CONTINUITY IN BITOPOLOGICAL SPACES

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Abstract

In this paper we discuss the topological property of the separation axioms in the special bitopological spaces (X,T,T^{α}) after the definition of continuous mapping and some theorems of its in this space.

Introduction

After we define the special Bitopological space (X,T,T^{α}) where T^{α} is the α-topological space and the open set in the space (X,T,T^{α}) define by using the definition a subset A is open set in (X,T,T^{α}) if and only if there exist T^{α}-open set U such that $A\subseteq U$, $A\subseteq cl_T(U)\cap int_T(U)$ see [Mahdi,2006] ,we will define the continuity and homeomorphism in this space and also study the topological property of the separation axioms see[Mahdi,2006], graph and closed graph mapping study in this paper see [Al-Swaidi,1990].

1-**The definition and auxilary results**

<u>1-1.Definition :-</u> Let (X,T,T^{α}) and (Y,ξ,ξ^{α}) are two bitopological spaces a mapping $f:(X,T,T^{\alpha})\to(Y,\xi,\xi^{\alpha})$ is λ-continuous at a point x_0 in X if and only if to every λ_Y-nbd. M of $f(x_0)$ there exist λ_X–nbd N of x_0 such that $f(N)\subseteq M$. also f is say to be λ-continuous if and only if its continuous at each point of X see [Sharma,1977].

<u>1-2.Example:-</u>

Let $X=\{a,b,c\}$,$T=\{X,\phi,\{a\}\}$
$Y=\{1,2,3\}$,$\xi=\{Y,\phi,\{2,1\}\}$
Let $f:(X,T,T^{\alpha})\to(Y,\xi,\xi^{\alpha})$ be a mapping defined by $f(x)=1$ for each $x\in X$ then f is λ-continuous.

<u>1-3.Theorem:-</u> Let $(X,T,T^{\alpha})$and (Y,ξ,ξ^{α}) be two bitopological spaces a mapping $f:(X,T,T^{\alpha})\to(Y,\xi,\xi^{\alpha})$ is say to be λ-continues if and only if $f^{-1}(U)$ is λ-open in X for each U is ξ-open in Y.

<u>1-4.Theorem:-</u> Let (X,T,T^{α}) and (Y,ξ,ξ^{α}) be bitopological space.A mapping $f:X\to Y$ is λ-continues if and only if the inverse image of every ξ-closed in Y is λ-closed in X.

<u>1-5.lemma:-</u>Every mapping from (X,T)into $(X ,T^{\alpha})$is continues.

Proof:- From the fact that $T\subset T^{\alpha}$, clearly that this mapping is continues.

<u>1-6.Theorem:-</u> If $f:(X,T)\to(Y,\xi)$ is closed and continues function then f from a bitopological space X into another bitopological space Y is λ-continues if and only if $f(cl(A))\subset cl_{\xi}(f(A))$ for every $A\subset X$.

<u>proof:</u> Let f be λ-continues and A be a subset of X , since f is closed f(A) is ξ-closed in Y , $cl_{\xi}(f(A))$ is ξ-closed , then $f^{-1}(cl_{\xi}(f(A))$ is λ-closed in X,
$cl(f^{-1}(cl_{\xi}(f(A)))=f^{-1}(cl_{\xi}(f(A))$, since $f(A)\subset cl_{\xi}(f(A))$ then
$A\subset f^{-1}(cl_{\xi}(f(A))$, then $cl(A)\subset cl(f^{-1}(cl_{\xi}(f(A))=f^{-1}(cl_{\xi}(f(A))$.
Then $f(cl(A))\subset cl_{\xi}(f(A))$.

Conversely, let $f(cl(A))\subset cl(f(A))$ and B is ξ-closed in Y , $cl(B)=B$, $f^{-1}(B)\subset X$ $f(cl(f^{-1}(B))\subset cl(f(f^{-1}(B)))\subset cl(B)=B$, then $cl(f^{-1}(B))\subset f^{-1}(B)$ and since $f^{-1}(B)\subset cl(f^{-1}(B))$
we get $cl(f^{-1}(B))=f^{-1}(B)$, from that we get $f^{-1}(B)$ is T-closed and then λ-closed .

1-7.Remark :- Let X,Y,Z are three bitopological spaces and $f:X\to Y$, $g:Y\to Z$ be λ-continuous mapping then gof is not necessary λ-continuous.

1-8.Theorem :- Let $(X,T,T^{\alpha}),(Y,\xi,\xi^{\alpha})$ be bitopological spaces, if $f:(X,T)\to(Y,\xi)$ is continuous then the mapping $f:(X,T,T^{\alpha})\to(Y,\xi,\xi^{\alpha})$ is λ-continuous if and only if $f^{-1}(int_{\xi}(B))\subset int(f^{-1}(B))$, for every B ξ-open in Y.

proof:- Let f be λ-continuous and B be ξ-open set then $f^{-1}(int_{\xi}(B))$ is λ-open set in X , $int(f^{-1}(int_{\xi}(B))=f^{-1}(int_{\xi}(B))$, since $int_{\xi}(B)\subset B$ then $f^{-}int_{\xi}(B))\subset f^{-1}(B)$ and $int(f^{-1}(int_{\xi}(B))\subset int(f^{-1}(B))$ from that, we get $f^{-1}(int_{\xi}(B))\subset int(f^{-1}(B))$.

Conversely, let A is ξ-open set in Y then $int_{\xi}(A)=A$, by hypothesis $f^{-1}(int_{\xi}(A))\subset int(f^{-1}(A))$, $f^{-1}(A)\subset int(f^{-1}(A))$, since $int(f^{-1}(A))\subset f^{-1}(A)$ we get $int(f^{-1}(A))=f^{-1}(A)$ and then $f^{-1}(A)$ is T-open , since every T-open is λ-open.
Then f is λ-continues.

1-9.Lemma:- Let (X,T),(Y,ξ) be a topological spaces such that $f:X\to Y$ is continuous then $f:(X,T,T^{\alpha})\to(Y,\xi,\xi^{\alpha})$ is λ-continuous .

Proof:- let U is ξ-open set in Y ,since f is continuous $f^{-1}(U)$ is T-open set in X , since every T-open set is λ-open set ,$f^{-1}(U)$ is λ-open set in X , therefore f is λ-continuous

1-10.Theorem:- If $f:(X,T)\to(Y,\xi)$ is continuous mapping. A mapping $f:(X,T,T^{\alpha})\to(Y,\xi,\xi^{\alpha})$ is λ-continues if and only if $cl(f^{-1}(B))\subset f^{-1}(cl_{\xi}(B))$, for each ξ-closed set B.

Proof:- let f be λ-continuous, B be ξ-closed set in Y then $cl_{\xi}(B)$ is ξ-closed set , since f is λ-continuous $f^{-1}(cl_{\xi}(B)$ is λ-closed set in X,
$cl(f^{-1}(cl_{\xi}(B))= f^{-1}(cl_{\xi}(B))$, now since $B\subset cl_{\xi}(B)$ then $f^{-1}(B)\subset f^{-1}(cl_{\xi}(B))$ then $cl(f^{-1}(B))\subset cl(f^{-1}(cl_{\xi}(B)))=f^{-1}(cl(B))$ .Then $cl(f^{-1}(B))\subset f^{-1}(cl(B))$.

Conversely, let A be ξ-closed set in Y , $cl_{\xi}(A)=A$, by assume

$cl(f^{-1}(A))\subset f^{-1}(cl_{\xi}(A))=f^{-1}(A)$,then

$cl(f^{-1}(A))\subset f^{-1}(A)$ (1)

since $f^{-1}(A)\subset cl(f^{-1}(A)$ (2)

by (1)and (2)we get, $cl(f^{-1}(A))=f^{-1}(A)$, then $f^{-1}(A)$ is T-closed then λ-closed set in X.Therefor f is λ-continuous.

1-11.Definition:- Let $(X,T,T^{\alpha})$and $(Y,\xi\xi^{\alpha})$ be bitopological spaces and f be a mapping of X into Y, then

1- f is said to be λ- open mapping if and only if f(U) is λ_{ξ}-open whenever U is λ_T-open.
2- f is said to be λ-closed mapping if and only if f(H) is λ_{ξ}-closed whenever H is λ_T-closed.
3- f is said to be λ- bicontinuous if and only if f is λ-open and λ-continuous .
4- f is said to be λ-homeomorphism if and only if f is bijection ,λ-continuous and f^{-1} is λ-continuous.

1-12.Theorem:- Let (X,T,T^{α}) , (Y,ξ,ξ^{α}) be bitopological spaces , then

1- a mapping $f:X\to Y$ is λ-open if and only if $f(int(A))\subset int(f(A))$, for every $A\subset X$.
2- a mapping $f:X\to Y$ is λ-closed if and only if $cl(f(A))\subset f(cl(A))$, for every $A\subset X$.

Proof:- see [Sharma,1977] with replacing open set by λ-open set

1-13.Example:- Let $X=\{a,b,c\}$,$T=\{X,\phi,\{a\}\}$

$Y=\{1,2,3\}$, $\xi=\{y,\phi,\{1\},\{2,3\}$

$T^{\alpha}=\{X,\phi,\{a\},\{b,c\}\}$, $\xi^{\alpha}=\{Y,(,\{1\},\{2,3\}\}$

Let f:X→Y be a mapping defined by f(a)=1, f(b)=2=f(c) then f is λ-open mapping . When we define f as follow f(a)=2,f(b)=1,f(c)=3, we get f is λ-close mapping

1-14.Theorem:- The mapping f: $(X,T,T^{\alpha}) \rightarrow (X,T)$ is λ-continuous.

Proof:- Since the λ-open set is finer than T, f is λ-continuous.

1-15.Theorem:-If:$(X,T)\rightarrow(X,T^{\alpha})$ is continuous then f:$(X,T,T^{\alpha})\rightarrow(X,T^{\alpha})$ is λ-continuous.

Proof:- let U is T^{α}-open ,since f is continues then $f^{-1}(U)$ is T-open and since every T-open is λ-open .Then f is λ-continuous.

1-16.Theorem:- The mapping f: $(X,T^{\alpha})\rightarrow(X,T,T^{\alpha})$ is continuous.

Proof:- Let U is λ-open set, since f from (X,T) into (X,T,T^{α}) is continuous we have $f^{-1}(U)$ is T-open and since every T-open is T^{α}-open .Then f is continuous.

1-17.Theorem:- If f:$(X,T,T^{\alpha})\rightarrow(X,T)$ is λ-continuous and g:$(X,T)\rightarrow(X,T^{\alpha})$ is continuous then gof :$(X,T,T^{\alpha})\rightarrow(X,T^{\alpha})$ is λ-continuous.

Proof:- Let U is T^{α}-open , $(f^{-1}og^{-1})(U)=f^{-1}(g^{-1}(U))=f^{-1}(W)$ since g is continuous and since f is λ-continuous then $f^{-1}(W)$ is λ-open and $f^{-1}og^{-1}=(gof)^{-1}$, we get gof is λ-continuous.

1-18.Theorem:- If f is a continuous mapping from (X,T) into $(X\xi,\xi^{\alpha})$, then f:$(X,T,T^{\alpha})\rightarrow(X,\xi,\xi^{\alpha})$ is λ-continuous.

Proof:- Let U is λ_{ξ}-open ,$f^{-1}(U)$ is T-open since f is continuous, and every T-open is λ_T-open .Then f is λ-continues.

2- topological property:-

2-1.Definition:- The bitopological space (X,T,T^{α}) is say to be λ- T_0-space if and only if for each two points x,y in X there exist λ -open set U such that $x\in U, y\notin U$.

2-2.Definition- The bitopological space (X,T,T^{α}) is say to be λ -T_1-space if and only if for each two points x,y in X there exist two λ- open sets U,W such that $x\in U$,$y\notin U$ and $x\notin W$, $y\in W$, $U\cap W=\emptyset$

2-3.Definition- The bitopological space (X,T,T^{α}) is say to be λ -T_2- space if and only if for each two points x,y in X there exist two λ -open sets U,W such that $x\in U$,$y \in W$, $U \cap W= \emptyset$

2-4.Definition- The bitopological space (X,T,T^{α}) is say to be λ- regular space if and only if for every T^{α}-closed set A and $x\notin A$ there exist λ-open set U,W such that $A\subset U$, $x\in W$, $U\cap W= \emptyset$

2-5.Definition- The bitopological space (X,T,T^{α}) is say to be λ- normal space if and only if for each T^{α} -closed sets U,W , $U\cap W= \emptyset$ there exist two λ-open set M,N such that $U\subset M$, $W\subset N$ and $M\cap N= \emptyset$

2-6.Theorem:- λ-T_0,λ-T_1 and λ-T_2 are topological property.

Proof:- The proof is clear from the definition of λ-T_0,λ-T_1 and λ-T_2 space .

2-7.Theorem:-if f:$(X,T)\rightarrow(Y,\xi^{\alpha})$is onto and continuous mapping, then λ-regular is a topological property with respect to the same mapping f.

Proof: Let f:$(X,T,T^{\alpha})\rightarrow(Y,\xi,\xi^{\alpha})$ is λ- homeomorphism.

Let A be a ξ^{α}-closed set and $y\in Y$, $y\in A$,since f is onto then there exist $x\in X$ there exists f(x)=y, $x=f^{-1}(y)$, since f is continues then $f^{-1}(A)$ is T-closed and since every T-closed is λ-closed then $f^{-1}(A)$ is λ-closed ,$f^{-1}(y) \in f^{-1}(A)$.

Now, since (X,T,T^{α}) is λ-regular space there exist two λ-open sets U,V such that $f^{-1}(y)\in U$, $f^{-1}(A) \subset V$ and $U \cap V=\phi$ from that, we get $y\in f(U)$ and $A\subset f(V)$, since f is λ-open f(U), f(V) are λ-open in Y , f(U)∩ f(V)=f(U∩ V)=ϕ.Then (Y,ξ,ξ^{α}) is λ-regular.

2-8.Theorem:- If $f:(X,T)\to(Y,\xi^{\alpha})$ is continuous and onto mapping then λ-T_3 is a topological property.

Proof:- Since λ-T_1 space is a topological property and by theorem(2-7) λ-T_3 space is a topological property.

2-9.Theorem :- If $f:(X,T)\to(Y,\xi^{\alpha})$ is onto and continues function then λ-normal is a topological property with respect to the same function f.

Proof:- Let A,B be two ξ^{α}-closed sets such that A∩B=ϕ ,since f is continuous then $f^{-1}(A),f^{-1}(B)$ are T-closed sets and since every T- closed is λ-closed then $f^{-1}(A),f^{-1}(B)$ are λ-closed such that $f^{-1}(A)\cap f^{-1}(B)=\phi$,now since X is λ-normal then there exist two λ-open sets U,W ⊂X such that $f^{-1}(A)\subset U$, $f^{-1}(B)\subset W$, U∩W=ϕ, from that we get A⊂f(U), B⊂f(W). Since f is λ-open mapping, f(U),f(W) are λ-open in Y also f(U)∩f(W)=ϕ .Then (Y,ξ,ξ^{α}) is λ-normal.

2-10.Theorem:- If $f:(X,T)\to(Y,\xi^{\alpha})$ is onto continuous mapping then λ-T_4 space is a topological property.

Proof:- Since λ-T_1 is a topological property and by theorem (2-9) λ-T_4 space is a topological property.

3- Graph function:-

3-1.Definition:- Let $f:(X,T,T)\to(Y,\xi,\xi)$ be a mapping , the mapping $g:X\to X\times Y$ defined by $g(x)=(x,f(x))$, for each x in X is called the graph of f and denoted by $G(f)=\{(x,f(x)),x\in X\}$

3-2.Definition :- The graph G(f) is called λ-closed graph if and only if $x\in X$, $y\in Y$ such that $y\neq f(x)$ there exist λ-open set U containing x in X and λ-open set V containing y in Y such that $(U\times cl(V))\cap G(f)=\phi$.

3-3.Example:- Let X={a,b,c}, T={X,ϕ,{a}}, T^{α}={X,ϕ,{a},{a,b},{a,c}}

Y={1,2,3}, ξ={Y,ϕ,{1},{2,3}, ξ^{α}={Y,ϕ, {1},{2,3}} then

λ_X-open={X,ϕ,{a}{b,c}},

λ_Y-open={Y,ϕ,{1}{2}{3}{1,2}{1,3}{2,3}}

Let $f:(X,T,T)\to(Y,\xi,\xi)$ be a mapping defined by f(a)=1,f(b)=f(c)=2, then G(f)={(a,1),(b,2),(c,2)}} is the graph of f and this graph is λ-closed.

3-4.Theorem:-Let $f(X,T)\to(Y,\xi)$ has a closed graph G(f) , if $f:(X,T,T^{\alpha})\to(Y,\xi,\xi^{\alpha})$ has a graph G(f) then G(f) is λ-closed graph .

Proof:- let $x\in X$, $y\in Y$ such that $y\neq f(x)$, since G(f) is closed graph there exist T-open set U, ξ-open set W such that $x\in U, y\in W$, $(U\times cl(W))\cap G(f)=\phi$, since every T-open set and ξ-open set is λ-open in X, Y respectively U,W are λ-open sets.Then G(f) is λ-closed graph.

References

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ملخص البحث:-

في هذا البحث ناقشنا بعض الصفات التبولوجية لبديهيات الفصل وذلك بعد مناقشة الاستمرارية في الفضاءات ثنائية التبولوجي (X,T,T^{α}) مع دراسة بعض نظريات الاستمرارية في هذا الفضاء.