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A General Formule for Characteristic Polynomials of Some Special Graphs

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ABSTRACT
The calculation of characteristic polynomials (Ch. Poly.) of graphs of any size, especially for the large number of vertices n is an extremely tedious problem if used the traditional methods, so in this paper, the general formulas of the characteristic polynomial of some graphs, such as, path, complete, circle and star graphs are introduced. It is constructed based on adjacency and Laplacian matrices. The efficiency of the proposed method is demonstrated in terms of complexity to show an improvement over traditional methods.
Keywords: Graph, Characteristic polynomial, Adjacency matrix,Laplacian matrix.

INTRODUCTION
The characteristics polynomials have several applications such as, in chemical, dynamics of oscillatory reactions and quantum chemistry for determination of the stabilities of reaction network, lattice statistics, where the characteristic polynomials play an important role in lattice statistics. They are useful in constructing matching polynomials, which generate the number of possible ways of placing a given number of dimers on lattices [1].
In 1950, Coulson [4] indicated that the coefficients of the polynomial are related to a count of pertinent subgraphs of the molecular orient (in the case of π-electron calculations of conjugated hydrocarbons, the relevant part of the skeleton is the structure formed by carbon atoms alone). The characteristic polynomial is imperative to study the structure properties of the graph based on linear algebra through its coefficients [2]. In 1984, Beezer [5] introduced a characteristic polynomial on a path graph based on adjacency matrix. In 2003, Dafonseca [6] introduced the distribution polynomial of the path graph based on adjacency matrix of the complete graph. In 1992,Ivanciuc [7] computed the characteristic polynomial of a molecular graph by the propagation diagram algorithm. In 2002, Oliveira, et al. [8] introduced a general form of the coefficients of the characteristic polynomial of the combinatorial Laplacian matrix. In 2003, Lipton et al. [9] introduced general formula of the characteristic polynomial of two isomorphic graphs based on adjacency matrix. In 2010, Dehmer et al. [10] introduced graph polynomials (called information polynomial). It is used either for describing combinatorial graph invariants or to characterize chemical structures by using the coefficients or the zeros of a graph polynomial. In 2013, Nuha A. Rajabet al. [11] have introduced some properties of the characteristic polynomial of the complete graph based on adjacency matrix and signless Laplacian matrix.

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In this paper, the general formulas of the characteristic polynomial of some graphs, such as, path, complete, circle and star graphsare introduced. These graphs play an important role in many applications [12]. The relationship between the graph and the characteristic polynomials is formulated according to the adjacency matrix and Laplacian matrix of these graphs.

Basic concepts of graph theory

A **graph**$\boldsymbol{G}(\boldsymbol{V},\boldsymbol{E})$ is defined by an order pair $(V(G),E(G))$, where $V(G)$is a nonempty set whose elements are called points or vertices, and $E(G)$is a set of unordered pairs distinct elements of $V(G)$. The elements of $E(G)$ are called lines or edges of G,for example, let $V(G)=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$and $E(G)=\{e_1,e_4,e_5,e_6,e_7,,e_{10},e_8,e_3,e_2,e_9\}$ then they construct the graph $G\ (V,E)$shown in Figure 1.

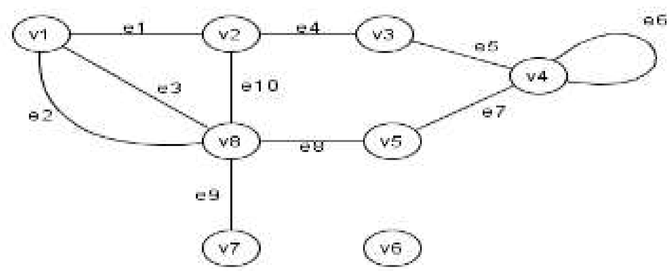
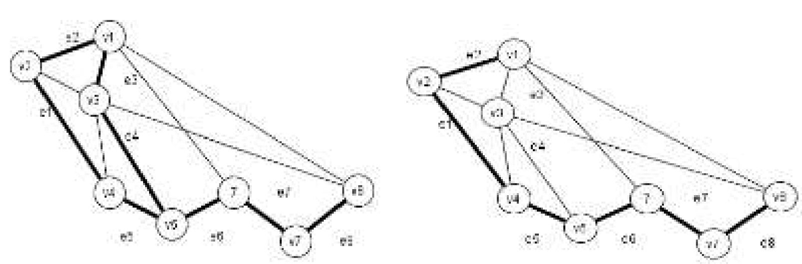


Figure (1).Graph $\boldsymbol{G}(\boldsymbol{V},\boldsymbol{E})$

Each edge in G is associated with a set consisting of either one or two vertices called its **endpoints**. From Figure 1, an edge that contains the same endpointis called a **loop** $\{e_6\}$.Two distinct edges with the same set of endpoints are called **multiple edges** $\{e_3,e_2\}$**.** If two vertices connect by an edge, they are called **adjacent.** An edge is said to be **incident** on each of its endpoints. If two edges are incident on the same vertex they are called **adjacent**. A vertex incident on a loop edge is called **adjacency itself**.The **order** of G is the number of vertices n such that $|V(G)|=n$. For example, in Figure 1,$|V(G)|=8$, the **size** of a graph $|E(G)|$ is the number of edges$m$, here$|E(G)|=10$. The **degree** of a vertex $\boldsymbol{d}(\boldsymbol{v}_i)$is the number of the edges that are incident with the vertex, for example$d(v_1)=3$. The vertex of degree zero is called **isolated vertex**,for example,$\boldsymbol{v}_6$.

Definitions of Some Graphs

Definition 1[14]:A walk is an alternating sequence of vertices and edges, which begins and ends with a vertex and each vertex is incident on the two edges expect first and last vertices. A walk is called closed if its first and last vertices are the same, and it is called open if they are different. Figure 2. (A) is an example of a walk is$\{e_1,e_2,e_3,e_4,e_5,\ e_1,e_2,e_3,e_4,e_6,e_8\}$. A walk in which all the edges are distinct is a trail. Figure 2. (B) is an example of a trail is$\{e_2,e_1,e_5,e_3,e_6,e_8\}$.



(A) (B)

Figure (2).(A) Walk and(B) trail of graph G.

Definition 2[14]: The **Path**is a trail with all vertices are distinct. It is denoted by P_n, where n is the number of vertices. The path P_n has $(n-1)$ edges. Figure 3, shows some examples of paths.

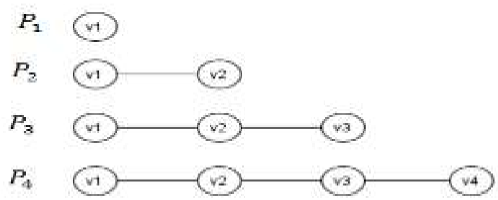


Figure (3).Paths (P_1, P_2, P_3, P_4).

Definition3[14]: The **Cycle** is a path graph that its initial vertex v_1 equal to its terminal vertexv_n. It is denoted by C_n, where $n \geq 3$. Figure 4, is an example of a cycle graph.

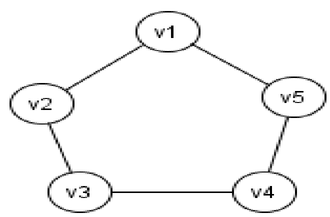


Figure (4). Cycle graph of order 5

Definition4[14]: A **complete graph** with n vertices denoted by k_n is a graph where every pair of vertices is adjacent. Thus $k_n$has $\binom{n}{2}$edges. Figure 5(a, b, c & d) shows the complete graphs of order 1, 2, 3, and 4, respectively.

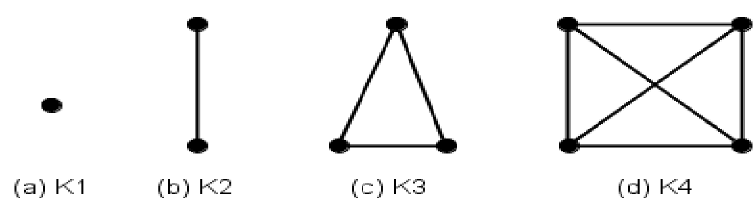


Figure (5). Complete graphs

Definition5[14]: The **Star graph** is a set of vertices joined by a common vertex; it is denoted by S_n. Figure 6 shows the star graphS_4.

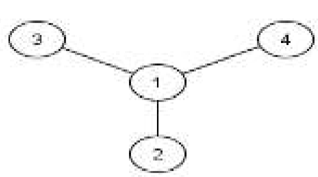


Figure (6). Star graph of order 4

Matrices of graph

In this section, some matrices used in this paper are introduced, such as adjacency matrix $A(G)$, degree matrix $D(G)$ and Laplacian matrix $L(G)$.

Adjacency matrix [14]

A simple graph can be represented by an ***adjacency matrix***. If a graph has $n$vertices $v_1, \dots, v_n$, then its $n \times n$ adjacency matrix is defined as;

$$A(G) = (a_{ij})_{n\times n} \text{where} a_{ij} = \begin{cases} 1 & \text{if} v_i \ and \ v_j \text{are adjacent} \\ 0 & otherwise \end{cases} \quad \dots(1)$$

Degree matrix [14]

A simple graph can be represented by the ***degree matrix***. If a graph has $n$vertices $v_1, \dots, v_n$, then an$n \times n$ degree matrix is defined as;

$$D(G) = (a_{ij})_{n\times n} \text{where} a_{ij} = \begin{cases} d(v_i) & if \quad i = j \\ 0 & otherwise \end{cases} \dots(2)$$

The Laplacian matrix [13]

The Laplacian matrix $L(G)$ of a graph G with n vertices and m edges is defined to be the $n \times n$ symmetric matrix.

$$L(G) = D(G) - A(G) \quad \dots (3)$$

The characteristic polynomial [14]

Let $A = (a_{ij})$ be an$n \times n$ matrix. If $Au = \lambda u$, then λ and $u \neq 0$ are called the eigenvalue and eigenvector of A, respectively. The eigenvalues of A are the roots of the characteristic polynomials given in (4)

$$\mathcal{P}_G(A, \lambda) = |\lambda I - A| \quad \dots(4)$$

The eigenvectors are the solution to the following homogenous system.

$$(\lambda I - A)u = 0 \quad \dots(5)$$

Frame's method

In 1984, Balasubramanian K. [3] introduced a method to find the general form of the coefficients of the characteristicpolynomial; this method is called Frame's method. To find the coefficients, he depends on the trace of the matrices. The general form of his method is described as follows:

Let A be a matrix of order n. Then the characteristic polynomial corresponding to $A$is:

$$\mathcal{P}_G(A, \lambda) = |A - \lambda I| = C_0\lambda^n - C_1\lambda^{n-1} - \cdots - C_n = 0 \quad \dots (6)$$

where $C_0 = 1$

Now, the matrices B_k and the coefficients C_k are constructed as follows:

$$B_1 = A(A - C_1 I), \quad C_1 = trace(A) \quad \dots(6.1)$$

$$C_2 = \frac{1}{2} trace(B_1) \quad \dots(6.2)$$

$$B_2 = A(B_1 - C_2 I) \quad \dots(6.3)$$

$$C_3 = \frac{1}{3} trace(B_2) \quad \dots(6.4)$$

In general,

$$B_{n-1} = A(B_{n-2} - C_{n-1} I) \quad \dots(6.5)$$

$$C_n = \frac{1}{n} trace(B_{n-1}) \quad \dots (6.6)$$

The Frame's method is applied to find the Ch. Poly. For any matrices.

The Proposed Method

In this section, by following the classical procedure to find the characteristic polynomial of graphs $\mathcal{P}_G(A,\lambda)$, we introduced a new efficient method for this purpose. Since the general form of $\mathcal{P}_G(A,\lambda)$is defined in (4), then the characteristic polynomial of$P_n$with one vertex basedon the adjacency matrix$A(P_1)$ is:

$$\mathcal{P}_{P_1}(A,\lambda) = |\lambda| = \lambda \quad \dots(7)$$

For $n = 2$, $A(P_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Therefore, $\mathcal{P}_{P_2}(A,\lambda) = \left|\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right| = \left|\begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix}\right| = \lambda^2 - 1 \dots(8)$

For $n = 3$,

$A(P_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, Therefore,

$$\mathcal{P}_{P_3}(A,\lambda) = \left|\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right| = \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} = \lambda^3 - 2\lambda \quad \dots(9)$$

Some coefficients$(a_0, a_1, \dots, a_9)$ of the characteristic polynomialsare inserted in Table 1, for $n =$ 4,5,6,7,8,9.

Table 3.1. The coefficient of the characteristic polynomials of P_n based on $A(P_n)$.

This table helps to find the general formula of the characteristic polynomials for any P_n, and as follows:

$$\mathcal{P}_{P_n}(A,\lambda)=\binom{n}{n}\lambda^{n}+0\lambda^{n-1}-\binom{n-1}{(n-1)-1}\lambda^{n-2}+(0)\lambda^{n-3}+\binom{n-2}{(n-2)-2}\lambda^{n-4}+0\lambda^{n-5}$$
$$-\binom{n-3}{(n-3)-3}\lambda^{n-6}+\cdots+(\lambda^{0})\begin{cases}1 & if\ n=4,8,12,\ldots\\ 0 & if\ \ n\ odd\\ -1 & otherwise\end{cases}$$
$$=\sum_{i=0}^{n-1}(-1)^{\frac{i}{2}}\binom{n-\frac{i}{2}}{n-i}\lambda^{n-i}+(\lambda^{0})\begin{cases}1 & if\ n=4,8,12,\ldots\\ 0 & if\ \ n\ odd\\ -1 & otherwise\end{cases} \quad \ldots (10)$$

where $n\geq 1$.

Proof: From (4), we have

$$\mathcal{P}_{P_n}(A,\lambda)=|\lambda I-A|=\begin{vmatrix}\lambda & -1 & 0 & 0 & 0 & \cdots & 0\\ -1 & \lambda & -1 & 0 & 0 & \cdots & 0\\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & -1 & \lambda & -1 & 0\\ 0 & 0 & \cdots & 0 & -1 & \lambda & -1\\ 0 & 0 & \cdots & \cdots & 0 & -1 & \lambda\end{vmatrix} \ldots(10.1)$$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
4	1	0	-3	0	1					
5	1	0	-4	0	3	0				
6	1	0	-5	0	6	0	-1			
7	1	0	-6	0	10	0	-4	0		
8	1	0	-7	0	15	0	-10	0	1	
9	1	0	-8	0	21	0	-20	0	5	0
n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
4	1	0	-3	0	1					
5	1	0	-4	0	3	0				
6	1	0	-5	0	6	0	-1			
7	1	0	-6	0	10	0	-4	0		
8	1	0	-7	0	15	0	-10	0	1	
9	1	0	-8	0	21	0	-20	0	5	0
n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
4	1	0	-3	0	1					
5	1	0	-4	0	3	0				
6	1	0	-5	0	6	0	-1			
7	1	0	-6	0	10	0	-4	0		
8	1	0	-7	0	15	0	-10	0	1	
9	1	0	-8	0	21	0	-20	0	5	0
n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
4	1	0	-3	0	1					
5	1	0	-4	0	3	0				
6	1	0	-5	0	6	0	-1			
7	1	0	-6	0	10	0	-4	0		
8	1	0	-7	0	15	0	-10	0	1	
9	1	0	-8	0	21	0	-20	0	5	0

Then, we have

$$\mathcal{P}_{P_n}(A,\lambda)=\lambda\begin{vmatrix}\lambda & -1 & 0 & 0 & 0 & \cdots & 0\\ -1 & \lambda & -1 & 0 & 0 & \cdots & 0\\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0\\ \vdots & \cdots & \ddots & \ddots & \ddots & & \vdots\\ 0 & \cdots & 0 & -1 & \lambda & -1 & 0\\ 0 & \cdots & 0 & 0 & -1 & \lambda & -1\\ 0 & \cdots & 0 & 0 & 0 & -1 & \lambda\end{vmatrix}_{n-1\times n-1}$$

$$+\begin{vmatrix}-1 & -1 & 0 & 0 & 0 & \cdots & 0\\ 0 & \lambda & -1 & 0 & 0 & \cdots & 0\\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0\\ \vdots & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & -1 & \lambda & -1 & 0\\ 0 & \cdots & 0 & 0 & -1 & \lambda & -1\\ 0 & \cdots & 0 & 0 & 0 & -1 & \lambda\end{vmatrix}_{n-1\times n-1}$$

The determent for the second matrix represent the characteristic polynomials for $(n-2)$, then

$$\mathcal{P}_{P_n}(A,\lambda)=\lambda\mathcal{P}_{P_{n-1}}(A,\lambda)-\mathcal{P}_{P_{n-2}}(A,\lambda) \qquad \ldots (10.2)$$

Now, to proof (10), the principle of mathematical induction is used and as follows:-

Step 1: If $n=1$, then (10) is true, since

$$\mathcal{P}_{P_1}(A,\lambda)=\lambda \qquad \ldots(10.3)$$

Step2: Suppose (10) is true for some $n = k \geq 1$,

Step3: Toprove that (10) is true for $n = k + 1$, is performed as follows:-

By using (10.2), we have

$$\mathcal{P}_{P_{k+1}}(A,\lambda)=\lambda\mathcal{P}_{P_k}(A,\lambda)-\mathcal{P}_{P_{k-1}}(A,\lambda)$$

Substitute (10) in (10.2) to obtain,

$$=\lambda\Bigg[\binom{k}{k}\lambda^{k}+0\lambda^{k-1}-\binom{k-1}{(k-1)-1}\lambda^{k-2}+(0)\lambda^{k-3}+\binom{k-2}{(k-2)-2}\lambda^{k-4}+0\lambda^{k-5}$$
$$-\binom{k-3}{(k-3)-3}\lambda^{k-6}+0\lambda^{k-7}+\binom{k-4}{(k-4)-4}\lambda^{k-8}+(0)\lambda^{k-9}+\cdots$$
$$+\left((\lambda^0)\begin{cases}1\ if\ 4,8,12,\ldots\\ 0\ if\ n\ odd\\ -1\ otherwise\end{cases}\right)\Bigg]$$
$$-\Bigg[\binom{k-1}{k-1}\lambda^{k-1}+0\lambda^{k-2}-\binom{k-2}{(k-2)-1}\lambda^{k-3}+(0)\lambda^{k-4}$$
$$+\binom{k-3}{(k-3)-2}\lambda^{k-5}+0\lambda^{k-6}-\binom{k-4}{(k-4)-3}\lambda^{k-7}+0\lambda^{k-8}$$
$$+\binom{k-5}{(k-5)-4}\lambda^{k-9}+(0)\lambda^{k-10}+\cdots+\left((\lambda^0)\begin{cases}1\ \ if\ \ n=4,8,12,\ldots\\ 0\ \ if\ \ \ n\ odd\\ -1\ \ \ otherwise\end{cases}\right)\Bigg]$$

$$= \left[\binom{k}{k}\lambda^{k+1} + 0\lambda^{k} - \left(\binom{k-1}{(k-1)-1} + \binom{k-1}{k-1} \right)\lambda^{k-1} + (0)\lambda^{k-2} + \left(\binom{k-2}{(k-2)-2} + \binom{k-2}{(k-2)-1} \right)\lambda^{k-3} + 0\lambda^{k-4} - \left(\binom{k-3}{(k-3)-3} + \binom{k-3}{(k-3)-2} \right)\lambda^{k-5} + 0\lambda^{k-6} + \left(\binom{k-4}{(k-4)-4} + \binom{k-4}{(k-4)-3} \right)\lambda^{n-7} + (0)\lambda^{k-8} + \cdots + \left((\lambda^{0}) \begin{cases} 1 & if \;\; n = 4,8,12,\dots \\ 0 & if \;\; n\ odd \\ -1 & otherwise \end{cases} \right) \right] \quad \dots(10.4)$$

Since by the combinatorics properties we have:

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$…(10.5)

This was helped to conclude that (10) is true.

The main results

The other results in this section are summarized as follows:

The characteristic polynomial of path graphs based on Laplacian matrix

The general formula of the characteristic polynomials of P_n based on $L(P_n)$ is:

$$\mathcal{P}_{P_n}(L,\mu) = \binom{n}{n}\mu^{n} - \binom{2(n-1)}{2(n-1)-1}\mu^{n-1} + \binom{2(n-1)-1}{2(n-1)-3}\mu^{n-2} - \binom{2(n-1)-2}{2(n-1)-5}\mu^{n-3} + \binom{2(n-1)-3}{2(n-1)-7}\mu^{n-4} - \binom{2(n-1)-4}{2(n-1)-9}\mu^{n-5} + \cdots + (-1)^{i}\binom{n}{1}\lambda$$

$$= \sum_{i=0}^{n-1} (-1)^{i} \binom{(2(n-1)-(i-1))}{(2(n-1)-j)} \mu^{n-i},\ n \geq 1 \quad \dots(11)$$

where$j = 2(i-1)+1$

Proof: By (4), we have

$$\mathcal{P}_{P_n}(L,\mu) = |\lambda I - L| = \begin{vmatrix} \mu-1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & \mu-2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \mu-2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & \mu-2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \mu-2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \mu-1 \end{vmatrix} \quad \dots(11.1)$$

By computing the determent of (11.1), we have

$$= (\mu-1)\begin{vmatrix} \mu-2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & \mu-2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \mu-2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & \mu-2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \mu-2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \mu-1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mu-2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \mu-2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & \mu-2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \mu-2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \mu-1 \end{vmatrix} \quad \ldots(11.2)$$

$$= (\mu-1)(\mu-2)\begin{vmatrix} \mu-2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & \mu-2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \mu-2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & \mu-2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \mu-2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \mu-1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mu-2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \mu-2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & \mu-2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \mu-2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \mu-1 \end{vmatrix}$$

Then, we have:

$\mathcal{P}_{P_n}(L,\mu) = (\mu-2)\mathcal{P}_{P_{n-1}}(L,\mu) - \mathcal{P}_{P_{n-2}}(L,\mu) n \geq 3$ …(11.3)

Now, to proof (11), the principle of mathematical induction is used and as follows:-

Step 1: If $n = 1$, then (11) is true, since

$\mathcal{P}_{P_1}(A,\lambda) = \lambda$ ...(11.4)

Step2: Suppose (11) is true for some $n = k \geq 1$,

Step3: Toprove that (11) is true for $n = k + 1$, is performed as follows:-
Substitute (11) in (11.3) to obtain:

$$\mathcal{P}_{P_{k+1}}(L,\mu) = (\mu - 2)\mathcal{P}_{P_k}(L,\mu) - \mathcal{P}_{P_{k-1}}(L,\mu)$$

Then by simple calculated and combinatorics properties, where$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ we have (11) is true. ■

The characteristic polynomial of complete graph based on adjacency matrix

The general formula of the characteristic polynomials of k_n based on $A(k_n)$ is:

$\mathcal{P}_{k_n}(A,\lambda) = \binom{n}{n}\lambda^n - \binom{n}{n-2}\lambda^{n-2} - 2\binom{n}{n-3}\lambda^{n-3} - 3\binom{n}{n-4}\lambda^{n-4} + \cdots (-)(n-1)\lambda^0 = \binom{n}{n}\lambda^n - \left(\sum_{i=2}^{n-1}(i-1)\binom{n}{n-i}\lambda^{n-i}\right) - (n-1)\lambda^0 (12)$

where $n \geq 1$.

Proof: From (4), we have

$$\mathcal{P}_{k_n}(A,\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & \cdots & -1 & \lambda & -1 \\ -1 & \cdots & -1 & -1 & \lambda \end{vmatrix} \qquad \ldots(12.1)$$

By calculating the determent of (12.1), we have

$$\mathcal{P}_{k_n}(A,\lambda) = (\lambda - (n-1))(\lambda + 1)^{n-1} \qquad \ldots(12.2)$$

Then

$$\mathcal{P}_{k_n}(A,\lambda) = \big(\lambda - (n-1)\big)\left(\binom{n-1}{0}\lambda^{n-1} + \binom{n-1}{1}\lambda^{n-2} + \binom{n-1}{2}\lambda^{n-3} + \binom{n-1}{3}\lambda^{n-4} + \cdots + \binom{n-1}{n-1}\right) \qquad (12.3)$$

$$= \left(\binom{n-1}{0}\lambda^n + \binom{n-1}{1}\lambda^{n-1} + \binom{n-1}{2}\lambda^{n-2} + \binom{n-1}{3}\lambda^{n-3} + \cdots + \binom{n-1}{n-1}\lambda\right) + \left(-(n-1)\binom{n-1}{0}\lambda^{n-1} - (n-1)\binom{n-1}{1}\lambda^{n-2} - (n-1)\binom{n-1}{2}\lambda^{n-3} - (n-1)\binom{n-1}{3}\lambda^{n-4} - \cdots - (n-1)\binom{n-1}{n-1}\right)$$

$$= \binom{n-1}{0}\lambda^n + \left(\binom{n-1}{1} - (n-1)\binom{n-1}{0}\right)\lambda^{n-1} + \left(\binom{n-1}{2} - (n-1)\binom{n-1}{1}\right)\lambda^{n-2} + \left(\binom{n-1}{3} - (n-1)\binom{n-1}{2}\right)\lambda^{n-3} + \cdots + \left(\binom{n-1}{n-1} - (n-1)\binom{n-1}{n-2}\right)\lambda + \left((n-1)\binom{n-1}{n-1}\right)$$

Since by the combinatorics properties we have (12) is true. ■

The characteristic polynomial of complete graph based on Laplacian matrix

The general formula of the characteristic polynomials of k_n based on $L(k_n)$ is:

$$\mathcal{P}_{k_n}(L,\mu) = \left(\binom{n-1}{0}\mu^n - n\binom{n-1}{1}\mu^{n-1} + n^2\binom{n-1}{2}\mu^{n-2} - n^3\binom{n-1}{3}\mu^{n-3} + \cdots + (-1)^i n^{n-1}\binom{n-1}{n-1}\mu\right), n \geq 1 \qquad ...(13)$$

Proof:following the same procedure given above, this formula can be proved.

The characteristic polynomial of cycle graph based on adjacency matrix

The general formula of the characteristic polynomials of C_n based on $A(C_n)$ is:

$\mathcal{P}_{C_n}(A,\lambda) = \lambda\mathcal{P}_{P_{n-1}}(A,\lambda) - 2\mathcal{P}_{P_{n-2}}(A,\lambda) - 2\ldots$ (14)

Proof: By (4), we have

$$\mathcal{P}_{c_n}(A,\lambda) = |\lambda I_n - A| = \begin{vmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & \lambda & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & \lambda & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & \lambda & -1 \\ -1 & \cdots & 0 & 0 & 0 & -1 & \lambda \end{vmatrix}_{(n\times n)} \qquad (14.1)$$

Then (14.1) become

$\mathcal{P}_{c_n}(A,\lambda) = \lambda X + Y + Z$(14.2)

$$\text{where} X = \begin{vmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 \\ -1 & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & \cdots & -1 & \lambda & -1 & 0 \\ \vdots & \cdots & 0 & -1 & \lambda & -1 \\ 0 & \cdots & 0 & 0 & -1 & \lambda \end{vmatrix}_{(n-1\times n-1)},$$

$$Y = \begin{vmatrix} -1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & \cdots & -1 & \lambda & -1 & 0 \\ \vdots & \cdots & 0 & -1 & \lambda & -1 \\ -1 & \cdots & 0 & 0 & -1 & \lambda \end{vmatrix}_{(n-1\times n-1)}$$

and

$$Z=\begin{vmatrix} -1 & \lambda & -1 & 0 & \cdots & 0 \\ 0 & -1 & \lambda & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & \lambda & -1 \\ \vdots & \cdots & 0 & 0 & -1 & \lambda \\ -1 & \cdots & 0 & 0 & 0 & -1 \end{vmatrix}_{(n-1\times n-1)}.$$

Now, to show that the determine of$X$is $\mathcal{P}_{P_{n-1}}(A,\lambda)$, and $Y=Z$.
If we use the first row in Y to calculate its determinant, we found that it is equal to the calculation of the determinant of Z based on the last row, as well as, after simple row permutation, we see that both of them are equal, and they are equal to the path matrix where the number of rows equal $n-2$. Hence, from Y and Z we have:

$$Y+Z=-2\mathcal{P}_{P_{n-2}}(A,\lambda)-2 \qquad \ldots(14.3)$$

Therefore, (14.2) can be written as:

$$\mathcal{P}_{C_n}(A,\lambda)=\lambda\mathcal{P}_{P_{n-1}}(A,\lambda)-2\mathcal{P}_{P_{n-2}}(A,\lambda)-2$$

We also conclude that by (14), we have an ability to find a relationship between path and cycle graph based on characteristic polynomials.

The characteristic polynomial of cycle graph based on Laplacian matrix

The general formula of the characteristic polynomials of C_n based on $L(C_n)$ is:

$$\mathcal{P}_{C_n}(L,\mu)=(\mu-2)\mathcal{P}_{C_{n-1}}(L,\mu)-\mathcal{P}_{C_{n-2}}(L,\mu)+(-1)^{n-1}2\mu \ldots(15)$$

Proof: By (4) we have

$$\mathcal{P}_{C_n}(L,\mu)=\begin{vmatrix} \mu-2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & \mu-2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \mu-2 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \mu-2 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & \mu-2 & 1 \\ 1 & \cdots & 0 & 0 & 0 & 1 & \mu-2 \end{vmatrix} \qquad (15.1)$$

By computing the determent of (15.1) the formula in (15) can be easily proved ■

The characteristic polynomial of star graph based on adjacency matrix

The general formula of the characteristic polynomials of S_n based on $A(S_n)$ is:

$$\mathcal{P}_{S_n}(A,\lambda)=\lambda^n-(n-1)\lambda^{n-2},\ n\geq 2 \qquad \ldots(16)$$

Proof: Using (4) we have:

$$\mathcal{P}_{S_n}(A,\lambda) = \begin{vmatrix} \lambda & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \lambda & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & \lambda & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & \lambda & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \lambda \end{vmatrix}_{n\times n} \quad \ldots(16.1)$$

Then $\mathcal{P}_{S_n}(A,\lambda) = \lambda(\mathcal{P}_{S_{n-1}}(A,\lambda)) - \lambda^{n-2}$ **…**(16.2)

The proof is performed by substituting (16) in (16.2).■

The characteristic polynomial of Star graph based on Laplacian matrix

The general formula of the characteristic polynomials of S_n based on $L(S_n)$ is:

$\mathcal{P}_{S_n}(L,\mu) = (\mu-(n-1))(\mu-1)^{n-1} - (n-1)(\mu-1)^{n-2}$ …(17)

Computational complexity

The concept of computational complexity measures the running time of the algorithms. Since the proposed method is a direct substitution then the computational complexity of our method $O(n)$ whereas,the computation complexity of the classical methods such as Frame's method is $O(n^4)$.We conclude that the proposed method is performed better.

CONCLUSION

We proposed a new approach to find the characteristic polynomial for some important graphs, which is proven efficient and accurate for this purpose. In the proposed method, it is computed directly, whereas, in the traditional methods it is computed recursively. In this paper, we have obtained a general form for the characteristic polynomial for some special graphs (path, complete, circle, and star) via two types of matrices (adjacencyand Laplacian). The proposed method shows an improvement over traditional method from the computational complexity point of view; it is an easy and fast method.

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