac{60}{h^3}y_{i-5/2} + \frac{120}{h^3}y_{i-3/2} + \frac{36}{h^2k^5}\csc(\theta)\, S_{i-1/2} - \frac{60}{h^3}y_{i-1/2} \\
-\frac{3}{h}M_{i-1/2} + \frac{3}{hk^4}S_{i-1/2} + \frac{24}{h^2}Z_{i-1/2} - \frac{60}{h^3k^6}S_{i-1/2} - \frac{1}{k^3}\csc(\theta)\, S_{i-1/2}
\end{array}\right\} = 0, \qquad (12)$$

$$\left.\begin{array}{l} -\frac{168}{h^3k^5}cot(\theta)\,S_{i+5/2}+\frac{168}{h^3k^5}cot(\theta)\,S_{i+1/2}-\frac{24}{h^2k^4}S_{i+5/2}+\frac{24}{h^2}M_{i+5/2} \\ -\frac{192}{h^3k^5}csc(\theta)\,S_{i+5/2}-\frac{168}{h^3}Z_{i+5/2}+\frac{360}{h^4k^6}S_{i+5/2}+\frac{360}{h^4}y_{i+5/2} \\ +\frac{24}{h^2k^4}S_{i+1/2}-\frac{24}{h^2}M_{i+1/2}+\frac{192}{h^3k^5}csc(\theta)\,S_{i+1/2}-\frac{168}{h^3}Z_{i+1/2} \\ -\frac{384}{h^3}Z_{i+3/2}-\frac{360}{h^4k^6}S_{i+1/2}-\frac{360}{h^4}y_{i+1/2} \end{array}\right\}=0, \qquad (13)$$

$$\left.\begin{array}{l} -\frac{168}{h^3k^5}cot(\theta)\,S_{i+1/2}+\frac{168}{h^3k^5}cot(\theta)\,S_{i-3/2}-\frac{24}{h^2k^4}S_{i+1/2}+\frac{24}{h^2}M_{i+1/2} \\ -\frac{192}{h^3k^5}csc(\theta)\,S_{i+1/2}-\frac{168}{h^3}Z_{i+1/2}+\frac{360}{h^4k^6}S_{i+1/2}+\frac{360}{h^4}y_{i+1/2} \\ +\frac{24}{h^2k^4}S_{i-3/2}-\frac{24}{h^2}M_{i-3/2}+\frac{192}{h^3k^5}csc(\theta)\,S_{i-3/2}-\frac{168}{h^3}Z_{i-3/2} \\ -\frac{384}{h^3}Z_{i-1/2}-\frac{360}{h^4k^6}S_{i-3/2}-\frac{360}{h^4}y_{i-3/2} \end{array}\right\}=0, \qquad (14)$$

$$\left.\begin{array}{l} -\frac{168}{h^3k^5}cot(\theta)\,S_{i+7/2}+\frac{168}{h^3k^5}cot(\theta)\,S_{i+3/2}-\frac{24}{h^2k^4}S_{i+7/2}+\frac{24}{h^2}M_{i+7/2} \\ -\frac{192}{h^3k^5}csc(\theta)\,S_{i+7/2}-\frac{168}{h^3}Z_{i+7/2}+\frac{360}{h^4k^6}S_{i+7/2}+\frac{360}{h^4}y_{i+7/2} \\ +\frac{24}{h^2k^4}S_{i+3/2}-\frac{24}{h^2}M_{i+3/2}+\frac{192}{h^3k^5}csc(\theta)\,S_{i+3/2}-\frac{168}{h^3}Z_{i+3/2} \\ -\frac{384}{h^3}Z_{i+5/2}-\frac{360}{h^4k^6}S_{i+3/2}-\frac{360}{h^4}y_{i+3/2} \end{array}\right\}=0, \qquad (15)$$

$$\left.\begin{array}{l} -\frac{168}{h^3k^5}cot(\theta)\,S_{i-1/2}+\frac{168}{h^3k^5}cot(\theta)\,S_{i-5/2}-\frac{24}{h^2k^4}S_{i-1/2}+\frac{24}{h^2}M_{i-1/2} \\ -\frac{192}{h^3k^5}csc(\theta)\,S_{i-1/2}-\frac{168}{h^3}Z_{i-1/2}+\frac{360}{h^4k^6}S_{i-1/2}+\frac{360}{h^4}y_{i-1/2} \\ +\frac{24}{h^2k^4}S_{i-5/2}-\frac{24}{h^2}M_{i-5/2}+\frac{192}{h^3k^5}csc(\theta)\,S_{i-5/2}-\frac{168}{h^3}Z_{i-5/2} \\ -\frac{384}{h^3}Z_{i-3/2}-\frac{360}{h^4k^6}S_{i-5/2}-\frac{360}{h^4}y_{i-5/2} \end{array}\right\}=0, \qquad (16)$$

$$\left.\begin{array}{l} \frac{1}{k}csc(\theta)\,S_{i+1/2}+\frac{1}{k}cos(\theta)\,cot(\theta)\,S_{i+1/2}-\frac{2}{k}cot(\theta)\,S_{i+3/2}-\frac{120}{h^3k^4}S_{i+3/2} \\ +\frac{60}{h^3k^4}S_{i+1/2}-\frac{60}{h^3}M_{i+1/2}+\frac{120}{h^3}M_{i+3/2}+\frac{360}{h^4k^5}csc(\theta)\,S_{i+1/2}-\frac{720}{h^4k^5}csc(\theta)\,S_{i+3/2} \\ -\frac{360}{h^4}Z_{i+1/2}+\frac{1440}{h^5k^6}S_{i+3/2}-\frac{720}{h^5k^6}S_{i+1/2}-\frac{720}{h^5}y_{i+1/2}+\frac{1440}{h^5}y_{i+3/2}+\frac{60}{h^3k^4}S_{i+5/2} \\ -\frac{60}{h^3}M_{i+5/2}+\frac{360}{h^4k^5}csc(\theta)\,S_{i+5/2}+\frac{360}{h^4}Z_{i+5/2}-\frac{720}{h^5k^6}S_{i+5/2}-\frac{720}{h^5}y_{i+5/2} \\ -\frac{720}{h^4k^5}cot(\theta)\,S_{i+3/2}+\frac{360}{h^4k^5}cot(\theta)\,S_{i+1/2}+\frac{360}{h^4k^5}cot(\theta)\,S_{i+5/2}+\frac{1}{k}cs\,c\,S_{i+5/2} \end{array}\right\}=0 \qquad (17)$$

$$\left.\begin{array}{l}\frac{1}{k}csc(\theta)\,S_{i-3/2}+\frac{1}{k}cos(\theta)\,cot(\theta)\,S_{i-3/2}-\frac{2}{k}cot(\theta)\,S_{i-1/2}-\frac{120}{h^3k^4}S_{i-1/2}\\ +\frac{60}{h^3k^4}S_{i-3/2}-\frac{60}{h^3}M_{i-3/2}+\frac{120}{h^3}M_{i-1/2}+\frac{360}{h^4k^5}csc(\theta)\,S_{i-3/2}-\frac{720}{h^4k^5}csc(\theta)\,S_{i-1/2}\\ -\frac{360}{h^4}Z_{i-3/2}+\frac{1440}{h^5k^6}S_{i-1/2}-\frac{720}{h^5k^6}S_{i-3/2}-\frac{720}{h^5}y_{i-3/2}+\frac{1440}{h^5}y_{i-1/2}+\frac{60}{h^3k^4}S_{i+1/2}\\ -\frac{60}{h^3}M_{i+1/2}+\frac{360}{h^4k^5}csc(\theta)\,S_{i+1/2}+\frac{360}{h^4}Z_{i+1/2}-\frac{720}{h^5k^6}S_{i+1/2}-\frac{720}{h^5}y_{i+1/2}\\ -\frac{720}{h^4k^5}cot(\theta)\,S_{i-1/2}+\frac{360}{h^4k^5}cot(\theta)\,S_{i-3/2}+\frac{360}{h^4k^5}cot(\theta)\,S_{i+1/2}+\frac{1}{k}csc(\theta)S_{i+1/2}\end{array}\right\}=0 \qquad (18)$$

$$\left.\begin{array}{l}\frac{1}{k}csc(\theta)\,S_{i+3/2}+\frac{1}{k}cos(\theta)\,cot(\theta)\,S_{i+3/2}-\frac{2}{k}cot(\theta)\,S_{i+5/2}-\frac{120}{h^3k^4}S_{i+5/2}\\ +\frac{60}{h^3k^4}S_{i+3/2}-\frac{60}{h^3}M_{i+3/2}+\frac{120}{h^3}M_{i+5/2}+\frac{360}{h^4k^5}csc(\theta)\,S_{i+3/2}-\frac{720}{h^4k^5}csc(\theta)\,S_{i+5/2}\\ -\frac{360}{h^4}Z_{i+3/2}+\frac{1440}{h^5k^6}S_{i+5/2}-\frac{720}{h^5k^6}S_{i+3/2}-\frac{720}{h^5}y_{i+3/2}+\frac{1440}{h^5}y_{i+5/2}+\frac{60}{h^3k^4}S_{i+7/2}\\ -\frac{60}{h^3}M_{i+7/2}+\frac{360}{h^4k^5}csc(\theta)\,S_{i+7/2}+\frac{360}{h^4}Z_{i+7/2}-\frac{720}{h^5k^6}S_{i+7/2}-\frac{720}{h^5}y_{i+7/2}\\ -\frac{720}{h^4k^5}cot(\theta)\,S_{i+5/2}+\frac{360}{h^4k^5}cot(\theta)\,S_{i+3/2}+\frac{360}{h^4k^5}cot(\theta)\,S_{i+7/2}+\frac{1}{k}csc(\theta)S_{i+7/2}\end{array}\right\}=0 \qquad (19)$$

$$\left.\begin{array}{l}\frac{1}{k}csc(\theta)\,S_{i-5/2}+\frac{1}{k}cos(\theta)\,cot(\theta)\,S_{i-5/2}-\frac{2}{k}cot(\theta)\,S_{i-3/2}-\frac{120}{h^3k^4}S_{i-3/2}\\ +\frac{60}{h^3k^4}S_{i-5/2}-\frac{60}{h^3}M_{i-5/2}+\frac{120}{h^3}M_{i-3/2}+\frac{360}{h^4k^5}csc(\theta)\,S_{i-5/2}-\frac{720}{h^4k^5}csc(\theta)\,S_{i-3/2}\\ -\frac{360}{h^4}Z_{i-5/2}+\frac{1440}{h^5k^6}S_{i-3/2}-\frac{720}{h^5k^6}S_{i-5/2}-\frac{720}{h^5}y_{i-5/2}+\frac{1440}{h^5}y_{i-3/2}+\frac{60}{h^3k^4}S_{i-1/2}\\ -\frac{60}{h^3}M_{i-1/2}+\frac{360}{h^4k^5}csc(\theta)\,S_{i-1/2}+\frac{360}{h^4}Z_{i-1/2}-\frac{720}{h^5k^6}S_{i-1/2}-\frac{720}{h^5}y_{i-1/2}\\ -\frac{720}{h^4k^5}cot(\theta)\,S_{i-3/2}+\frac{360}{h^4k^5}cot(\theta)\,S_{i-5/2}+\frac{360}{h^4k^5}cot(\theta)\,S_{i-1/2}+\frac{1}{k}csc(\theta)S_{i-1/2}\end{array}\right\}=0 \qquad (20)$$

By solving equations. (3) - (19) together, we can remove Z's and M's . The obtained values of Z's and M's are used in Eq. (3), which result, after lengthy calculations the following recurrence relation ,

$$y_{i-7/2}-6y_{i-5/2}+15y_{i-3/2}-20y_{i-1/2}+15y_{i+1/2}-6y_{i+3/2}+y_{i+5/2}$$
$$=h^6\{\alpha\left(S_{i-7/2}+S_{i+5/2}\right)+\beta\left(S_{i-5/2}+S_{i+3/2}\right)+\gamma\left(S_{i-3/2}+S_{i+1/2}\right)+\delta S_{i-1/2} \qquad (21)$$

where

$$\alpha=\frac{1}{120}\left(-\frac{120}{\theta^6}-\frac{20}{\theta^3 sin\theta}+\frac{1}{\theta sin\theta}+\frac{120}{\theta^5 sin\theta}\right)$$

$$\beta=\frac{1}{120}\left(-\frac{40}{\theta^3 sin\theta}-\frac{2cos\theta}{\theta sin\theta}+\frac{720}{\theta^6}-\frac{480}{\theta^5 sin\theta}-\frac{240cos\theta}{\theta^5 sin\theta}+\frac{26}{\theta sin\theta}+\frac{40cos\theta}{\theta^3 sin\theta}\right)$$

$$\gamma = \frac{1}{120}\left(\frac{100}{\theta^3 sin\theta} + \frac{67}{\theta sin\theta} + \frac{960cos\theta}{\theta^5 sin\theta} + \frac{480}{\theta^5 sin\theta} - \frac{1800}{\theta^6} - \frac{52cos\theta}{\theta sin\theta} + \frac{80cos\theta}{\theta^3 sin\theta}\right)$$

$$\delta = \frac{1}{120}\left(\frac{2400}{\theta^6} - \frac{960}{\theta^5 sin\theta} - \frac{80}{\theta^3 sin\theta} - \frac{132cos\theta}{\theta sin\theta} - \frac{240cos\theta}{\theta^3 sin\theta} + \frac{52}{\theta sin\theta} - \frac{1440cos\theta}{\theta^5 sin\theta}\right) \quad (22)$$

for $i = 4, 5, 6, \dots, n-3$.

Here $\lim_{\theta\to 0}(\alpha, \beta, \gamma, \delta) = \frac{1}{5040}(1, 120, 1191, 2416)$, which is the polynomial case as mentioned in Eq. (3) .

The local truncation error t_i, associated with the scheme developed in (21) is ,

$$t_i = C_6 h^6 y_i^6 + C_7 h^7 y_i^7 + C_8 h^8 y_i^8 + C_9 h^9 y_i^9 + C_{10} h^{10} y_i^{10} + C_{11} h^{11} y_i^{11}$$
$$+C_{12} h^{12} y_i^{12} + C_{13} h^{13} y_i^{13} + C_{14} h^{14} y_i^{14} + O(h^{15}), \quad (23)$$

where

$$C_6 = -(-1 + 2\alpha + 2\beta + 2\gamma + \delta),$$

$$C_7 = \frac{(-1 + 2\alpha + 2\beta + 2\gamma + \delta)}{2},$$

$$C_8 = -\frac{(-3 + 74\alpha + 34\beta + 10\gamma + \delta)}{8},$$

$$C_9 = \frac{(-7 + 218\alpha + 98\beta + 26\gamma + \delta)}{48},$$

$$C_{10} = \frac{(121 + 5302\alpha + 706\beta + 82\gamma + \delta)}{1920},$$

$$C_{11} = \frac{(-77 + 13682\alpha + 2882\beta + 242\gamma + \delta)}{3840},$$

$$C_{12} = \frac{(6227 + 2133274\alpha + 16354\beta + 730\gamma + \delta)}{967680},$$

$$C_{13} = \frac{(-3353 + 3745418\alpha + 75938\beta + 2186\gamma + \delta)}{1935360},$$

$$C_{14} = -\frac{(-4681 + 6155426\alpha + 397186\beta + 6562\gamma + \delta)}{10321920} \quad (24)$$

So for different choices of $\alpha, \beta, \gamma, \delta$ in scheme (21), methods of different order are obtained. The relation (21) gives $n-6$ algebraic equations in *n* unknowns $y_{m+1/2}, m = 0,1,\dots,n-1$. We require six more equations, three at each end of the range of integration, for the direct computation of $y_{m+1/2}$. These equations are developed by Taylor series and method of undetermined coefficients . "Generally from the boundary equations of the main scheme is as following :

$$20y_0 - 35y_{1/2} + 21y_{3/2} - 7y_{5/2} + y_{7/2} + \tau y_{9/2}$$
$$= \alpha_1 h^2 y_0^{(2)} + \beta_1 h^4 y_0^{(4)} + h^6\left(a_1 y_{1/2}^{(6)} + b_1 y_{3/2}^{(6)} + c_1 y_{5/2}^{(6)} + d_1 y_{7/2}^{(6)} + e_1 y_{9/2}^{(6)} + g_1 y_{11/2}^{(6)}\right) \quad (25)$$

$$-10y_0 + 21y_{1/2} - 21y_{3/2} + 15y_{5/2} - 6y_{7/2} + y_{9/2}$$
$$= \alpha_2 h^2 y_0^{(2)} + \beta_2 h^4 y_0^{(4)} + h^6\left(a_2 y_{1/2}^{(6)} + b_2 y_{3/2}^{(6)} + c_2 y_{5/2}^{(6)} + d_2 y_{7/2}^{(6)} + e_2 y_{9/2}^{(6)} + g_2 y_{11/2}^{(6)}\right) \quad (26)$$

$$2y_0 - 7y_{1/2} + 15y_{3/2} - 20y_{5/2} + 15y_{7/2} - 6y_{9/2} + y_{11/2}$$
$$= \alpha_3 h^2 y_0^{(2)} + \beta_3 h^4 y_0^{(4)} + h^6\left(a_3 y_{1/2}^{(6)} + b_3 y_{3/2}^{(6)} + c_3 y_{5/2}^{(6)} + d_3 y_{7/2}^{(6)} + e_3 y_{9/2}^{(6)} + g_3 y_{11/2}^{(6)}\right) \quad (27)$$

The remaining three equations at the other end can be obtained from (25), (26) and (27) by writing them in reverse order. The constants $a_i, b_i, c_i, d_i, e_i, g_i, \alpha_i, \beta_i$, and τ are parameters which must be chosen so that the local truncation errors of (25)-(27) are identical with (21) .

3- Convergence

The family of numerical methods is described by the Eqs. (21) boundary equations and the solution vector $Y = [y_{1/2}, y_{3/2}, \dots, y_{n-1/2}]^T$. *T* denoting transpose, is obtained by solving a nonlinear algebraic system of order *n* which has the form :

$$A_0 Y + h^6 B f(x, Y) - C = 0, \quad (28)$$

Now we investigate the error analysis of the seven-degree non-polynomial spline method described in Numerical Methods to do so we let ,

$y = \left(y(x_{i+1/2})\right), Y = (y_{i+1/2}), C = (c_i), T = (t_i)$ and $E = (e_{i+1/2})$, are the n-dimensional column vectors. where $e_{i+1/2} = y_{i+1/2} - y_{i+1/2}$ represent the discretization error for $i = 4, \dots, n-4$. So, the introduced method can be write in the matrix form:

$$A_0 y - h^6 B f(x, y) - C = 0, \text{ such that,}$$

$$A_0 = \begin{bmatrix} -35 & 21 & a & 0 & 0 \\ 21 & -21 & b & 0 & 0 \\ a^T & b^T & M & b^{TR} & a^{TR} \\ 0 & 0 & b^R & -21 & 12 \\ 0 & 0 & a^R & 21 & -35 \end{bmatrix}, \qquad (29)$$

where $a = [-7,1,0,\dots,0], b = [15,-6,1,0,\dots,0]$. Where T indicate the operation for transposition and for a row vector $\mathbf{v} = (v_1, v_2, \dots, v_n), \mathbf{v}^R = (v_n, v_{n-1}, \dots, v_1)$. Further, *a, b* are (*n-4*) dimensional row vectors. The matrix $M = (m_{i,j})$ is a seven-band matrix of order (*n-4*) given by :

$$m_{i,j} = \begin{cases} -20, & i = j = 1,2,\dots,n-4, \\ 15, & |i-j| = 1, \\ -6, & |i-j| = 2 \\ 1, & |i-j| = 3 \\ 0, & |i-j| > 3 \end{cases} \qquad (30)$$

$$B = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & g_1 & & & & & \\ a_2 & b_2 & c_2 & d_2 & e_2 & g_2 & & & & & \\ a_3 & b_3 & c_3 & d_3 & e_3 & g_3 & & & & & \\ \alpha & \beta & \gamma & \delta & \beta & \alpha & & & & & \\ & \ddots & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & & \alpha & \beta & \gamma & \delta & \beta & \alpha \\ & & & & g_3 & e_3 & d_3 & c_3 & b_3 & a_3 \\ & & & & g_2 & e_2 & d_2 & c_2 & b_2 & a_2 \\ & & & & g_1 & e_1 & d_1 & c_1 & b_1 & a_1 \end{bmatrix} \qquad (31)$$

The column vector C is given by

$$C = \begin{bmatrix} -20A_0 + \alpha_1 h^2 A_2 + \beta_1 h^4 A_4 \\ 10A_0 + \alpha_2 h^2 A_2 + \beta_2 h^4 A_4 \\ -2A_0 + \alpha_3 h^2 A_2 + \beta_3 h^4 A_4 \\ 0 \\ \\ \\ 0 \\ -2B_0 + \alpha_3 h^2 B_2 + \beta_3 h^4 B_4 \\ 10B_0 + \alpha_2 h^2 B_2 + \beta_2 h^4 B_4 \\ -20B_0 + \alpha_1 h^2 B_2 + \beta_1 h^4 B_4 \end{bmatrix} \tag{32}$$

The vector $y = [y_{1/2}, y_{3/2}, \dots, y_{n-1/2}]^T$ satisfies

$$A_0 Y + h^6 B f(x,Y) - C = 0,$$

Here $T = [t_1, t_2, \dots, t_n]$ represent the vector of local truncation errors, and a conventional convergence analysis shows that the norm of the vector

$$E = y - Y$$

satisfies:

$$||\mathrm{E}||_\infty \leq \frac{(b-a)^6}{46080-(61(b-a)^6+175h^2(b-a)^4+259h^4(b-a)^2+225h^6)B^*F^*} \times \{C_6 h^6 V_6 + C_7 h^7 V_7 + C_8 h^8 V_8 + C_9 h^9 V_9 + C_{10} h^{10} V_{10} + C_{11} h^{11} V_{11} + C_{12} h^{12} V_{12} + \cdots\}, \tag{33}$$

where $V_i = \max_{a \le x \le b} |\frac{d^i y(x)}{dx^i}|$ for $i = 1,2,\dots,$ $B^* = ||B||$, and $F^* = \max_{a \le x \le b} |\frac{\partial f}{\partial y(x)x}|$.

The order of convergence of the numerical method is, thus *p*, if C_{p+6} is the first non-vanishing constant on the right hand side of (23) provided

$$F^* < \frac{(b-a)^6}{\left(61(b-a)^6 + 175h^2(b-a)^4 + 259h^4(b-a)^2 + 225h^6\right)} \tag{34}$$

4- Numerical Result and Discussion

We tested our numerical methods on the nonlinear and linear problems as following:

Firstly: Nonlinear problem

Let the nonlinear sixth-order BVP,

$$y^{(6)}(x) = e^{-x}y^2(x), \quad 0 < x < 1, \tag{35}$$

with to the following boundary conditions:

$$y(0) = y^{(2)}(0) = y^{(4)}(0) = 1, \; y(1) = y^{(2)}(1) = y^{(4)}(1) = e \tag{36}$$

The theoretical solution for this problem is

$$y(x) = e^x. \tag{37}$$

Secondly: linear problems

Consider the linear sixth-order BVP,

$$y^{(6)}(x) + xy = -(24 + 11x + x^3)e^x(x), \quad 0 \le x \le 1 \tag{38}$$

With boundary conditions:

$$y(0) = 0 = y(1), y^{(2)}(0) = 0, y^{(2)}(1) = -4e, y^{(4)}(0) = -8, y^{(4)}(1) = -16e \tag{39}$$

The analytical solution of the above differential equation is

$$y(x) = x(1-x)e^x, \tag{40}$$

The results of maximum errors (in absolute value) for two presented problems are listed in table (1), where we show that our new method (21) that we have used to resolve the problem (1) was better than [14,5,2] in its performance after comparing with the results of [14,5,2] . Figs. (1) and (2) show graphs of exact and approximate solutions for various of *n* . Error graphs are shown in Figs. (3), (4) .

Table (1)

n	Sixth order method	Wazwaz [14]	Boutayeb and Twizall[5]	Akram and Siddiqi[2]
1	4.55×10^{-5}	2.33×10^{-3}	2.41×10^{-5}	1.04×10^{-4}
3	3.577×10^{-7}	1.299×10^{-4}	3.77×10^{-6}	9.2×10^{-5}
5	5.96×10^{-10}	4.42×10^{-7}	7.56×10^{-10}	6.58×10^{-6}
7	2.68×10^{-11}	3.39×10^{-8}	2.25×10^{-10}	6.86×10^{-7}
10	1.97×10^{-13}	7.00×10^{-11}	4.55×10^{-13}	7.33×10^{-8}

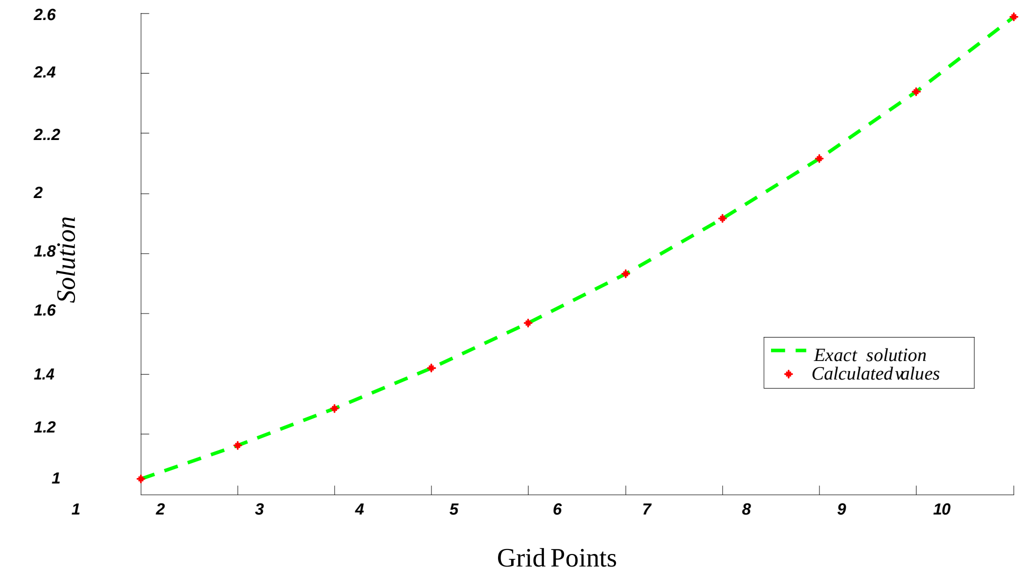


Figure (1): Sketch for solution for problem (1) , $n = 1,2,3, \dots ,10$

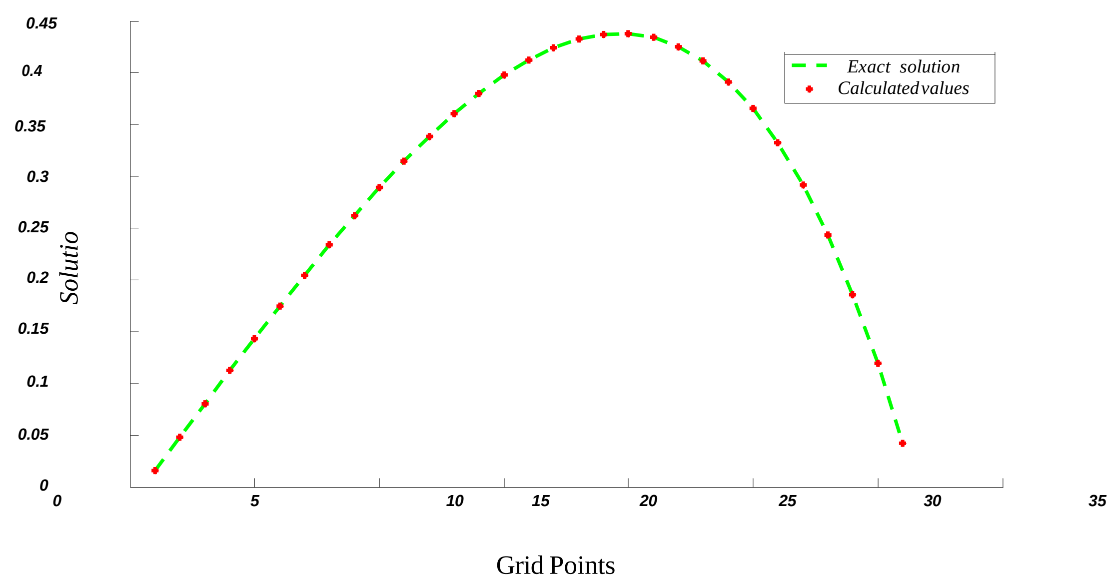


Figure (2): Sketch of solution for nonlinear problem (38) , $n = 1,2,3, \dots ,31$

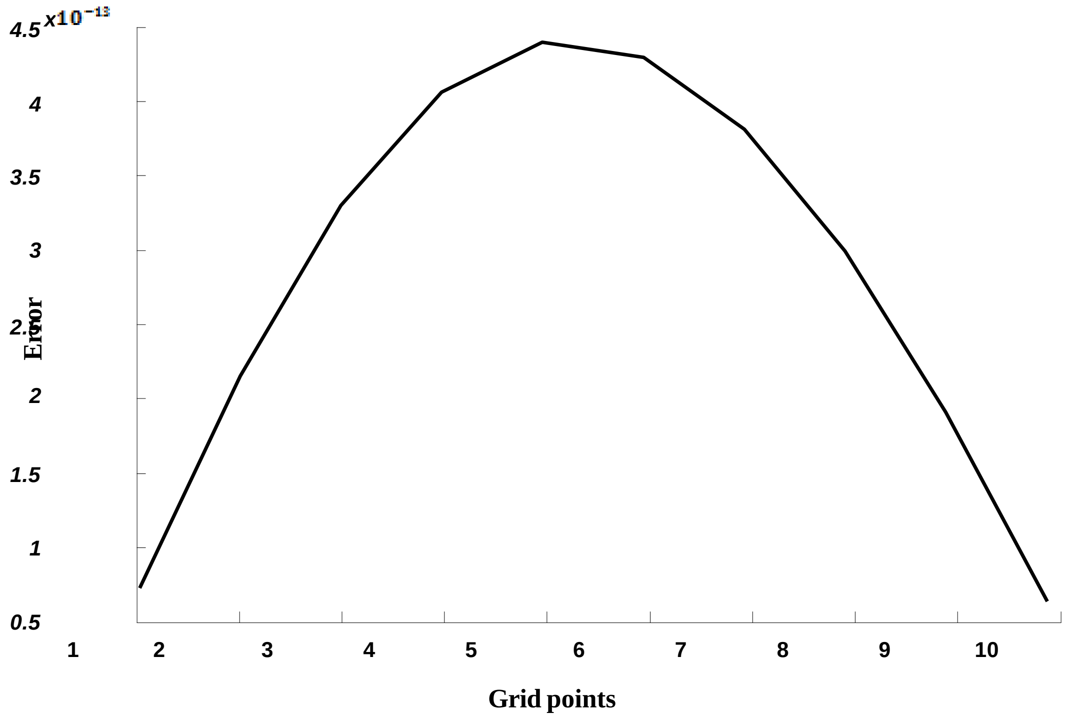
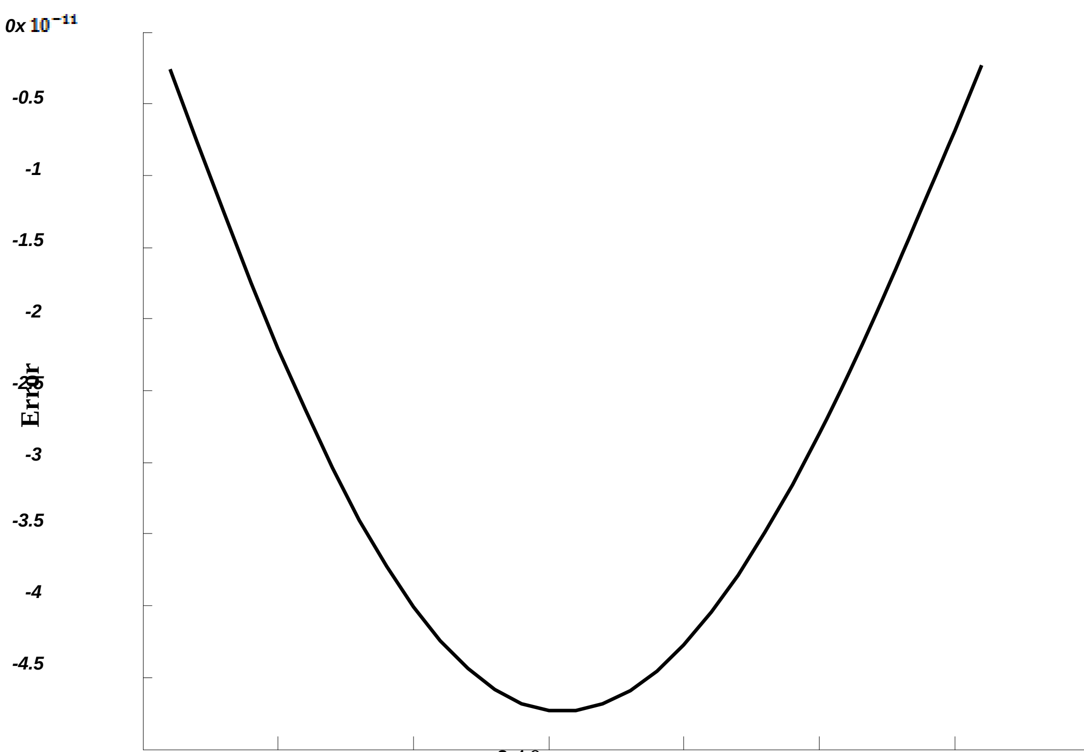


Figure (3): Error Sketch for problem (1) , $n = 1,2,3, \dots,10$



-5

0 5 10 15 20 25 30 35

Grid points

Figure(4): Sketch of solution for nonlinear problem (38) , $n = 1,2,3, \dots ,31.$

4 Conclusion

If we used the Non-polynomial spline functions to develop a class of numerical methods to approximate solution of sixth-order linear and non-linear BVPs , with two point boundary conditions. Sixth- order convergence is obtained. It has been shown that the relative errors in absolute value confirm the theoretical convergence .

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