T-Operation on Topological Space and Separation Axioms

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Abstract

In this paper we introduce a new kind of operation, t, on a topological space (X,τ) , and study the concepts of t-open , t- semi open together with its corresponding t-closure ,t-interior ,t-semi closure and t-semi interior operators. Also we study the concepts of t-T_i spaces and t-semi T_i spaces (i=0,½,1,2) and study the relations between them. Finally ,we study the concepts of t-T_b and t-T_d spaces using the concepts of t-gs.closed sets in a topological space and investigate the relation between them.

الخلاصة

في هذا البحث عرفنا نوعا جديدا من المؤثر على الفضاء التوبولوجي أسميناه مؤثر t ودرسنا مفاهيم المجموعات t – المفتوحة و t –شبه المفتوحة مع المفاهيم المرتبطة بها مثل t–انغلاق ,t–داخلية ,t–شبه انغلاق وt–شبه داخلية.كذلك درسنا الفضاءات t-T_i و t-semi T_i (i=0,½,1,2) وحددنا العلاقات بينها.

وأخيرا درسنا الفضاءات t-T_b و t-T_d بالاعتماد على مفهوم المجموعات t-gs. المغلقة وحددنا العلاقات بينها وبين بعض بديهيات الفصل الأخرى.

1. Introduction.

Jankovic 1983 defined the concept of operation on topological spaces and introduce the concept of α-closed graphs of an operation. Ogata 1991 called the operation α as γ-operation and introduce the notion of τ_γ which is the collection of all γ-open sets in a topological space (X,τ) .Further he introduce the concept of γ-T_i spaces (i =0,½,1,2) and characterized γ-T_i by the notion of γ-closed set or γ-open sets.

Krishnan and Balachandran 1998 introduced the concept of γ-semi open sets in a topological space (by using the definition of operation γ as a mapping from τ into P(X) with the condition $U \subseteq \gamma(U)$ for each $U \in \tau$) and the concepts of γ-semi T_i spaces (i =0,½,1,2) and studied the relation between them, also they introduced the concepts of γ-T_b and γ-T_d spaces using the concept of γ-gs.closed sets in a topological space.

In this paper ,in section 2, the author give a deferent definition of operation on a topological space (X,τ) and call it t-operation (as a mapping from P(X) into τ with the condition $t(U) \subseteq U$ for each $U \subseteq X$ and t(X)=X), and study the concepts of t-open, t-closed ,t-g.closed, t-semi open , t-semi closed ,t- semi g.closed and t-g.semi closed sets in (X,τ) together with its corresponding t-closure ,t-interior ,t-semi closure and t-semi interior operators in the sense of the new definition of operation t. Also in section 2 the author prove many results about the previous concepts.

In section 3 the author study the concepts of t- T_i , t-semi T_i (i =0,½,1,2) , t-T_b and -T_d spaces using the t-operation on a topological space and investigate the relationship between them.

Through out sections 2 and 3 the symbols clA and intA denote the closure and interior sets of the set A (respectively)with respect to the topology τ.

2 T-operation on a topological space.

2.1 Definition. Let (X,τ) be a topological space. An operation t on (X,τ) is a mapping from P(X) into τ such that $t(V) \subseteq V$ for each $V \subseteq X$ and t(X)=X.

2.2 Definition. Let t be an operation on a topological space (X,τ), $A\subseteq X$.

2.3

A is said to be a t-open set if for each x in A ,there exists a subset G of X such that $x\in t(G)\subseteq A$. The set of all t-open sets in (X,τ) is denoted by t-O(X).

2.4 Remark. t-O(X) $\subseteq\tau$, $t(U)\in$ t-O(X) for each $U\subseteq X$, but a member of t-O(X) need not be t(U) for some U.

2.5 Remark. If t is an operation on a topological space (X,τ) defined by t(A)=intA , for each $A\subseteq X$, then t-O(X)=τ.

2.6 Remark. By 2.2 and 2.3, if t is an operation on a topological space (X,τ), then

i) φ, X $\in$ t-O(X)

ii) If $A_\alpha\in$ t-O(X), $\alpha\in\Lambda$, then $\bigcup_{\alpha\in\Lambda}A_\alpha\in$ t-O(X).

iii) If A,B $\in$ t-O(X), it is not necessary that $A\cap B\in$ t-O(X).

(i.e. t-O(X) dose not form a topology on X in general)

2.7 Example. Let $X=\{a,b,c\}$, $\tau=\{\varphi,X,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$, t , defined by ,t(A)=intA if intA$\neq\{c\}$, t(A)= φ if intA=$\{c\}$.Then

t-O(X)= τ-$\{\{c\}\}$.

If $A=\{a,c\}$,$B=\{b,c\}$, then A,B $\in$ t-O(X) ,but $A\cap B\notin$ t-O(X).

2.8 Definition. A subset A of X is said to be a t-closed set if its complement is t-open.

2.9 Remark. By 2.5 and 2.7 , if t is an operation on a topological space (X,τ), then

i) φ and X are t-closed sets.

ii) The intersection of any family of t-closed sets is a t-closed set.

iii) The union of two t-closed sets need not be t-closed set.

2.10 Definition. The t-closure set of a subset A of X, denoted by cl_tA, is defined by $cl_tA=\{x\in X|\ A\cap t(U)\neq\varphi$ for each set U such that $x\in t(U)\}$.

2.11 Remark. For each subset A of X , $A\subseteq clA\subseteq cl_tA$.(since $t(U)\in\tau$)

2.12 Remark. In Example 2.6 if E={a,b,d}, then clE=E where $cl_tE=X$.

2.13 Remark. If t is an operation on a topological space (X,τ), then:

i) $A\subseteq B\subseteq X$ implies $cl_tA\subseteq cl_tB$.

ii) $\cup cl_tA_\alpha\subseteq cl_t\cup A_\alpha$

iii) $cl_t\cap A_\alpha\subseteq\cap cl_tA_\alpha$ (where $\{A_\alpha\}$is an arbitrary family of subsets of X).

iv) $cl_tA=\cap\{F|$ F is a t-closed set and $A\subseteq F\}$.

v) A is a t-closed set if and only if A= cl_tA.

vi) $cl_t(cl_tA)=cl_tA$,i.e cl_tA is a t-closed set.

proof: (i) follows from Definition 2.9.

(ii) and (iii) follow from (i) since $A_\alpha\subseteq\cup A_\alpha$ and $\cap A_\alpha\subseteq A_\alpha$ for each α.

iv) If $x\notin cl_tA$ then by 2.9 there exists U such that $x\in t(U)$ and $t(U)\cap A=\varphi$.

Hence $A\subseteq X$- t(U), $x\notin X$-t(U) and $t(U)\in$ t-O(X),

i.e $x\notin\cap\{F|$ F is a t-closed set and $A\subseteq F\}$.

On the other hand if $x\notin\cap\{F|$ F is a t-closed set and $A\subseteq F\}$ then there exists a t-closed set F such that $A\subseteq F$ and $x\notin F$, which implies $x\in(X-F)\in$ t-O(X) and $A\cap(X-F)=\varphi$, i.e $x\notin cl_tA$(Definition 2.9).

v) If A is t-closed , then by (iv) $cl_tA\subseteq A$ and by 2.10 $A\subseteq cl_tA$, so, A= cl_tA.

Now if A= cl_tA, then by (iv) and 2.8 (ii) A is a t-closed set.

vi) By 2.10,$cl_tA\subseteq cl_t(cl_tA)$.If $x\notin cl_tA$ then by Definition 2.9, there exists U such that $x\in t(U)$ and $t(U)\cap A=\varphi$, assume that $t(U)\cap(cl_tA)\neq\varphi$ and let $y\in t(U)\cap cl_tA$, therefore $y\in t(U)$ and $y\in cl_tA$, so by Definition 2.9, $t(U)\cap A\neq\varphi$, which is a contradiction. Hence $x\notin cl_t(cl_tA)$.

2.14 Definition. Let t be an operation on a topological space (X,τ), $A\subseteq X$. A is said to be a t-generalized closed set (shortly t-g.closed set) if $cl_tA\subseteq U$, whenever $A\subseteq U$ and U is a t-open set.

2.15 Remark. Any t-closed set in (X,τ) is a t-g.closed set ,but the converse is not true.

2.16 Example. Let $X=\{a,b,c,d\}$, $\tau=\{\varphi,X,\{a\},\{a,b\},\{a,b,c\}\}$, $t:P(X)\rightarrow\tau$, defined by $t(A)=intA$, then $t\text{-}O(X)=\tau$.
$A=\{a,b,d\}$is not t-closed ,where the only t-open set containing A is X ,which contains $clA=X$. i.e A is a t-g.closed set.

2.16 Theorem. Let t be an operation on a topological space (X,τ). Then For each $x\in X$, $\{x\}$ is t-closed set or $X\text{-}\{x\}$ is a t-g.closed set in (X,τ).

Proof : If $\{x\}$ is not a t- closed set then $X\text{-}\{x\}$ is not a t-open set, and so X is the only t-open set containing $X\text{-}\{x\}$which is containing $cl_t(X\text{-}\{x\})$ too. i.e $X\text{-}\{x\}$ is a t-g.closed set.

2.17 Definition. Let t be an operation on a topological space (X,τ) and $A\subseteq X$**.** The set t-int(A) is defined by, $t\text{-}int(A)=\cup\{U|U\in t\text{-}O(X) \text{ and } U\subseteq A\}$.

2.18 Remark. By 2.3 and 2.17 $t\text{-}int(A)\subseteq int(A)\subseteq A$.

2.19 Remark. Let t be an operation on a topological space (X,τ), $A,B\subseteq X$ and $\{A_\alpha\}$ an arbitrary family of subsets of X, then

i) $A\subseteq B$ implies $t\text{-}int(A)\subseteq t\text{-}int(B)$.

ii) $\cup(t\text{-}intA_\alpha)\subseteq t\text{-}int(\cup A_\alpha)$.

iii) $t\text{-}int(\cap A_\alpha)\subseteq\cap(t\text{-}int\, A_\alpha)$.

iv) A is t-open if and only if A=t-intA.

Proof: i) Follows from definition.

ii) By (i) and since $A_\alpha\subseteq\cup A_\alpha$ for each α.

iii) Also by (i) and since $\cap A_\alpha\subseteq A_\alpha$ for each α.

iv) If A is t-open then , for each $x\in A$, $x\in t(G)\subseteq A$ for a subset G of X (Definition 2.2), and by Remark 2.3 $t(G)\in t\text{-}O(X)$,hence $A\subseteq t\text{-}int(A)$, (Definition 2.17), but $t\text{-}int(A)\subseteq A$,(Remark 2.18) ,i.e A= t-int(A).
If A= t-int(A),then by 2.17 and 2.5(ii) A is a t-open set.

2.20 Remark. The equality in 2.19 (ii)and(iii) is not true. Return to Example 2.6 , if $A=\{a\}$and $B=\{c\}$then $t\text{-}intA\cup t\text{-}intB=\{a\}\cup\varphi=\{a\}$,where $t\text{-}int(A\cup B)=\{a,c\}$. Also if $A=\{a,c\}$and $B=\{b,c\}$, then $t\text{-}intA\cap t\text{-}intB=\{a,c\}\cap\{b,c\}=\{c\}$, where $t\text{-}int(A\cap B)=t\text{-}int(\{c\})=\varphi$.

2.21 Definition. Let t be an operation on a topological space (X,τ). A subset A of X is said to be t-semi open set if there exists a t-open set U such that, $U\subseteq A\subseteq cl_t(U)$. The set of all t-semi open sets in (X,τ) is denoted by t-SO(X). The complement of a t-semi open set is called a t-semi closed set.

2.22 Theorem. Let (X,τ) be a topological space, t an operation on (X,τ),$A\subseteq X$.

i) A is a t-semi open set if and only if $A\subseteq cl_t(t\text{-}intA)$.

ii) A is a t-semi colsed set if and only if $t\text{-}int(cl_tA)\subseteq A$.

Proof. i) If $A\subseteq cl_t(t\text{-}intA)$, then by Definition 2.17 and Remark 2.5(ii), U=t-intA is a t-open set and $U\subseteq A$,and so, $U\subseteq A\subseteq cl_t(U)$.Hence A is a t-semi open set.

Conversely, given A a t-semi open set , then $U\subseteq A\subseteq cl_tU$, for some t-open set U,by Remark 2.19(iv) t-intU=U ,hence $A\subseteq cl_t(t\text{-}intU)$, and $U\subseteq A$ by 2.12(i),2.19(i) implies $cl_t(t\text{-}intU)\subseteq cl_t(t\text{-}intA)$,therefore $A\subseteq cl_t(t\text{-}intA)$.

ii) A is a t-semi colsed set if and only if X-A is a t-semi open set ,if and only if $(X\text{-}A) \subseteq cl_t(t\text{-}int(X\text{-}A))$.

But $t\text{-}int(X\text{-}A)= \cup\{U|U\in t\text{-}O(X) \text{ and } U\subseteq X\text{-}A\}=\cup\{U|U\in t\text{-}O(X) \text{ and } A\subseteq X\text{-}U\}$ $=X\text{-}\cap\{X\text{-}U|U\in t\text{-}O(X) \text{ and } A\subseteq X\text{-}U\}=X\text{-}(cl_tA)$.

Similarly $cl_t(X\text{-} cl_tA)=X\text{-}(t\text{-}int(cl_tA))$.

Hence $(X\text{-}A) \subseteq X\text{-}(t\text{-}int(cl_tA))$, and so $t\text{-}int(cl_tA) \subseteq A$.

2.23 Remark. $t\text{-}O(X) \subseteq t\text{-}SO(X)$.

Proof: Follows from 2.19(iv),2.10 ,and 2.22.

2.24 Remark. The converse of 2.23 is not true ,for example ,if $X=\{a,b,c\}$, $\tau=\{\varphi,X,\{a\},\{a,b\}\}$and $t:P(X)\rightarrow \tau$,defined by $t(A)= (intA)\text{-}\{b\}$,if $A\neq X$, and $t(X)=X$. Then $t\text{-}O(X) =\{\varphi,X,\{a\}\}$ and $t\text{-}SO(X)= \{\varphi,X,\{a\},\{a,b\},\{a,c\}\}$.

2.25 Remark. The concepts of semi open and t-semi open sets are independent. See the following examples.

2.26 Examples.

i) Let $X=\{a,b,c\}$**,** $\tau=\{\varphi,X,\{a\},\{c\},\{a,b\},\{a,c\}\}$**,** and define $t: P(X)\rightarrow \tau$ by $t(A)= (intA)\text{-}\{b\}$,if $A\neq X$, and $t(X)=X$. Then $t\text{-}O(X) =\{\varphi,X,\{a\},\{c\},\{a,c\}\}$. The set $\{b,c\}$ is a t-semi open set but it is not a semi open set in (X,τ).

ii) Let $X=\{a,b,c\}$**,** $\tau=\{\varphi,X,\{a\}\}$, and define $t: P(X)\rightarrow \tau$ by $t(A)=\varphi$ if $A\neq X$, and $t(X)=X$. Then $t\text{-}O(X) =\{\varphi,X\}$, and the set $\{a\}$is a semi open set ,but it is not t-semi open set.

2.27 Remark. Let (X,τ) be a topological space, t an operation on (X,τ).

i) φ and X are t-semi open sets.

ii) The union of any family of t-semi open subsets of X is a t-semi open set. The intersection of two t-semi open subsets of X need not be a t-semi open set.

Proof: i) Is obvious.

ii) Follows from 2.22(i),2.12(ii) and 2.19(ii).

iii) In Example 2.6 A and B are t-semi open sets ,but $A\cap B=\{c\}$ is not a t-semi open set.

2.28 Remark. By Remark 2.23 and Definition 2.21, any t-closed set is a t-semi closed set. The converse is not true.

2.29 Remark. By Definition 2.21and Remark 2.27 we have:

i) φ and X are t-semi closed sets.

ii) The intersection of any family of t-semi closed subsets of X is a t-semi closed set.

iii) The union of two t-semi closed subsets of X need not be a t-semi closed set.

2.30 Definition. Let t be an operation on a topological space $(X,\tau),A\subseteq X$**.** Then the set ,scl_tA is defined by

$scl_tA=\cap\{F|F \text{ is a t-semi closed set and } A\subseteq F \}$.

2.31 Remark. By Definition 2.30 and Remark 2.29(ii), scl_tA is the smallest t-semi closed set containing A, and $scl_t(scl_tA)= scl_tA$. Also A is a t-semi closed set if and only if $A= scl_tA$.

2.32 Remark. By Remarks 2.23,2.28 and Definitions 2.21,2.30, $A\subseteq scl_tA \subseteq cl_tA$.

2.33 Definition. Let t be an operation on a topological space $(X,\tau),A\subseteq X$**.** A is said to be t-semi generalized closed set (shortly t-sg.closed set) if $scl_tA \subseteq U$, whenever $A\subseteq U$ and U is a t-semi open set in (X,τ).

2.34 Remark. By Remark 2.31 every t-semi closed set is a t-sg.closed set.

2.35 Remark. The converse of 2.34 is not true. In example 2.26 (ii), any subset other than φ and X is a t-sg.closed set but not t-semi closed.

2.36 Remark. The concepts of t-g.closed and t-semi closed sets are independent.

2.37 Examples. i) Let X= {a,b,c}, τ={φ,X}, and t an operation on (X,τ) defined by t(A)=intA. Then any subset of X other than φ and X is g.closed but not t-semi closed set.

ii) Let X = {a,b,c,d}, τ={φ,X,{a},{b,c},{a,b,c}}, and t an operation on (X,τ) defined by t(A)=intA. Then {a} is a t-semi closed set but not t-g.closed.

2.38 Theorem. Let t be an operation on a topological space (X,τ). Then For each $x\in X$, {x} is a t-semi closed set or X-{x} is a t-sg.closed set in (X,τ).

Proof: Similar to the proof of 2.16.

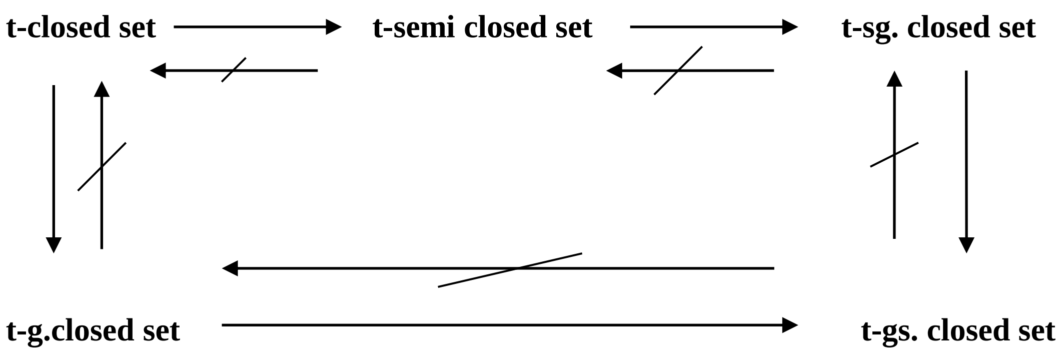
2.39 Definition. Let t be an operation on a topological space (X,τ),$A\subseteq X$. A is said to be t- generalized semi closed set (shortly t-gs.closed set) if $scl_tA \subseteq U$, whenever $A\subseteq U$ and U is a t- open set in (X,τ).

2.40 Remark. By Remark 2.23 and Definitions 2.33,2.37, every t-sg.closed set is a t-gs.closed set.

2.41 Remark. The converse of 2.40 is not true. Let X={a,b,c}, τ={φ,X,{a}}, and t an operation on (X,τ) ,defined by t(A)=intA, then the set {a,b} is a t-gs.closed set but not t-sg.closed.

2.42 Remark. By Definitions 2.13 ,2.39 and Remark 2.32 every t-g.closed set is a t- gs.closed set. The converse is not true, in Example 2.37 (ii) ,{a} is a t- gs.closed set but not t-g.closed set.

We close this section by the following diagram which explain the relations between the concepts that we defined in the section:



Where P ⟶ Q represents P imply Q , P ⟶̸ Q represents P does not imply Q.

3 Separation axioms and t-operation

3.1 Definition. Let t be an operation on a topological space (X,τ).Then (X,τ) is said to be:

i) a t-T_0 space if for each distinct points x and y in X there exists a t-open set U containing one of them but not containing the other.

ii) a t-T_1 space if for each distinct points x and y in X there are two t-open sets U and V containing x and y respectively such that $y\notin U$ and $x\notin V$.

iii) a t-T_2 space if for each distinct points x and y in X there are two disjoint t-open sets U and V containing x and y respectively.

iv) a t-$T_{½}$ space if every t-g.closed subset of (X,τ) is a t-closed set.

v) a t-semi T_0 space if for each distinct points x and y in X there exists a t-semi open set U containing one of them but not containing the other.

vi) a t-semi T_1 if for each distinct points x and y in X there are two t-semi open sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.

vii) a t-semi T_2 space if for each distinct points x and y in X there are two disjoint t-semi open sets U and V containing x and y respectively.

viii) a t-semi $T_{½}$ space if every t-semi g.closed subset of (X,τ) is a t- semi closed set.

ix) a t-T_b space (t- T_d space) if every t-gs.closed set is t-closed (t-g.closed).

In the remaining of this part we always assume that (X,τ) is a topological space and t is an operation on (X,τ) .

3.2 Remark. If (X,τ) is a t- T_i space ,then (X,τ) is a T_i and t-semi T_i space(for i=0,1,2). The converse is not true.

Proof: It follows from the facts that t-O(X) $\subseteq \tau$ (Remark2.3) and t-O(X) $\subseteq$ t-SO(X) (Remark2.23).

For the converse see the following examples.

3.3 Example. i) Let $X=\{a,b,c\}$,τ=P(X) and t:P(X)$\to\tau$ be defined by, t(A)=φ if $A\neq X$ and t(X)=X. Then t-O(X)=$\{\varphi,X\}$ and so (X,τ) is a T_i space but not t-T_i space for i=0,1,2.

ii) In the example of Remark 2.24 (X,τ) is a t-semi T_0 but not t-T_0 space.

iii) Let $X=\{a,b,c\}$, $\tau=\{\varphi,X,\{a\},\{b\},\{a,b\},\{a,c\}\}$ and t:P(X)$\to\tau$ is defined by t(A)=int(A) for each A, then t-O(X)= τ and t-SO(X)= $\tau\cup\{\{b,c\}\}$. Hence (X,τ) is a t-semi T_i space but not t-T_i space for i=1and i=2.

3.4 Remark. If (X,τ) is a t-T_i space (t-semi T_i space) then (X,τ) is a t-T_{i-1} space (t-semi T_{i-1} space) for i=1 and 2.The converse is not true.

Proof: Follows from Definition 3.1.For the converse see the following examples.

3.5 Example. i) Let $X=\{a,b\}$,$\tau=\{\varphi,X,\{a\}\}$ and t:P(X)$\to\tau$ is defined by t(A)=int(A) for each A. Then t-O(X)= τ=t-SO(X) and (X,τ) is a t-T_0 space and t-semi T_0 space but neither t-T_1 space nor t-semi T_1 space.

ii) If X=N (the set of natural numbers), $\tau=\{U|X-U \text{ is finite}\}\cup\{\varphi\}$and t is defined by t(A)=int(A) for each A. Then t-O(X) = τ = t-SO(X) and (X,τ) is a t-T_1 space and t-semi T_1 space but neither t-T_2 space nor t-semi T_2 space.

3.6 Remark. (X,τ) is a t-T_0 space (t-semi T_0 space) if and only if for each x and y in X, $x\neq y$, $cl_t\{x\}\neq cl_t\{y\}$($scl_t\{x\}\neq scl_t\{y\}$).

Proof: If for some x ,y in X ,$x\neq y$, $cl_t\{x\}=cl_t\{y\}$ then $\{x\}\subseteq cl_t\{y\}$which implies that $x\in\cap\{F|F \text{ is t-closed and } y\in F\}$,that is $x\notin\cup\{U|U \text{ is t-open and } y\notin U\}$,that is $x\notin U$ for each t-open set U such that $y\notin U$. Similarly $y\notin V$ for each t-open set V such that $x\notin V$. Hence (X,τ) is not a t-T_0 space.

On the other hand if $cl_t\{x\}\neq cl_t\{y\}$ for each x and y in X, $x\neq y$, then either $x\notin cl_t\{y\}$ and so $x\in X- cl_t\{y\}$and $y\notin X- cl_t\{y\}$,where $X- cl_t\{y\}$ is a t-open set, or $y\notin cl_t\{x\}$ and so $y\in X- cl_t\{x\}$and $x\notin X- cl_t\{x\}$,where $X- cl_t\{x\}$ is a t-open set. that is, (X,τ) is a t-T_0 space.

The proof of the (semi) case is similar.

3.7 Lemma. A subset A of (X,τ) is a t-g.closed (t-sg.closed) set if and only if $cl_t(\{x\})\cap A\neq\varphi$($scl_t\{x\}\cap A\neq\varphi$) , holds for every $x\in cl_tA$(scl_tA).

Proof : Assume that A is a t-g.closed set, and $x\in cl_tA$ such that $cl_t(\{x\})\cap A=\varphi$, then $A\subseteq X- cl_t(\{x\})$, but cl_tA is not a subset of $X- cl_t(\{x\})$.Since $X- cl_t(\{x\})$

is a t-open set , this contradicts the assumption that A is a t-g.closed set.
Now if A is not a t-g.closed set , then there exists a t-open set U such that $A\subseteq U$ and cl_tA is not a subset of U, therefore there is an element $x\in cl_tA$ and $x\notin U$, hence $cl_t(\{x\})\subseteq X-U$, i.e $cl_t(\{x\})\cap A=\varphi$.
The proof of the (semi) case is similar.

3.8 Lemma. If $cl_t(\{x\})\cap A\neq\varphi$ ($scl_t\{x\}\cap A\neq\varphi$), holds for every $x\in cl_tA(scl_tA)$, then $(cl_tA)-A$ ($(scl_tA)-A$)does not contain a non empty t-closed (t-semi closed) set.
Proof: Assume that $cl_t(\{x\})\cap A\neq\varphi$,for each $x\in cl_tA$**,** and assume that B is a non empty t-closed subset of $(cl_tA)-A$**.** Let $x\in B$, then $x\in cl_tA$ and $x\notin A$, also $cl_t(\{x\})\subseteq B$ (since B is a t-closed set containing {x}), therefore $x\in cl_tA$ and $cl_t(\{x\})\cap A=\varphi$, which contradicts the assumption
The proof of the (semi) case is similar.

3.9 Theorem. (X,τ) is a t-$T_{½}$ space (t-semi $T_{½}$ space) if and only if for each x in X, {x} is either t-closed (t-semi closed) or t-open (t-semi open) set.
Proof: Let (X,τ) be a t-$T_{½}$ space, $x\in X$. If {x}is not t-closed, then by Theorem 2.16 X-{x} is a t-g.closed set, and by Definition 3.1(iv) it is t-closed, that is {x} is a t-open set.
Conversely assume that for each x in X , {x} is either t-closed or t-open set .
Let A be a t-g.closed set in (X,τ)**,** $x\in cl_tA$ and $x\notin A$.
<u>Case1.</u> {x} is t-closed, in this case {x} is a non empty t-closed set contained in $(cl_tA)-A$ which contradicts Lemma 3.7 and 3.8(since A is a t-g.closed set).
<u>Case2.</u> {x} is t-open , $A\subseteq X-\{x\}$ and $cl_tA\subseteq X-\{x\}$(since X-{x} is a t-closed set), which contradicts the assumption that $x\in cl_tA$**.** Therefore $x \in cl_tA$ implies $x\in A$, that is A is t-closed set. Hence (X,τ) is a t-$T_{½}$ space.
The proof of the (semi) case is similar.

3.10 Corollary. If (X,τ) is a t-$T_{½}$ space then (X,τ) is a t-semi $T_{½}$ space. The converse is not true.
Proof: By Theorem 3.9, if (X,τ) is a t-$T_{½}$ space then for each x in X,{x} is either t-closed or t-open set, and by Remark 2.23,{x} is either t-semi closed or t-semi open set, and finally by 3.9 (X,τ) is a t-semi $T_{½}$ space.
For the converse see Example 2.24 in which (X,τ) is a t-semi $T_{½}$ space but not t- $T_{½}$ space.

3.11 Theorem. (X,τ) is a t-T_1 space (t-semi T_1 space) if and only if for each x in X, {x} is a t-closed (t-semi closed) set.
Proof: Let (X,τ) be a t-T_1 space and $x\in X$. Therefore for each $y\in X$, $y\neq x$, there exits a t-open set U_y such that $x\notin U_y$ and $y\in U_y$.
Let $V=\bigcup_{y\neq x}\left\{U_y \,\middle|\, U_y \textit{ is a t-open set } y\in U_y \textit{ and } x\notin U_y\right\}$, then V is a t-open set (Remark 2.5(ii)), and V=X-{x},hence {x} is a t-closed set.
On the other hand if {x} is a t-closed set for each x in X, then if $x\neq y$, then U= X-{y} and V=X-{x} are t-open sets containing x and y respectively with $y\notin U$ and $x\notin V$, i.e (X,τ) is a t-T_1 space.
The proof of the (semi) case is similar.

3.12 Theorem. If (X,τ) is a t-T_i space (t-semi T_i space) then (X,τ) is a t-$T_{i-½}$ space (t-semi $T_{i-½}$ space), for i=½ and 1.The converse is not true.
Proof: (i=½) Let (X,τ) be a t-$T_{½}$ space, and let $x,y\in X$, $x\neq y$. By Theorem 3.9 {x} is either t-open (and in this case $x\in\{x\}$,$y\notin\{x\}$) or t-closed (in this case X-{x}is t-open ,$x\notin X-\{x\}$,where $y\in X-\{x\}$). Therefore (X,τ) is a t-T_0 space .
(i=1) If (X,τ) is a t-T_1 space, then by Theorem 3.11 for each x in X, {x} is a

t-closed set and hence by Theorem 3.9 (X,τ) is a t-$T_{½}$ space.
The proof of the (semi) cases is similar.
For the converse see the following examples.

3.13 Example. In the following examples we assume that t: P(X)→ τ is defined by t(A)= int A, and so t-O(X) = τ.

i) Let X=N,the set of natural numbers, τ={U⊆N|1∈U and N-U is finite}∪{φ}, then t-SO(X)=t-O(X)=τ, and (X,τ) is a t-T_0 space and t-semi T_0 space but neither t-$T_{½}$ space nor t-semi $T_{½}$ space , since {1} is not t-open, not t-closed, not t-semi open and not t-semi closed set.

ii) Let X={a,b,c}, τ={φ,X,{a}, {b},{a,b}}, then t-O(X)=τ , and (X,τ) is a t-$T_{½}$ space, but not t-T_1 space, since {a}and {b} are not t-closed sets.

iii) Let X={a,b,c}, τ={φ,X,{a}}, then t-SO(X)={φ, X, {a}, {a,b}, {a,c}}, hence (X,τ) is a t-semi $T_{½}$ space but not t-semi T_1 space since {a}is not t-semi closed set.

3.14 Theorem. (X,τ) is a t-T_b space if and only if (X,τ) is a t-$T_{½}$ and t-T_d space.

Proof: Let (X,τ) be a t-T_b space ,if A is a t-g.closed set in (X,τ), then by Remark 2.42, A is a t-gs.closed and by Definition 3.1(ix) it is a t-closed set, that is, (X,τ) is a t-$T_{½}$ space. A is a t-closed set implies A is at-g.closed set(Remark 2.14), that is, (X,τ) is a t-T_d space.

Conversely, if (X,τ) is a t-$T_{½}$ and t-T_d space, A is a t-gs.space, then A is a t-g.closed set(Definition 3.1(ix) since (X,τ) is a t-T_d space),hence it is a t-closed set (Definition 3.1(iv) since (X,τ) is a t-$T_{½}$space) . Therefore (X,τ) is a t-T_b space (Definition3.1(ix)).

3.15 Remark. **i)** If (X,τ) is a t-$T_{½}$, then (X,τ) need not be a t-T_b space.
ii) If (X,τ) is a t-T_d, then (X,τ) need not be a t-T_b space.
iii) The concepts of t-$T_{½}$ and t-T_d are independent.
iv) The concepts of t-T_1 and t-T_b are independent.

See the following examples.

3.16 Examples.

i) In Example 3.13(ii), (X,τ) is a t-$T_{½}$, but it is not t-T_b space since {a}(and {b}) is a t-gs.closed set but not t-closed set.

ii) In Example 3.13(i) , (X,τ) is not a t-$T_{½}$ space and so it is not a t-T_b space . The t-closed sets are all the finite subsets of X not containing 1,and X . Hence cl_tA=Aor X, for each A, also scl_tA=Aor X(by Remark2.32). The only t-gs.closed set which is not t-closed is X-{1}, and it is a t-g.closed set ,that is (X,τ) is a t-T_d space.

iii) Let X=R(the set of real numbers) , τ be the usual topology on R , the operation t is defined by t(A)=intA for each A then
(X,τ) is a t-T_1 space

3.17 Theorem. i) If (X,τ) is a t-T_b space then for each x in X, {x}is a t-semi closed or t-open set.

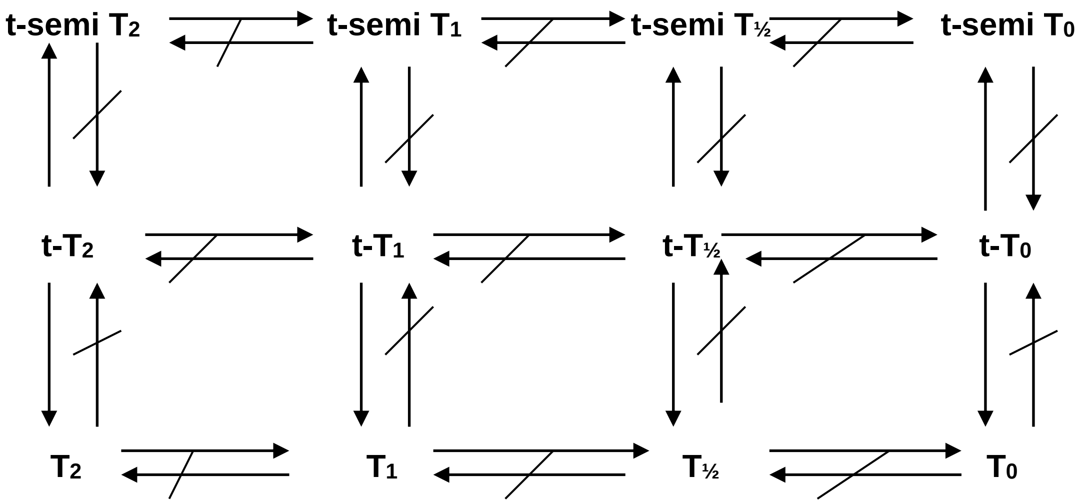
ii) If (X,τ) is a t-T_d space then for each x in X, {x}is a t- closed or t-g.open set.

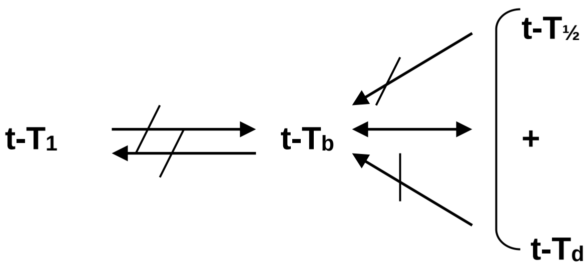
Proof: i) Let (X,τ) be a t-T_b space, x∈X. If {x}is not t-semi closed, then X-{x} is not t-semi open and so the only t-semi open set containing X-{x}is X and scl_t(X-{x})⊆X, that is, X-{x}is a gs.closed set and since (X,τ) is a t-T_b space, X-{x}is a t-closed set. Hence {x}is a t-open set.

ii) Let (X,τ) be a t-T_d space , x∈X. If {x}is not t- closed, then X-{x} is not t- open and so the only t- open set containing X-{x}is X and scl_t(X-{x})⊆X, that is, X-{x}is a gs.closed set. Since (X,τ) is a t-T_d space, X-{x}is a t-g.closed set, hence {x} is a t-g.open set.

3.18 Remark. The converse of theorem3.18 is not true. In Example 3.13(ii), (X,τ) is neither t-T_b space nor t-T_d space, on the other hand {a},{b}and {c}are semi closed sets , {c} is t-closed , {a}and {b} are t-g.open sets.

Finally, the following two diagrams summarize the conclusions of this section about the relationships between the deferent concepts appeared in Definition 3.1.





Where P ⟶ Q represents P imply Q , P ↛ Q represents P does not imply Q.

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