

# Characterizations of $\alpha$ -Light Mappings

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## Abstract:

In this work, we give the definition of  $\alpha$ -Light mapping for the first time (to the best of our knowledge) and investigate some of its several properties and characterization .Also, we study the relation between this concept and the concept of light mapping and we proved that the pull back of the  $\alpha$ -light mapping is also  $\alpha$ -light mapping.

## الخلاصة

في هذا العمل، قدمنا مفهوماً جديداً حسب علمنا (التطبيق الواهن- $\alpha$ ) مع دراسة العديد من الخصائص والمميزات الخاصة بهذا المفهوم. أيضاً درسنا العلاقة بين هذا المفهوم والتطبيق الواهن وبرهنا بأن السحب الخلفي للتطبيق الواهن- $\alpha$  يكون أيضاً تطبيقاً واهناً- $\alpha$ .

## 1- Introduction

Njastad (1965) gives the definition of  $\alpha$ -open set and studies the properties of it. Also, (Mashhour *et al.*,1983), give the definition of  $\alpha$ -continuous and  $\alpha$ -open mapping and study the properties of it. Throughout this work,  $(X,\tau)$  simply space  $X$  always means topological space . A subset  $A$  of  $(X,\tau)$  is called  $\alpha$ -open (Njastad, 1965) if  $A \subseteq (\bar{A}^\circ)^\circ$ . The complement of an  $\alpha$ -open set is called an  $\alpha$ -closed set. The intersection of all  $\alpha$ -closure sets containing  $A$  is called the  $\alpha$ -closure of  $A$ , denoted by  $(\bar{A})^\alpha$ . A subset  $A$  is called  $\alpha$ -closed if and only if  $A = (\bar{A})^\alpha$  a point  $x \in X$  is said to be an  $\alpha$ -interior point of  $A$  if there exists an  $\alpha$ -open set  $U$  containing  $x$  such that  $U \subseteq A$ . The set of all  $\alpha$ -interior points of  $A$  is said to be  $\alpha$ -interior of  $A$  (Mashhour *et al.*,1983) and denoted by  $(A^\circ)^\alpha$ . It is clear that every open is  $\alpha$ -open but the converse is not true. We denote the family of all  $\alpha$ -open sets of  $(X,\tau)$  by  $\tau^\alpha$ . It is shown in (Njastad ,1965; Ohba and Umehara, 2000)) that each of  $\tau \subseteq \tau^\alpha$  and  $\tau^\alpha$  is a topology on  $X$ . A mapping  $f:(X,\tau) \rightarrow (Y,\sigma)$  is said to be  $\alpha$ -continuous iff for  $\tau^\alpha f^{-1}(V) \in \sigma$  every  $V \in \sigma$ , and, equivalently, if for each  $x \in X$  and each open set  $V$  of  $Y$  containing  $f(x)$ , there exists  $U \in \tau^\alpha$  with  $x \in U$  such that  $f(U) \subseteq V$  (Latif, 2006)).  $f:(X,\tau) \rightarrow (Y,\sigma)$  is said to be  $\alpha$ -open if each  $\alpha$ -open set  $U$  in  $X$ ,  $f(U)$  is  $\alpha$ -open set in  $Y$ . and  $f$  is said to be  $\alpha$ -closed if each  $\alpha$ -closed set  $U$  in  $X$ ,  $f(U)$  is  $\alpha$ -closed set in  $Y$  (Latif, 2006)).

## 2- $\alpha$ - connected spaces

Now we introduce the following definitions:-

### 2-1 Definitions:

1- let  $X$  be a space and  $A, B$  be non empty  $\alpha$ -open sets , then we say that  $A/B$  is  $\alpha$ -disconnection to  $X$  if  $X = A \cup B$  and  $A \cap B = \emptyset$ .

2- let  $X$  be a space ,  $X$  is said to be  $\alpha$ -disconnected if there exists a disconnection  $A/B$  to  $X$ , other wise we say that  $X$  is  $\alpha$ -connected.

### 2-2 Definition:

Let  $X$  be a space and let  $G$  a non empty subset of  $X$  and  $A, B$  non empty  $\alpha$ -open sets in  $X$ , then  $A/B$  is said to be  $\alpha$ -disconnection of  $G$  if

$$1 - (A \cup B) \cap G = G$$

$$2 - (A \cup B) \cap G = \emptyset$$

other wise  $G$  is  $\alpha$ -connected set.

### 2-3 Example:

let  $X = \{a, b, c\}$  and  $\tau$  is a discrete topology defined on  $X$  note that  $\{a\}/\{b, c\}$  is a  $\alpha$ -disconnection to  $X$ . Also  $\{a\}/\{b\}$  is a  $\alpha$ -disconnection to the subset  $\{a, b\}$  of  $X$ .

The following theorem is given in (Mashhour *et al.*,1983).

**2-4 Theorem:**

Every  $\alpha$ -connected space is connected. but the converse may not be true.

**2-5 Definition:**

A space  $X$  is said to be totally  $\alpha$ -disconnected if for each pair of points  $a, b \in X$  there is an  $\alpha$ -disconnection  $A/B$  of  $X$  with  $a \in A$  and  $b \in B$ .

**2-6 Remark:**

Each totally disconnected space is totally  $\alpha$ -disconnected space. but the converse is not true for example let  $X = \{a, b, c\}$  and  $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$ . It is clear that  $X$  is totally  $\alpha$ -disconnected space but  $X$  is not totally disconnected space because there is no exists a disconnection of  $b, c \in X$ .

The following example is given in (Khalid, 2004)).

**2-7 Example:**

The set  $Q$  of rational numbers with the usual topology is totally  $\alpha$ -disconnected set. (since  $Q$  is totally disconnected set with the usual topology.

Now we introduce the following definition:

**2-8 Definition:**

A mapping  $f: X \rightarrow Y$  is said to be  $\alpha$ -homeomorphism if

- 1-  $f$  is bijective mapping.
- 2-  $f$  is  $\alpha$ -continuous mapping.
- 3-  $f$  is  $\alpha$ -open or  $\alpha$ -closed mapping.

Now the following theorem shows that the totally  $\alpha$ -disconnectedness is a topological property.

**2-9 Theorem:**

let  $X$  and  $Y$  be two spaces, and let  $f: X \rightarrow Y$  be a  $\alpha$ -homeomorphism .If  $X$  or  $Y$  is a totally  $\alpha$ -disconnected so is the other.

**Proof:**

that  $X$  is totally  $\alpha$ -disconnected and let  $y_1, y_2 \in Y$  such that  $y_1 \neq y_2$  .since  $f$  is bijective mapping then there exist two points  $x_1, x_2 \in X$  such that  $x_1 \neq x_2$  and  $f(x_1) = y_1$  , and  $f(x_2) = y_2$  since  $X$  is totally  $\alpha$ -disconnected space , then there is a  $\alpha$ -disconnection  $U/V$  such that  $x_1 \in U, x_2 \in V$  since  $f$  is  $\alpha$ -homeomorphism then each of  $f(U)$  and  $f(V)$  are  $\alpha$ -open sets in  $Y$  but  $f(U) \cup f(V) = f(U \cup V) = f(X) = Y$  and since  $f$  is one-to-one then  $f(U) \cap f(V) = f(U \cap V) = f(\emptyset) = \emptyset, \in X$  and is totally  $Y$  hence ,  $y_1 \in f(U), y_2 \in f(V)$   $\alpha$ -disconnected space .

**3-  $\alpha$ -Light mapping**

The following definition is given in (Whyburn,1942))

**3-1 Definition:**

A mapping  $f: X \rightarrow Y$  is called Light mapping if  $f^{-1}(y)$  , is totally disconnected set (relative topology) for all  $y \in Y$  with (i-e each component  $f^{-1}(y)$  is singleton)

**3-2 Example:**

let  $(Q, \tau_U)$  be the space such that  $\tau_U$  is the usual topology defined on the set of rational numbers  $Q$  and let  $(\{k\}, \tau_i)$  be indiscrete topology for  $k \in R$ . and let  $f: (Q, \tau_U) \rightarrow (\{k\}, \tau_i)$  mapping defined as follows :  $f(x) = k$  for all  $x \in Q$  note that  $f^{-1}(k) = Q$  and since  $Q$  is totally disconnected set (2-7) , then  $f$  is Light mapping.

Now we introduce the following definition:

**3-3 Definition:**

A mapping  $f: X \rightarrow Y$  is called  $\alpha$ -Light mapping if  $f^{-1}(y)$ , is totally  $\alpha$ - disconnected set (relative topology) for all  $y \in Y$ .

**3-4 Example:**

The example (3-2) we note that  $f$  is  $\alpha$ -Light mapping it easy from (2-6).

**3-5 Theorem:**

Every Light mapping is  $\alpha$ -Light mapping.

**Proof:**

It is easy from (2-6), that if  $f^{-1}(y)$  is totally disconnected set in  $X$  then  $f^{-1}(y)$  is totally  $\alpha$ -disconnected set in  $X$ .

**3-6 Proposition:**

let  $f: X \rightarrow Y$  be a  $\alpha$ -Light mapping and let  $G \subset X$ , then the restriction mapping  $f|_G : G \rightarrow f(G)$ , is also  $\alpha$ -Light mapping.

**Proof:**

to show that for all  $y \in f(G)$ ,  $f^{-1}(y) \cap G$  is totally  $\alpha$ -disconnected set in  $G$ . let  $a, b \in f^{-1}(y) \cap G$  then  $a, b \in f^{-1}(y)$  and since  $f$  is  $\alpha$ -Light map then for all  $y \in Y$ ,  $f^{-1}(y)$  is totally  $\alpha$ -disconnected set in  $X$ , that is there exists a  $\alpha$ -disconnection  $A/B$  such that  $(A \cap f^{-1}(y)) \cup (B \cap f^{-1}(y)) = f^{-1}(y)$  and

$$(A \cap f^{-1}(y)) \cap (B \cap f^{-1}(y)) = \emptyset$$

Such that  $A, B$  are disjoint  $\alpha$ -open subsets of  $X$ , and  $a \in A, b \in B$  now to show that  $A/B$  is  $\alpha$ -disconnection to  $f^{-1}(y) \cap G$  also .since

$$((G \cap f^{-1}(y)) \cap A) \cup ((G \cap f^{-1}(y)) \cap B) = (G \cap ((f^{-1}(y) \cap A)) \cup (G \cap (f^{-1}(y) \cap B)))$$

$$= G \cap [(f^{-1}(y) \cap A) \cup (f^{-1}(y) \cap B)] = G \cap f^{-1}(y) \text{ and}$$

$$((G \cap f^{-1}(y)) \cap A) \cap ((G \cap f^{-1}(y)) \cap B) = (G \cap (f^{-1}(y) \cap A)) \cap (G \cap (f^{-1}(y) \cap B))$$

$$= G \cap [(f^{-1}(y) \cap A) \cap (f^{-1}(y) \cap B)] = G \cap \emptyset = \emptyset$$

Such that  $(G \cap f^{-1}(y)) \cap A, (G \cap f^{-1}(y)) \cap B$  are two disjoint  $\alpha$ -open sets, hence  $f^{-1}(y) \cap G$  is totally  $\alpha$ -disconnected set,  $f|_G$  is  $\alpha$ -Light mapping.

Now we introduce the following definitions:

**3-7 Definition:**

A mapping  $f: X \rightarrow Y$  is said to be totally  $\alpha$ -disconnected mapping if each totally  $\alpha$ -disconnected set  $U$  in  $X$ ,  $f(U)$  is totally  $\alpha$ -disconnected set in  $Y$ .

**3-8 Definition:**

A mapping  $f: X \rightarrow Y$  is said to be inversely totally  $\alpha$ -disconnected mapping if each totally  $\alpha$ -disconnected set  $U$  in  $Y$ ,  $f^{-1}(U)$  is totally  $\alpha$ -disconnected set in  $X$ .

**3-9 Proposition:**

let  $f_1: X \rightarrow Y$  and  $f_2: Y \rightarrow Z$  be two mappings, then  $f = f_2 \circ f_1: X \rightarrow Z$  is  $\alpha$ -Light mapping if  $f_1$  is inversely totally  $\alpha$ -disconnected and  $f_2$  is  $\alpha$ -Light mapping.

**Proof:**

to prove that for  $z \in Z$ ,  $f^{-1}(z)$  is totally  $\alpha$ -disconnected set in  $X$ . let  $z \in Z$  then  $f^{-1}(z) = (f_2 \circ f_1)^{-1}(z) = f_1^{-1}(f_2^{-1}(z))$  and since  $f_2$  is  $\alpha$ -light mapping then  $f_2^{-1}(z)$  is totally  $\alpha$ -disconnected set in  $Y$  and since  $f_1$  is inversely totally  $\alpha$ -disconnected mapping then  $f_1^{-1}(f_2^{-1}(z))$  is totally  $\alpha$ -disconnected set in  $X$ , but  $f^{-1}(z) = f_1^{-1}(f_2^{-1}(z))$ , then  $f^{-1}(z)$  is totally  $\alpha$ -disconnected set in  $X$ .

**3-10 Theorem:**

let  $f: X \rightarrow Y$  be the composition  $f = f_2 \circ f_1$  of two mapping  $f_1: X \rightarrow Y'$  and  $f_2: Y' \rightarrow Y$ , then

- 1- If  $f_2$  is an bijective mapping and  $f_1$  is a  $\alpha$ -Light mapping then  $f$  is a  $\alpha$ -Light mapping.
- 2- If  $f$  is a  $\alpha$ -Light mapping and  $f_2$  is an injective mapping then  $f_1$  is a  $\alpha$ -Light map.
- 3- If  $f$  is a  $\alpha$ -Light mapping and  $f_1$  is a totally  $\alpha$ -disconnected mapping then  $f_2$  is a  $\alpha$ -Light mapping.

**Proof:**

- 1- let  $y \in Y$ , since  $f_2$  is a bijective mapping then there exists one point  $y' \in Y$  such that  $f_2(y')=y$  and since  $f^{-1}(y) = (f_2 \circ f_1)^{-1}(y) = f_1^{-1}(f_2^{-1}(y)) = f_1^{-1}(f_2^{-1}(f_2(y'))) = f_1^{-1}(y')$  and  $f_1$  is  $\alpha$ -light mapping then  $f_1^{-1}(y')$  is totally  $\alpha$ -disconnected set in  $X$ , so  $f_1^{-1}(y)$  is totally  $\alpha$ -light disconnected set in  $X$  and hence  $f$  is  $\alpha$ -light mapping.
- 2- Let  $y' \in Y'$ , then  $f_2(y') \in Y$  and since  $f$  is  $\alpha$ -light mapping then  $f^{-1}(f_2(y'))$  is a totally  $\alpha$ -disconnected set in  $X$  but  $f^{-1}(f_2(y')) = f_1^{-1}(f_2^{-1}(f_2(y'))) = f_1^{-1}(y')$ , then  $f_1$  is a  $\alpha$ -light mapping.
- 3- let  $y \in Y$ , since  $f$  is a  $\alpha$ -light, then  $f^{-1}(y)$  is a totally  $\alpha$ -disconnected set in  $X$  and since  $f_1$  is totally  $\alpha$ -disconnected mapping then  $f_1(f^{-1}(y))$  is a totally  $\alpha$ -disconnected set in  $Y'$ , but  $f_1(f^{-1}(y)) = f_1((f_2 \circ f_1)^{-1}(y)) = f_1(f_1^{-1}(f_2^{-1}(y))) = f_2^{-1}(y)$ . Then  $f_2^{-1}(y)$  is a totally  $\alpha$ -disconnected set in  $Y'$ , so  $f_2$  is  $\alpha$ -light mapping.

Now we will show when the product of two mappings is a  $\alpha$ -light mapping.

### 3-11 Theorem:

let  $f_1: X_1 \rightarrow Y_1$ ,  $f_2: X_2 \rightarrow Y_2$  be two mappings, then the product mapping  $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$  is a  $\alpha$ -light mapping if  $f_1$  is an bijective mapping and  $f_2$  is a surjective  $\alpha$ -light mapping.

**Proof:**

let  $(y_1, y_2) \in Y_1 \times Y_2$ , then  $(f_1 \times f_2)^{-1}(y_1, y_2) = (f_1^{-1} \times f_2^{-1})(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2)$  and since  $f_1$  is bijective then there exists  $x_1 \in X_1$  such that

$f_1^{-1}(y_1) = f_1^{-1}(f_1(x_1)) = x_1$ , then  $(f_1 \times f_2)^{-1}(y_1, y_2) = x_1 \times f_2^{-1}(y_2)$ , and since  $f_2$  is a  $\alpha$ -light mapping, then  $f_2^{-1}(y_2)$  is a totally  $\alpha$ -disconnected set in  $X_2$  and since  $x_1 \times f_2^{-1}(y_2)$  is  $\alpha$ -homeomorphic to  $f_2^{-1}(y_2)$ , then from (2-9)  $x_1 \times f_2^{-1}(y_2)$  is totally  $\alpha$ -disconnected, so,  $(f_1 \times f_2)^{-1}(y_1, y_2)$  is totally  $\alpha$ -disconnected set in  $X_1 \times X_2$  and hence  $f_1 \times f_2$  is  $\alpha$ -light mapping.

### 3-12 Corollary:

let  $f_1: X_1 \rightarrow Y_1$ ,  $f_2: X_2 \rightarrow Y_2$  be two mappings such that if  $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$  is a  $\alpha$ -light mapping, then if one of them is a surjective, then the other is a  $\alpha$ -light mapping.

The following definition is given in (Spainer, E.H, (1966))

### 3-13 Definition:

let  $f: X \rightarrow Y$  and  $g: Y' \rightarrow Y$  be mappings and  $X'$  is the subspace of the product space  $X \times Y'$  defined as follows:-

$X' = \{(x, y') \in X \times Y' \mid f(x) = g(y')\}$ ,  $X'$  is called the fibered product of  $Y'$  and  $X$  over  $Y$ .

let  $f': X' \rightarrow Y'$  be the restriction of the second projection then  $f'$  is called the pull back of  $f$  by  $g$ .

### 3-14 Theorem:

The pull back of  $\alpha$ -light mapping is also  $\alpha$ -light mapping.

**Proof:** let  $f: X \rightarrow Y$  be  $\alpha$ -light mapping and  $f': X' \rightarrow Y'$  be a pull back of  $f$  by  $g: Y' \rightarrow Y$ , now let  $y' \in Y'$ ,  $g(y') \in Y$ , since  $f$  is  $\alpha$ -light mapping then  $f^{-1}(g(y'))$  is a totally  $\alpha$ -disconnected set in  $X$ . now for fixed  $y' \in Y'$ ,

$$f'^{-1}(y') = \{(x, y') \in X \times Y' \mid f'(x, y') = y'\} \text{ for all } x \in X \text{ and } f(x) = g(y')$$

$$= \{(x, y') \in X \times Y' \mid x \in f^{-1}(g(y'))\} = f^{-1}(g(y')) \times \{y'\} \text{ and since}$$

$f^{-1}(g(y')) \times \{y'\}$  is  $\alpha$ -homeomorphic to  $f^{-1}(g(y'))$  and  $f^{-1}(g(y'))$  is totally  $\alpha$ -disconnected then  $f^{-1}(g(y')) \times \{y'\}$  is totally  $\alpha$ -disconnected by (2-9)  $f'^{-1}(y')$ , is totally  $\alpha$ -disconnected, then  $f'$  is  $\alpha$ -light mapping.

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