Journal of education college no.3 vol.2 Aug/2012

On Best Trigonometric Approximation in $L_p(\mu)$-Space

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Abstract.

Let f be 2π-periodic bounded μ-measurable function, that is $f \in L_p(\mu), 1 \le p < \infty$. in this paper, we discuss the approximation of f by using Trigonometric polynomial and $q_{r,j}$ operator. Also, we will estimate the best approximation by Weighted Ditzian-Totik modulus o

f smoothness.

1. Introduction

Let $L_p(\mu)$, $1 \le p < \infty$ consists of all $\mu-$ measurable function f for which $\|f\|_{p,\mu} < \infty$, where

(1.1) $$\|f\|_{p,\mu} = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}} ,$$

Also , let f be 2π-periodic bounded μ-measurable function ,

Let us consider $S_n(f;x)$ be Trigonometric polynomial which has the representation

(1.2)

$$S_n(f;x) = \frac{1}{n}\sum_{i=1}^{2n} f(x_i)\, D_n(x - x_i) ,$$

where $x_i = \dfrac{2\pi i}{2n+1}$, $i = 0,1,2,\ldots,2n$

It is easy to get the following

(1.3)

$$S_n(f;x) = \frac{1}{\pi}\int_X f(x+t)\, D_n(t)\, d\mu(t)$$

Such that, $D_n(x)$ is Dirchlet Kernel of degree n, where $n = 0,1,\ldots,$ which defined as

Key words : D. T modulus of smoothness ,Trigonometric polynomial, Best Approximation

(1.4)

$$D_n(x) = 1 + 2\sum_{i=1}^{n}\cos(ix) \quad , \quad x \in \mathrm{R} .$$

Now, we define the following operator ,that dependent on Dirchlet Kernel, where

$$q_r(t) = \frac{2}{r+2}\sum_{i=1}^{r/2} D_{\frac{r}{2}+i}(t) \quad \text{where } r = 2n .$$

We construct the following polynomial of function $f \in L_p(\mu)$, such that

(1.5)

$$q_{r,j}(f,t) = \frac{2}{j+2}\sum_{i=0}^{j} f(x_i)\, q_r(x - x_i) \text{ , where}$$

$r = 2n$, $j = 3n$

and

(1.6)

$$q_{r,j}(f,t) = \frac{1}{\pi}\int_X f(u)\, q_r(x-u)\, d\mu(u) .$$

Since f is bounded $\mu-$ measurable function and $1 \le p < \infty$, then we have

(1.7) $$\|f\|_p \le C(p)\|f\|_{p,\mu} .$$

Let us consider the family of locally global norms for $\delta > 0$ and $1 \le p < \infty$, then

(1.8)

$$\|f\|_{\delta,p} = \left\{ \int_X \left(\sup\left\{ |f(u)| : u \in N(x,\delta) \right\}\right)^p dx \right\}^{\frac{1}{p}} ,$$

also ,if $f \in L_p(\mu)$,then we define the locally global norms of f and ϕ by a function such that , $\phi(u,\delta) = \delta\phi(x) + \delta^2$

(1.9)

$$\|f\|_{\delta,p,\mu} = \left\{ \int_X \left(\sup\left\{ |f(u)| : u \in N(x,\delta) \right\}\right)^p d\mu(x) \right\}^{\frac{1}{p}}$$

where ,

$N(x,\delta) = \left\{ u \in X : |x-u| \le \delta \right\}$, $\delta \in R^+$,

also see [1].

We will use the modulous of smoothness which are connected with difference of higher order, that is the rth symmetric difference of f which is given by

(1.10)

$$\Delta_h^k(f,x) = \begin{cases} \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right), & x \mp \frac{kh}{2} \in X \\ 0 & , \text{ o.w.} \end{cases}$$

Then the rth usual modulus of smoothness of $f \in L_p(\mu)$ is defined by

(1.11)

$$\omega_k(f,\delta)_{p,\mu} = \sup_{0<p\le\delta} \left\|\Delta_h^k(f,x)\right\|_{p,\mu} ,$$

and the Ditzian-Totik modulus of smoothness of f is defined by

(1.12)

$$\omega_k^{\phi}(f,\delta)_{p,\mu} = \sup_{0<p\le\delta} \left\|\Delta_{h\phi(.)}^k(f,x)\right\|_{p,\mu} ,$$

where ,in this applications the ϕ usually used

$\phi(.) = \phi(x) = (x(1-x))^{\frac{1}{2}}$ for $x \in [0,1]$.

The weighted Ditzian-Totik modulus of smoothness of f is defined by

(1.13)

$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} = \sup_{0<p\le\delta} \left\|\phi^r(x)\Delta_{h\phi(.)}^k(f,x)\right\|_{p,\mu} ,$$

where , k, r denote to the nonnegative integers and $k + r > 0$.

By [4] for a function $f \in L_p(X)$, $1 \le p < \infty$, so we have

(1.14) $$\omega_{k,r}^{\phi}(f,\delta)_p \approx \widetilde{K}_{k,r,\phi}(f,\delta)_p$$

where, $\widetilde{K}_{k,r,\phi}$ is the weighted Ditzian-Totik $\widetilde{K}$ - functional defined by

(1.15)
$$\widetilde{K}_{k,r,\phi}(f,\delta)_p = \inf_{\substack{P_n \in \Pi_n \\ n=\left[\frac{1}{t}\right]}} \left\{ \left\| \phi^r(x)(f-P_n) \right\|_p + \delta^k \left\| \phi^k P_n^{(k)} \right\|_p \right\}$$
.

Let $C^{\ell}(X)$ denoted the set of ℓ – times continuously differentiable functions on $[0,1]$.

Also, for $1 \le p < \infty$, the Sobolev space W_p^{ℓ} is a collection of all functions f on X, such that, $f^{(\ell-1)}$ is absolutely continuous and $f^{(\ell-1)} \in L_p(X)$.

2. Auxiliary Results

In this section we mention some basic results ,which will be used to prove the main results.

Lemma 2.1 [1]

Let μ be a non-decreasing function on R, satisfying: $\mu(y)-\mu(x) =$ constant, and $1 \le p < \infty$, We put:

$\omega_{\mu}(\delta) = \sup\limits_{0<y-x\le\delta} (\mu(y)-\mu(x)), \delta > 0$,and

$\left(\frac{1}{n} \sum_{k=0}^{n-1} \max_{x \in I_k} |P_n|^p \right)^{\frac{1}{p}} \le C(p) \|P_n\|_p$,where P_n is algebraic polynomials of degree at most n and $I_k = \left[\frac{k}{n}, \frac{k+1}{n} \right]$, then:

(2.1)
$$\|P_n\|_{p,\mu} \le C(p) \left(n\omega_{\mu}\left(\frac{1}{n}\right) \right)^{\frac{1}{p}} \|P_n\|_p .$$

Lemma 2.2 [3]

Let f be a bounded μ – measurable function then for $1 \le p < \infty$, we have:

(2.2)
$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} \le c\delta^k \left\| f^{(k)} \right\|_{p,\mu}$$

Lemma 2.3 [3]

Let f be a bounded μ – measurable function on X ,and $g_n \in \Pi_n \cap L_p(\mu)$,then we have

(2.3)
$$\left\| f - g_{k,n} \right\|_{\delta,p,\mu} \le C(r,k,p) n^{1/p} 2^r \, \omega_{k,r}^{\phi}(f,\delta)_{p,\mu} .$$

Lemma 2.4 [1]

Let f be a bounded μ – measurable function then for $1 \le p < \infty$, we have:

(i) $\|f\|_{\delta,p} \le C(p) \|f\|_{\delta,p,\mu}$,

(ii) $\|f\|_{\delta,p,\mu} \le C(p) \|f\|_{p,\mu}$.

Lemma 2.5 [3]

Let f and g be bounded μ – measurable functions for $1 \le p < \infty$, then we have

(2.4)
$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} \le \omega_{k,r}^{\phi}(f-g,\delta)_{p,\mu} + \omega_{k,r}^{\phi}(g,\delta)_{p,\mu}$$
.

Lemma 2.6 [2]

Let $T \in \mathrm{T}_n$,then we have

(2.5) $$\frac{1}{n}\sum_{i=1}^{2n} T(x_i)D_n(x-x_i) = T(x).$$

Lemma 2.7 [5]

Let $f \in L_p(X)$, $1 \le p < \infty$, $\alpha, \beta > 0$, $h \in R$,then we have

(i) $\Delta_h^{\alpha}(\Delta_h^{\beta}(f,x)) = \Delta_h^{\alpha+\beta}(f,x)$ for almost every α ,

(ii) $\left\|\Delta_h^{\alpha+\beta}(f,x)\right\|_p \le C(\alpha)\left\|\Delta_h^{\beta}(f,x)\right\|_p$.

Lemma 2.8 [1]

Let f and g be two functions defined on the same domain ,then we have

(i) $q_{r,j}(f+g) = q_{r,j}(f) + q_{r,j}(g)$,

(ii) $q_{r,j}(\alpha f) = \alpha q_{r,j}(f)$.

where α is a constant.

Lemma 2.9

Let f be a bounded $\mu-$measurable convex function then for $1 \le p < \infty$, we have

(2.6) $$E_n^{(2)}(f)_{p,\mu} \le c(p,k)n^{-1}\omega_{k,r}^{\phi}(f,\delta)_{p,\mu}$$

.

Proof:-

Let $f \in L_p(\mu) \cap \Delta^2$ and $T \in \mathrm{T}_n$, then we have

$E_n^{(2)}(f)_{p,\mu} = \inf_{T\in \mathrm{T}_n}\left\|f-T\right\|_{p,\mu}$,then by using Lemma 2.1, and the formulas (1. 13) and (1.7) we get

$$\begin{aligned} E_n^{(2)}(f)_{p,\mu} &\le c(p)n^{-1}\inf_{T\in \mathrm{T}_n}\left\|f-T\right\|_p \\ &\le c(p)n^{-1}E_n^{(2)}(f)_p \\ &\le c(p)\omega_{k,r}^{\phi}(f,\delta)_p \\ &\le c(p)n^{-1}\sup_{T\in \mathrm{T}_n}\left\|\phi^r\Delta_{h\phi}^k(f,x)\right\|_p \\ &\le c(p,k)n^{-1}\sup_{T\in \mathrm{T}_n}\left\|\phi^r\Delta_{h\phi}^k(f,x)\right\|_{p,\mu} \\ &\le c(p,k)n^{-1}\omega_{k,r}^{\phi}(f,\delta)_{p,\mu}. \end{aligned}$$

Lemma 2.10 [7]

Let f be a 2π -periodic bounded $\mu-$ measurable function then for $1 \le p < \infty$, we have:

(2.7)
$$K\left[f,\frac{1}{n};L_p,w_p^1,w_p^{-1}\right] \le \frac{c}{n}\sum_{s=0}^{n}\begin{cases} E_S^T(f)_{\frac{1}{s+1},p} + E_S^T(f)_{\frac{1}{s+1},p} & p=1, p=\infty \\ c_{(p)}E_S^T(f)_{\frac{1}{s+1},p} & 1<p<\infty \end{cases}$$

Lemma 2.11 [1]

f be a 2π -periodic bounded $\mu-$measurable function for $1 \le p < \infty$

(2.8) $$\left\|Q_{r,j}(f)\right\|_{p,\mu} \le c\left\|f\right\|_{p,\mu}.$$

Lemma 2.12

Let f be a bounded $\mu-$measurable function then for $1 \le p < \infty$, we have:

(2.8)

$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} \le c(p,k)\, n^{-1}\left\|\phi^r\ f\ \right\|_{p,\mu}.$$

Proof :-

By using (1.13) ,then we have

$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} = \sup_{0<p\le\delta}\left\|\phi^r(x)\Delta_{h\phi(.)}^{k}(f,x)\right\|_{p,\mu}$$

Then by Lemma 2.1 ,Lemma 2.6 and (1.7), we have

$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} \le c(p)n^{-1}\sup_{0<h\le\delta}\left\|\phi^r(x)\Delta_{h\phi(.)}^{k}(f,x)\right\|_{p}$$

,

$$\le c(p,k)n^{-1}\left\|\phi^r(x)f\right\|_{p},$$

$$\le c(p,k)n^{-1}\left\|\phi^r(x)f\right\|_{p,\mu}.$$

Now, we finished the proof .

Lemma 2.13 [1]

Let f and g be two functions defined on the same domain , It follows that

$$(i)\quad S_n(f+g;x) = S_n(f;x) + S_n(g;x),$$

$$(ii)\quad S_n(\alpha f) = \alpha S_n(f).$$

where α is a constant.

Lemma 2.14 (Minkowsk's Inequality) [6]

If

$p \ge 1$ and $f,g \in L_p(\mu)$, then $f+g \in L_p(\mu)$ and

(2.9)

$$\left[\int_X |f+g|^p\,d\mu\right]^{1/p} \le \left[\int_X |f|^p\,d\mu\right]^{1/p} + \left[\int_X |g|^p\,d\mu\right]^{1/p}$$

.

3.Main Results

We are review the main results and the following Theorem 3.1 and Theorem 3.4 represent the direct theorems for best approximation .On the other hand Theorem 3.2 and Theorem 3.3 represent the inverse Theorems.

Theorem 3.1

Let f be 2π-periodic bounded μ -measurable function for $1 \le p < \infty$ .then we have

(3.1)

$$\left\|f - q_{r,j}(f)\right\|_{p,\mu} \le cn^{1/2}2^r\,\omega_{k,r}^{\phi}(f,\delta)_{p,\mu}$$

with the equivalence constants depending only on k,r and p .

Theorem 3.2

Let f be 2π-periodic bounded μ -measurable function ,then we have

(3.2)

$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} \le cn^{1/p}\sum_{r=0}^{n}\left\|f - q_{r,j}(f)\right\|_{\frac{1}{r+1},p,\mu}$$

with the constants depending only on r and p .

Theorem 3.3

Let $S_n(f;x)$ be Trigonometric polynomial and f be 2π-periodic bounded μ -measurable function, then

(3.3)

$$\omega_{k,r}^{\phi}(S_n(f;x),\delta)_{p,\mu} \le c\delta^{-rp}\sum_{j=0}^{n-1}|J_j|^{rp}\|f-q_{r,j}(f)\|_{p,\mu}$$

where, c depending on k,r and p.

Theorem 3.4

Let $S_n(f;x)$ be Trigonometric polynomial and f be 2π-periodic bounded μ-measurable function, for $1 \le p < \infty$ then we have

(3.4)

$$\|f-S_n(f;x)\|_{p,\mu} \le c\frac{2n+1}{n}\omega_k^{\phi}(f,\delta)_{p,\mu},$$

with the constants depending only p.

4.Proof of The Main Results

Proof of Theorem 3.1

We may assume the Trigonometric polynomial $T \in \mathrm{T}_n$,and we introduce

$$\|f-T\|_{p,\mu} \le E_n^{(2)}(f)_{p,\mu} \text{ where}$$

$f \in L_p(\mu) \cap \Delta^2$

Also, by using Lemma 2.8 (i) (ii) ,Lemma 2.11 and Lemma 2.9, then we have

$$\|f-q_{r,j}(f)\|_{p,\mu} = \|f-T+T-q_{r,j}(f)\|_{p,\mu}$$

$$\le \|f-T\|_{p,\mu} + \|T-q_{r,j}(f)\|_{p,\mu}$$

$$\le \|f-T\|_{p,\mu} + \|q_{r,j}(T)-q_{r,j}(f)\|_{p,\mu}$$

$$\le \|f-T\|_{p,\mu} + \|q_{r,j}(T-f)\|_{p,\mu}$$

$$\le c\|f-T\|_{p,\mu}$$

since the best approximation $E_n^{(2)}(f)_{p,\mu} = \inf_{T\in\mathrm{T}_n}\|f-T\|_{p,\mu}$ then by (2.6) we have

$$\|f-T\|_{p,\mu} \le cE_n^{(2)}(f)_{p,\mu} \le cn^{-1}\omega_{k,r}^{\phi}(f,\delta)_{p,\mu},$$

where the constant c depends only on p and k.

Now we complete proof (3.1) ♦

Proof of Theorem 3.2

Suppose that any polynomial $h \in W_p^k \cap L_p(\mu)$,$1 \le p < \infty$, $\delta > 0$ then by using

Lemma 2.5 and Lemma 2.1, we have

$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} \le \omega_{k,r}^{\phi}(f-h,\delta)_{p,\mu} + \omega_{k,r}^{\phi}(h,\delta)_{p,\mu}$$

$$\le \mathrm{c}\left\{\|\phi^r(x)(f-h)\|_{p,\mu} + \delta^k\|h^{(k)}\|_{p,\mu}\right\}$$

$$\le \mathrm{c}n^{1/p}\left\{\|\phi^r(x)(f-h)\|_{p} + \delta^k\|\phi^r h^{(k)}\|_{p}\right\}$$

$$\le cn^{1/p}\tilde{\mathrm{K}}_{\mathrm{k,r},\phi}(f,\delta)_p$$

By using Lemma 2.10 and Lemma 2.4(i) , we get

$$E_r(f)_{\frac{1}{r+1},p} = \inf_{0\le r\le n}\|f-q_{r,j}\|_{\frac{1}{r+1},p}$$

then we have

$$\tilde{\mathrm{K}}_{\mathrm{k,r},\phi}(f,\delta)_p \le \sum_{r=0}^{n} c(p)E_r(f)_{\frac{1}{r+1},p}$$

$$\le c\sum_{r=0}^{n}\|f-q_{r,j}\|_{\frac{1}{r+1},p}$$

From (1.7) it follows that

$$\omega_{k,r}^{\phi}(f,\delta)_{p,\mu} \le cn^{1/p}\sum_{r=0}^{n}\left\| f - q_{r,j}(f)\right\|_{\frac{1}{r+1},p,\mu}$$

with the constants depending only on r and p and this complete the proof Theorem 3.2 ♦

Proof of Theorem 3.3

We assume that $S_n(f;x)$ is Trigonometric polynomial and $f \in L_p(\mu)$ and let $\phi(x) = \sqrt{x^2 - 1}$.

Now, using Lemmas 2.1, 2.7 , 2.13 and the formula (3.1) , we get the following

$$\left\|\phi^r(x)\Delta_{h\phi(x)}^k(S_n(f;x);x)\right\|_{p,\mu}$$

$$\le cn^{1/p}\left\|\phi^r(x)\Delta_{h\phi(x)}^k(S_n(f;x);x)\right\|_p$$

$$\le cn^{1/p}\sum_{j=0}^{n-1}\left\|\phi^r(x_j)\Delta_{h\phi(x_j)}^k(S_n(f;x_j),x_j)\right\|_p$$

$$\le cn^{1/p}\delta^{-rp}\sum_{j=0}^{n-1}\left|J_j\right|^{rp}\left\|\phi^r(x_j)\Delta_{h\phi(x_j)}^k(S_n(f;x_j)-q_{r,j},x_j)\right\|_p$$

$$\le cn^{1/p}\delta^{-rp}\sum_{j=0}^{n-1}\left|J_j\right|^{rp}\left\|\phi^r(x_j)\Delta_{h\phi(x_j)}^k(S_n(f;x_j)-S_n(q_{r,j}),x_j)\right\|_p$$

$$\le cn^{1/p}\delta^{-rp}\left(\frac{j}{n}\right)^r\sum_{j=0}^{n-1}\left|J_j\right|^{rp}\left\|\Delta_{h\phi(x_j)}^k(S_n(f-q_{r,j}),x_j)\right\|_p$$

$$\le cn^{1/p}\delta^{-rp}\left(\frac{j}{n}\right)^r\sum_{j=0}^{n-1}\left|J_j\right|^{rp}\left\|f-q_{r,j}\right\|_p$$

$$\le cn^{1/p}\left(\frac{j}{n\delta}\right)^{rp}\sum_{j=0}^{n-1}\left|J_j\right|^{rp}\left\|f-q_{r,j}\right\|_p,$$

where, for $0<\phi(x_j)\le \frac{j}{n}$ and by using sup to the right and left inequality and $0<h\le\delta$, then we have

$$\omega_{k,r}^{\phi}(S_n(f;x),\delta)_{p,\mu}$$

$$\le cn^{1/p}\left(\frac{j}{n\delta}\right)^{rp}\sum_{j=0}^{n-1}\left|J_j\right|^{rp}\left\|f-q_{r,j}\right\|_p$$

The constant c depends on k, p and r ♦

Proof of Theorem 3.4

We assume that $S_n(f;x)$ is a Trigonometric polynomial and $f \in L_p(\mu)$, $1\le p<\infty$ and by using Lemmas 2.14 , 2.14 , 2.1, 2.3 and the formula ,(1.1). Also , suppose that $q_{r,j}\in L_p(\mu)$,it follows that

$$\left\|f-S_n(f;x)\right\|_{p,\mu} = \left(\int_X\left|f(x)-S_n(f;x)\right|^p d\mu\right)^{1/p}$$

$$\le\left(\int_X\left|f(x)-q_{r,j}+q_{r,j}-S_n(f;x)\right|^p d\mu\right)^{1/p}$$

$$\le\left(\int_X\left|f(x)-q_{r,j}\right|^p d\mu\right)^{1/p}$$

$$+\left(\int_X\left|q_{r,j}-S_n(f;x)\right|^p d\mu\right)^{1/p}$$

$$\le\left\|f-q_{r,j}\right\|_{p,\mu}+\left(\int_X\left|q_{r,j}-\frac{1}{n}\sum_{j=1}^{n}f(x_j)D_n(x-x_j)\right|^p d\mu\right)^{1/p}$$

where , D_n is Dirichlet Kernel of degree n and since $D_n(x-x_j) \le 2n+1$, $\forall j$ and using (1.1) then we have

$$\|f - S_n(f;x)\|_{p,\mu}$$

$$\le \|f - q_{r,j}\|_{p,\mu} + \frac{2n+1}{n}\sum_{j=1}^{2n}\left(\int_X |q_{r,j} - f(x_j)|^p d\mu\right)^{1/p}$$

$$\le \|f - q_{r,j}\|_{p,\mu} + \frac{2n+1}{n}\sum_{j=1}^{2n}\|f - q_{r,j}\|_{p,\mu}$$

Furthermore, by using Theorem 3.1 and (1.7) then we have

$$\sum_{j=1}^{2n}\|f - q_{r,j}\|_{p,\mu} \le$$

$$cn^{-p}2^r\sum_{j=1}^{2n}\omega_{k,r}^{\phi}(f, n^{-1})_{p,\mu}$$

Now we complete the proof for $r \ge 1$, $k \ge 1$ and the constant depending on

k, r and p ♦

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المستخلص:

الهدف من هذا البحث هو دراسة درجة تقريب الدالة القابلة للقياس- μ **باستخدام متعددة الحدود المثلثية ومؤثر** $q_{r,j}$, وسوف نركز على حساب التقدير الأفضل باستخدام مقياس **النعومة المرجّح** لكل من دتيزين وتوتك $\omega_{k,r}^{\varphi}$.