On (k, n; f) – arcs of type (1, n) in PG(2, 9)

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Abstract

In this paper we discussed the existence of (k, n; f) – arcs of type (1, n) in the projective plane of order nine; when Im(f) = { 0, 1, ω } and 1< ω < n, where $(n-1)|9$, i.e. n = 4 or 10. For this purpose we use the technique in reference [12] and we deduced the example (19, 4; f) – arc of type (1, 4) when ω = 2, and we have an ordinary (13, 4) – arc of type (1, 4) in PG(2, 9), when ω = 3. Also we have the examples (46, 10; f) – arc when ω = 2 and the monoidal (11, 10; f) – arc when ω = 9, which is of type (1, 10) in PG(2, 9). Also we proved there are no (k, 10; f) – arcs of type (1, 10), in PG(2, 9), for other values of ω (ω ≠ 2 and 9) .

Introduction

In the last years the study of finite projective spaces has been developed in many different directions. Recently generalizations of the notions of (k,n) – arcs and (k,n) – caps were given and studied by various authors. In 1971, M. Tallini Scafati [13] introduced the notion of a graphic curve with "multiple points" of order n in a finite projective plane over $GF(q)$; a later paper [10] by A. D. Keedwell pursued further investigation about graphic arcs with "multiple points". In 1978, A. Barlotti [2] presented the notion of a $(k,n;\{w_i\})$– set of kind s relating this purely geometrical concept to some combinatorial questions. Among them there is the "packing problem" which is known to be important in the theory of factorial designs and in the theory of error correcting codes. The $(k,n;\{w_i\})$– set of kind 2 in a projective plane, also called $(k,n;\{w_i\})$– arcs, where studied by M. Barnabei [3]. Then, the study of the weighted arcs of type $(n-2,n)$ in a finite projective plane was developed by E. D'Agostini [4] and the study of the weighted arcs of type $(n-3,n)$ in a finite projective plane was developed by B. J. Wilson [15], F. K. Hameed [6] and M. Y. Abass [1], also the weighted arcs of type $(n-5,n)$ was developed by R. D. Mahmood [11] and M. Y. Abass at el [8]. The notion of weighted arc of type

$(1, n)$ introduced by G. Raguso and L. Rella [12]. Also, F. K. Hameed [7] generalize the results of the monoidal arcs in PG(2, q).

In this paper we investigated the weighted arcs of type $(1, n)$ and its properties in the finite projective plane of order nine.

1. Preliminaries

We will denote by $PG(2, q)$ the projective desarguesian plane of order $q = p^h$, by $\mathcal{P}$ the set of all points of the plane and by $\mathcal{R}$ the set of all lines of the plane. Then $PG(2, q)$ having $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Each line contains $q + 1$ points and through every point there pass $q + 1$ lines. Also the projective plane has the following properties:

(1) Any two points have exactly one line joining them and any two different lines meet at just one point.

(2) There exists four points, no three of them are collinear.

Definition 1.1. [9]

(k, n) – Arc in PG(2, q) is a set of k points no $n + 1$ of which are collinear, where $n \geq 2$. Write simply k – arc for (k, 2) – arc.

Definition 1.2. [9]

A line ℓ in PG(2, q) is an i – secant of a (k, n) – arc $\mathcal{K}$ if $|\ell \cap \mathcal{K}| = i$. Let τ_i denote the total number of i – secants to $\mathcal{K}$ in PG(2, q), then the type of $\mathcal{K}$ is defined by $(\tau_0, \tau_1, \dots, \tau_n)$.

Lemma1.1. [9]

For a (k, n) – arc $\mathcal{K}$, the following equations hold :

$$(1) \quad \sum_{i=0}^{n} \tau_i = q^2 + q + 1 \quad ; \qquad (2) \quad \sum_{i=1}^{n} i\tau_i = k \quad ;$$

$$(3) \quad \sum_{i=2}^{n} \frac{i(i-1)}{2} \tau_i = \frac{k(k-1)}{2} .$$

For any function f from $\mathcal{P}$ to the set of natural numbers N we will say that $f(P)$ is the weight of the point P . From such f we may define a function F from $\mathcal{R}$ to N in the following way:

and we will say that $F(r)$ is the weight of the line r . Moreover, if $F(r) = j$ we will also say that r is a " j – weighting " line.

Definition 1.3. [5]

A arc K in $PG(2,q)$ is a function $f : \mathcal{P} \rightarrow N$ such that

$k =$ | support of (the points of non – zero weight) | and $n = \max F$.

Let us remark that an ordinary

$(k,n)-$ arc is a arc with

Im .

Definition 1.4.[6]

For any arc K the underlying arc is the ordinary arc whose points are all the points of K .

Definition 1.5.[7]

A arc K is called monoidal if Im f = { 0, 1, ω } and $l_\omega = 1$.

Definition 1.6.[5]

The characters of a arc K are the

integers $j = 0, 1, \dots, n$.

Definition 1.7.[5]

The type of a arc K is the set of Im F . To write explicitly the type of K we can use the sequence

$(n_1, \dots, n_\rho)$ where and $n_1 < n_2 < \dots < n_\rho = n$.

Let us use the following notation:

$\omega = \max f$, $W = \sum_{P \in \mathcal{P}} f(P)$ and W will be called the weight of K .

$L_i = f^{-1}(i)$ and $l_i = |L_i|$, $i = 0, 1, \dots, \omega$.

indicates the set of all lines through the point M .

It is well known from [5] that:

(1.1)

.

(1.2)

.

(1.3) .

(1.4) .

A useful result, mentioned in [5], is the following:

If there exists a point of a arc such that every line through it is a – weighting line, then

(1.5)

(1.6) If and

, then .

Hence .

An arc with weight such that the equality holds, is called maximal. Of course, a maximal arc is also such that through a point of maximal weight there pass only – weighting lines.

Finally we shall recall [5] the following relations concerning the

characters of a arc :

(1.7)

(1.8)

(1.9)

2. (k, n; f) – arcs of type (m, n)

From now on, shall denote a

arc of type , where
. Let firstly state the following:

Lemma 2.1.[6]

The weight of a of type satisfies:

.

We call arcs for which the values in lemma (2.1) are attained, maximal and minimal of type respectively.

Theorem 2.1. [6]

Let be a of type and let and respectively the number of lines of weight and the number of lines of weight passing through a point of weight . Then

Theorem 2.2.[6]

A necessary condition for the existence of a of type is that:

;

3. (k, n; f) – arcs of type (1, n)

In this section we discussed (k, n; f) – arcs of type . Then we get the following properties as in [12]:

(a) Im f { 0, 1, ω };

(b) The weight of a (k, n; f) – arc satisfies:

.

(c)

(d) From theorem (2.1) we have:

and

.

(e) The characters of a (k, n; f) – arc of type are given by:

(f) The weight of the plane must be a root of the following equation of degree two:

(3.1)

Proof: (a) suppose that Im f { 0, 1, ..., ω }; then for and this contradicts with lines of weight 1.

(b) From Lemma (2.1) with .

(c) From Theorem (2.1) with

.

(d) From Theorem (2.2) with

.

(e) From the equations (1.7) and (1.8) with only.

(f) From equation (1.9) and part (e) with only and .

□

Proposition 3.1.

A necessary condition for the existence in a projective plane of order q of a (k, n; f) – arc of type with is that the total weight of the plane is

.

Proof : see [12, proposition (1)].

Also from [12], we have the following:

By (3.3) it follows that , and so

.

In this paper we take , and since or .

4. The case in which

In this case we have . Then .

If then we have the following theorem:

Theorem 4.1.

In a projective plane of order q the (k, 4; f) – arcs of type (1, 4) with are precisely those which have as points of weight 1 the points of a (2q-2, 4) – arc of type (0, 1, 2, 4) having three 0 – secants and 3(q-1) 2 – secants and as points of weight the points of intersection of the 0 – secants .

Proof : See [12, Proposition (5)]. □

Now, if , then from (3.3) we get:

Then Im f = { 0, 1 } and we have an ordinary (13, 4) – arc of with only 1 – secants and 4 – secants in PG(2, 9), which is represent as a subplane PG(2, 3) of PG(2, 9) as in [1, p.7 – 8 and p.74].

5. The case in which

In this section we discuss the (k, 10; f) – arcs of type (1, 10) in PG(2, 9). From theorem (2.2) we must have , and , this implies that take one value from the set { 2, 3, 4, 5, 6, 7, 8, 9 }.

If and , we have the following theorem:

Theorem 5.1.

In a projective plane of odd order q the (k, n; f) – arcs of type (1, n) with and are precisely those which have as points of weight 1 the points of an oval and as points of weight the interior points of .

Proof : See [12, Proposition (3)]. □

Now, if , we have the following:

In equation (3.3), substitute and , we get:

, and

.

Since , we have no more than three points of weight 3 on a line. Then the points of weight 3 form (21, 3) – arc in PG(2, 9), but Thas [14], shows that such arc does not exists in PG(2, 9). Then we obtain the following theorem:

Theorem 5.2.

There is no (31, 10, f) – arc of type (1, 10) in PG(2, 9), in which Im f = { 0, 1, 3 } and the points of weight 3 form (21, 3) – arc .

Now, let us take . But in this case does not occur that , because 4 does not divisor of 90. Hence we deduce the following theorem:

Theorem 5.3.

There is no (k, 10, f) – arc of type (1, 10) in PG(2, 9), in which Im f = { 0, 1, }.

The next value we must be take is , and for this value we have:

, and .

Also from equation (3.2), we get:

We note that there are at most two points of weight 5 on a line because . Then the points of weight 5 form 9 – arc and from Hirschfeld [9, Ch. 14], there is a unique projectively distinct 9 – arc in PG(2, 9), and we can take the 9 – arc as:

{ (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, ,

), (1, , 2), (1, ,), (1, ,

), (1, ,), (1,1,1) }, where

.

From Mohammed Yousif Abass [1], we have the projectivity which permutes the points of PG(2, 9), as

Let , then

.

Then the points of the 9 – arc take the following numbers respectively:

{ 1, 2, 3, 5, 9, 23, 43, 53, 63 }. Since there are only 1 – weighting lines and 10 – weighting lines, then the 2 – secants and 1 – secants of the 9 – arc ( ) are 10 – weighting lines of the (19, 10; f) – arc in PG(2, 9). Then from Lemma (1.1) with , we have the following:

Since 2 – secants and 1 – secants of are 10 – weighting lines of the (19, 10; f) – arc in PG(2, 9), then , but , and this contradiction. Hence we get the following theorem:

Theorem 5.4.

There is no (19, 10, f) – arc of type (1, 10) in PG(2, 9), in which Im f = { 0, 1, }.

Now, we discuss the case in which , then we have , and this implies that the line joining two points of weight 6 has weight 12, and this is contradiction because , the maximum weight of the line. Hence we obtain the following theorem:

Theorem 5.5.

There is no (16, 10, f) – arc of type (1, 10) in PG(2, 9), in which Im f = { 0, 1, }.

Also when or , there is no (k, 10, f) – arc of type (1, 10) in PG(2, 9), because 7 and 8, does not divisor .

The last case in which , then we have the following theorem:

Theorem 5.6.

The (k, n; f) – arcs of PG(2, q), of type (1, n) with , are precisely the monoidal (k, n; f) – arcs having as points of weight 1 those of a line .

Proof : see [12, proposition (6)]. □

References

[1] M. Y. Abass, "Existence and non – existence of (k, n; *f*) – arcs of type and monoidal arcs in PG(2, 9)", M. Sc. Thesis, University of Basrah, Iraq (2011).

[2] A. Barlotti, "Recent results in Galois geometries useful in coding theory ", Colloques Internationaux du C.N.R.S. N^0 276, THEORIE DE L 'INFORMATION, 1978, 185 – 187.

[3] M. Barnabei, " On arcs with weighted points " , Journal of statistical planning and Inference, 3 (1979), 279 – 286.

[4] E. D'Agostini, "Sulla caratterizzazione delle (k, n; *f*)-calotte di tipo (n –2, n)", Atti

Sem. Mat. Fis. Univ. Modena, XXIX, (1980), 26 - 275.

[5] E. D'Agostini, "ON WEIGHTED ARCS WITH THREE NON – ZERO CHARACTERS", Journal of Geometry, Vol. 50 (1994).

[6] F. K. Hameed, "Weighted (k, n) – arcs in the projective plane of order nine " , Ph.D. Thesis, University of London, England (1989).

[7] F. K. Hameed, "Monoidal (k, q+m; *f*) – arcs of type (m, q+m) in PG(2, q)", Basrah Journal of Science (A), Vol.30(1), 90 – 96, (2012).

[8] F. K. Hameed, M. Hussein and Mohammed. Y. Abass, "On (k, n; *f*)– arcs of type (n – 5, n) in PG(2, 5)", Journal of Basrah Researches ((Sciences)) Volume 37. Number 4. C ((2011)).

[9] J. W. P. Hirschfeld, "projective geometries over finite fields", Second Edition, Clarendon Press, Oxford, 1998.

[10] A. D. Keedwell, " When is a (k, n) – arc of PG(2, q) embeddable in a unique algebraic plane curve of order n ? ", Rend. Mat. (Roma) Serie VI, 12 (1979), 397 – 410.

[11] R. D. Mahmood, " (k, n; *f*) – arcs of type (n – 5, n) in PG(2, 5) ", M.Sc. Thesis, University of Mosul, Iraq (1990).

[12] G. Raguso and L. Rella, " Sui (k, n; *f*) – archi tipo (1, n) di un piano proiettivo finito ", Note di Matematica Vol. III, (1983), 307 – 320.

[13] M. Tallini Scafati, "Graphic Curves on a Galois plane ", Atti del convegno di Geometria combinatoria e sue Applicazioni Perugia, (1971), 413 – 419.

[14] J. A. Thas, " Some Results Concerning { (q+1)(n-1); n } – Arcs and { (q+1)(n-1)+1; n } – Arcs in Finite Projective Plane of Order q ", Journal of Combinatorial Theory (A) 19, (1975), 228 – 232.

[15] B. J. Wilson, " (k, n; *f*) – arcs and caps in finite projective spaces", Annals of Discrete mathematics 30 (1986), 355 – 362.

الخلاصة:

في المستوي الاسقاطي من الرتبة $(1, n)$ من النوع $(k, n; f)$ في هذا البحث ناقشنا وجود الأقواس -
$n = 10$ أو $n = 4$ أي إن و $1 < \omega < n$ و $Im(f) = \{0, 1, \omega\}$ التاسعة؛ حيث
$(1, 4)$ من النوع $(19, 4; f)$ و استنتجنا القوس- [12]. لهذا الغرض استخدمنا الأسلوب في المصدر
. كذلك حصلنا $PG(2, 9)$ في $\omega = 3$ عندما $(1, 4)$ العادي من النوع $(13, 4)$ والقوس- $\omega = 2$ عندما
من $\omega = 9$ الأحادي عندما $(11, 10; f)$ و على القوس - $\omega = 2$ عندما $(46, 10; f)$ على القوس -
في $(1, 10)$ من النوع $(k, 10; f)$. أيضا برهنا عدم وجود الأقواس -$PG(2, 9)$ في $(1, 10)$ النوع
$PG(2, 9)$ لقيم ω بين 2 و 9 9) ولا ($\omega \neq 2$.