Bifurcation of extremals in the analysis of bifurcation Solutions of Duffing equation

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الخلاصة

قمنا في هذا البحث بدراسة تفرع نقاط النهايات للدوال الملساء , حيث تم دراسة تفرع نقاط النهايات للدوال التي تمتلك عند نقطة الأصل بعد لحلقة القسمة يساوي ثلاث و كذلك للدوال التي تمتلك عند نقطة الأصل بعد لحلقة القسمة يساوي خمسة عشر. بالنسبة للدوال التي يكون عندها بعد حلقة القسمة يساوي ثلاث حددنا شروط وجود النقاط الحرجة لهذه الدوال بالإضافة إلى ذلك قمنا بإيجاد مخطط التفرع. أما بالنسبة للدوال التي يكون بعد حلقة القسمة يساوي خمسة عشر قمنا بدراسة حلول التفرع لهذه الدوال بالاعتماد على نظرية المتفردات الحدودية للدوال الملساء و قمنا كذلك بإيجاد وصف هندسي للكاوستيك مع إعطاء بعض التطبيقات لمعادلة دوفينك

Abstract. In the present paper we are interested in the study of bifurcation of extremals of smooth functions. We considered the functions that have codimensions three (hyperbolic umbilic) and fifteen at the origin. For the function of codimension three we gave the conditions of the existence of the critical points. Also, we found the bif-spreading of the critical points. For the function of codimension fifteen we used the boundary singularities of smooth maps to study the bifurcation analysis of this function, also we found geometrical description of the caustic.

1. Introduction.

It is known that many of the nonlinear problems that appear in Mathematics and Physics can be written in the form of operator equation,

$F(x,\lambda) = b,\ x \in O,\ b \in Y,\ \lambda \in \mathbb{R}^n$. (1.1)

The method used for these studies is the Lyapunov-Schmidt method [8]. Singularities of smooth maps play an where X, Y are real Banach spaces and O is open subset of X. For these problems, the method of reduction to finite dimensional equation,

$$\Theta(\xi,\lambda) = \beta, \xi \in \widehat{M}, \beta \in \widehat{N}. \quad (1.2)$$

can be used, where $\widehat{M}$ and $\widehat{N}$ are smooth finite dimensional manifolds. important role in the study of bifurcation solutions of BVPs. There are many studies of different types of smooth functions in one variable and more than one variables. Good review of these studies one can find in [9]. Bifurcation of extremely in the classical case (without boundaries) was studied in [11], [14] and [15]. The theory of Fredholm functionals on Banach manifolds and its applications have been studied recently in [1] , [4], [5], [7] and [12]. In the early years the study of singularities of smooth maps and its applications to the BVPs took an important character in the works of Sapronov and his group. For example, [6] used the Lyapunov-Schmidt reduction to find bifurcation solutions of the BVP,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3 = 0,$$

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0,$$

$$u = u(x),\ x \in [0,\pi]$$

by considering the following functional energy,

$$V(u,\lambda) = \int_0^\pi \left(\frac{(u'')^2}{2} - \alpha\frac{(u')^2}{2} + \beta\frac{u^2}{2} + \frac{u^4}{4}\right)dx,$$

which is reduced to the study of the following key function,

$$\widetilde{W}(\eta,\gamma) = \eta_1^4 + \eta_2^4 + 4\eta_1^2\eta_2^2 + \lambda_1\eta_1^2 + \lambda_2\eta_2^2 + o(|\eta|^4) + O(|\eta|^4)o(\delta).$$

In [1], the following problem has been studied,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^2 = 0,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

$$u(x_1) \ge 0,\ u(x_2) \ge 0,\ x_1, x_2 \in [0,1]$$

by considering the following functional energy,

$$V(u,\lambda) = \int_0^1 \left(\frac{(u'')^2}{2} - \alpha\frac{(u')^2}{2} + \beta\frac{u^2}{2} + \frac{u^3}{3}\right)dx,$$

which is reduced to the study of the following key function with boundaries,

$$W(\xi,\gamma) = \frac{\xi_1^3}{3} + \xi_1\xi_2^2 + \delta\xi_2^2 + \beta\xi_1 + o(|\xi|^3) + O(|\xi|^3)O(\gamma),$$

$$\xi_1 - a\xi_2 \ge 0, \qquad \xi_1 + b\xi_2 \ge 0.$$

In [2], the following problem has been studied,

$$\frac{d^4u}{dx^4} + \alpha\frac{d^2u}{dx^2} + \beta u + u^3 = \psi,$$

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0,$$

$$u = u(x), \qquad x \in [0,\pi]$$

by considering the following functional energy,

$$V(u,\lambda,\psi) = \int_0^\pi \left(\frac{(u'')^2}{2} - \alpha\frac{(u')^2}{2} + \beta\frac{u^2}{2} + \frac{u^4}{4} - \psi\, u\right) dx.$$

The problem reduced to study the bifurcation of extremals of the following key function,

$$\widetilde{W}(\eta,\gamma) = \eta_1^4 + \eta_2^4 + 4\eta_1^2\eta_2^2 + \lambda_1\eta_1^2 + \lambda_2\eta_2^2 + q_1\eta_1 + q_2\eta_2 + o(|\eta|^4) + O(|\eta|^4)o(\delta).$$

In [17], the following problem has been studied,

$$\delta\frac{d^4z}{dx^4} + \alpha\frac{d^2z}{dx^2} + \beta z + z^2 + z^3 = \psi,$$

$$z(0) = z(\pi) = z''(0) = z''(\pi) = 0,$$

$$u = u(x), \; x \in [0,\pi]$$

by considering the following functional energy,

$$V(z,\lambda,\psi) = \int_0^\pi \left(\delta\frac{(z'')^2}{2} - \alpha\frac{(z')^2}{2} + \beta\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} - z\,\psi\right) dx.$$

The problem reduced to study the bifurcation of extremals of the following key function,

$$\overline{W}(\eta,\gamma) = \eta_1^4 + \eta_2^4 + \eta_3^4 + 4\left(\eta_1^2\eta_2^2 + \eta_1^2\eta_3^2 + \eta_3^2\eta_2^2 + \eta_3\eta_1\eta_2^2 - \tfrac{1}{3}\eta_1^3\eta_3\right) + \frac{24\,r_1}{35}\eta_3^2\eta_1 - \frac{8\,r_1}{45}\eta_1^2\eta_3 + \frac{32\,r_1}{45}\eta_2^2\eta_1 + \frac{8\,r_1}{27}\eta_1^3 + \frac{8\,r_1}{81}\eta_3^3 + \frac{32\,r_1}{63}\eta_2^2\eta_3 + k_1\eta_1^2 + k_2\eta_2^3 + k_3\eta_3^2 - q_1\eta_1 - q_3\eta_3 + o(|\eta|^4) + O(|\eta|^4)O(\gamma).$$

In this paper we studied the bifurcation solutions of Duffing equation of type [10],

$$\alpha\,\ddot{u} + \lambda\,u + \frac{\partial U}{\partial u} = \psi,$$

$$u(0) = (1) = 0,$$
$$u = u(x),\ x \square [0,1]$$

by considering the following functional energy,

$$V(u,\lambda) = \int_0^1 \left(\alpha \frac{(\dot{u})^2}{2} + \lambda \frac{u^2}{2} + U(u) \right) dt,$$

where ψ is symmetric function with respect to the involution $I: \psi(t) \mapsto \psi(1-t)$ and $U(u)$ is the following potentials,

$$1)\ U(u) = \frac{u^3}{3},$$
$$2)\ U(u) = \frac{u^5}{5} + \frac{u^3}{3}.$$

2. Basic concepts of singularity theory.

Denote by ϵ_n the set of all germs at the origin of smooth functions on $\mathbb{R}^n$. The ring ϵ_n has an important ideal m_n consisting of all smooth function germs vanishing at the origin : $m_n = \{f \in \epsilon_n: f(0) = 0\}$. In fact this is a maximal ideal , and indeed the only maximal ideal in ϵ_n , which makes ϵ_n into a local ring as by definition a local ring is one with a unique maximal ideal .

Definition 2.1 [13] Two maps $f, g \square\ \mathbb{R}^n \to \mathbb{R}^p$ are said to be germ-equivalent at $p \square\ \mathbb{R}^n$ if p is in the domain of both and there is a neighborhood U of p such that the restrictions coincide: $f|_U = g|_U$.

Definition 2.2 [13] Two map germs $f, g \square (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are *C-equivalent*, or *contact equivalent*, if there exist,

(i) a diffeomorphism φ of the source $(\mathbb{R}^n, 0)$,

(ii) a matrix $M \in GL_p(\epsilon_n)$ such that

$$f \circ \varphi(x) = M(x)g(x),$$

where $f(x)$ *and* $g(x)$ are written as column vectors, $M(x)g(x)$ is the usual product of matrix times vector, and $GL_p(\epsilon_n)$ is the set of invertible $p \times p$ matrices whose entries are in ϵ_n.

Morse Lemma 2.1 [13] Let $p \square\ \mathbb{R}^n$ be a non-degenerate critical point of f of index k. Then there is a change of coordinates $x = \varphi(y)$ $(\varphi: \mathbb{R}^n \to \mathbb{R}^n)$ near p such that in these new coordinates y_i one has,

$$f(\phi(y)) = f(p) - y_1^2 - y_2^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2.$$

Implicit function theorem 2.1 [13] Let the left hand side of the equation

$$f(x,\lambda) = 0 \qquad (2.1)$$

be given by a smooth map $f\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ such that

(1) $f(0{,}0) = 0$;

(2) $\det\left(\frac{\partial f}{\partial x}(0{,}0)\right) \neq 0$,

then there exists a neighborhood $U=E \times \Lambda$ of the point (0,0) in $\mathbb{R}^n \times \mathbb{R}^p$ such that the set of solutions of equation (2.1) in U coincides with the graph of a smooth map $\Psi\colon \Lambda \to \mathbb{R}^n$ such that $\Psi(0) = 0$. In other words, if (x,λ) is close to the solution (0,0), then $f(x,\lambda) = 0 \Leftrightarrow x = \Psi(\lambda)$.

Definition 2.3 [13] (Local algebra). The local algebra Q_f of the singularity of f at the origin is the quotient of the algebra of function-germs by the gradient ideal of f:

$$Q_f = \varepsilon_n / \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right).$$

The *multiplicity* $\mu(f)$ of the critical point is the dimension of its local algebra:

$$\mu(f) = \dim Q_f$$

A critical point is said to be isolated if $\mu(f) < \infty$.

Theorem 2.2 [13] The multiplicity of an isolated critical point is equal to the number of Morse critical points into which it decomposes under a generic deformation of the function.

3. Fredholm functional.

Definition 3.1[8] Let E, F be real linear Banach spaces. Let $A\colon E \to F$ be a linear continuous operator. Then A is a Fredholm operator if the spaces Ker(A) and Coker(A) $=F/Im(A)$ are finite dimensional. The number ind(A) = dim Ker(A) − dim Coker(A) is called the Fredholm index of the operator A.

It follows from this definition that the image of A is closed in F and any subspace that directly complements *Ker(A)* in E is isomorphically mapped to *Im*(A) by A. For any Fredholm operators $A_1\colon E_1 \to E_2$ and $A_2\colon E_2 \to E_3$, the following index summation formula is valid:

$$ind(A_2A_1) = ind(A_1) + ind(A_2).$$

Definition 3.2 [8] A smooth nonlinear map $f: U \to F$, where U is a domain of a Banach space E is called Fredholm map if the Frēchet derivative $Df(x)$ is Fredholm operator from E to F for any $x \in U$. The index of nonlinear Fredholm map f is the index of the linear operator $Df(x)$ that is ; ind(f) =ind($Df(x)$) .

Definition 3.3 [8] A smooth functional $V: U \subset E \to \mathbb{R}$ (U is a domain of a Banach space E) , is called Fredholm functional, if

(1) there exist a Banach space F and a Hilbert space H such that the embedding $E \subset F \subset H$ are continuous and E is dense in H ;

(2) there exist a Fredholm map $f: U \to F$ such that its index is equal to zero and

$$\langle f(x), h\rangle_H = \frac{\partial V}{\partial x}(x)h, x \in U, h \in E$$

($\langle . , . \rangle_H$ is inner product on Hilbert space H) .

Definition 3.4 [16] A smooth map $f: U \subset E \to F$ is said to be has variational property, if there exist a functional $V: U \subset E \to \mathbb{R}$ such that $f = grad_H V$ or equivalently,

$$\langle f(x), h\rangle_H = \frac{\partial V}{\partial x}(x)h, x \in U, h \in E .$$

Suppose that $f: E \to F$ is a smooth Fredholm map of index zero , E, F are real Banach spaces and

$$\langle f(x), h\rangle_H = \frac{\partial V}{\partial x}(x)h, x \in U, h \in E,$$

where V is a smooth functional on E . Also it is assumed that the embedding $E \subset F \subset H$ are continuous and E is dense in H, then by using the method of finite dimensional reduction (Local method of Lyapunov-Schmidt) the problem,

$$f(x, \lambda) \to extr, \quad x \in E, \quad \lambda \in \mathbb{R}^n.$$

can be reduced into an equivalent problem,

$$W(\tau, \lambda) \to extr, \quad \tau \in \mathbb{R}^n.$$

the function $W(\tau, \lambda)$ is called Key function.

If $N=\{e_1, \dots, e_n\}$ is a subspace of E, where $e_1, \dots, e_n$ are orthonormal set in E, then the key function $W(\tau, \lambda)$ can be defined in the form,

$$W(\tau, \lambda) = \inf_{x:\langle x, e_i\rangle = \tau_i \forall i} V(x, \lambda),$$

$$\tau = (x_1, \dots, x_n).$$

The function W has all the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, *etc*) .The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of key function W.

It is easy to check that ,

$$\theta(\tau,\lambda) = grad\, W(\tau,\lambda)$$

(*here* $grad\, W(\tau,\lambda) = \nabla W(\tau,\lambda)$) . And $\theta(\tau,\lambda) = 0$ is called bifurcation equation .

4. Boundary Singularities of Fredholm functional.

To investigate the behavior of a Fredholm functional in a neighborhood of a boundary singular point, one uses the reduction to an analogous problem

$$W(\xi) \rightarrow extr$$

where

$$\xi \in \mathbb{R}^n_+,$$

$$\mathbb{R}^n_+ = \{\, \xi = (\xi_1, \xi_2, \ldots, \xi_n)^T \in \mathbb{R}^n : \ \xi_2 \geq 0\}$$

and W is a smooth (key) function. If we impose certain natural assumptions, then the function W inherits analytic and topological properties of the functional V .

A point $a = (a_1, \ldots, a_n)^T$ from $\mathbb{R}^n_+ \backslash \partial\mathbb{R}^n_+$ is called a *critical point of a smooth function* W if grad$W(a)$ = 0. Let $a \in \partial\mathbb{R}^n_+$ (the boundary of the set $\mathbb{R}^n_+$); the point a is called *critical* if grad$W(a)$ is orthogonal to the boundary at this point, i.e., if we have

$$\frac{\partial W}{\partial \xi_1}(a) = \frac{\partial W}{\partial \xi_3}(a) = \cdots = \frac{\partial W}{\partial \xi_n}(a) = 0.$$

If $€_a(\mathbb{R}^n)$ is the ring of germs of smooth functions on $\mathbb{R}^n$ at the point a and

$$I = \left(\frac{\partial W}{\partial \xi_1}, \xi_2 \frac{\partial W}{\partial \xi_2}, \frac{\partial W}{\partial \xi_3}, \ldots, \frac{\partial W}{\partial \xi_n}\right)$$

is the boundary Jacobi ideal [3,18], then the value $dim\bar{Q}$ is called the *multiplicity* of the boundary critical point a (and is denoted by $\bar{\mu}$), where $\bar{Q}$ denotes the set

$$\bar{Q} = €_a(\mathbb{R}^n)/I.$$

The multiplicity $\bar{\mu}$ of a boundary singularity is equal to the sum of multiplicities $\mu + \mu_0$, where μ is the (usual) multiplicity of W on $\mathbb{R}^n$, while μ_0 is the (usual) multiplicity of the restriction $W|\partial\mathbb{R}^n_+$. In the sequel, we assume (without loss of generality) that $a = 0$.

Let $\tilde{\mathbb{R}}^n$ be a space with coordinates $\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n$ such that it is a double covering of the space $\mathbb{R}^n$ with coordinates $\xi_1, \xi_2, \ldots, \xi_n$ established by the map

$$\pi: \tilde{\mathbb{R}}^n \to \mathbb{R}^n, \quad \xi_1 = \tilde{\xi}_1, \quad \xi_2 = \tilde{\xi}_2^2, \ldots, \quad \xi_n = \tilde{\xi}_n.$$

Then the function $W(\xi)$, ξ □ $\mathbb{R}^n$ lifts to the covering space by the relation

$$\tilde{W}(\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n) = W(\tilde{\xi}_1, \tilde{\xi}_2^2, \ldots, \tilde{\xi}_n).$$

The natural involution

$$J(\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n) = (\tilde{\xi}_1, -\tilde{\xi}_2, \ldots, \tilde{\xi}_n)$$

can be defined on the space $\tilde{\mathbb{R}}^n$.

The function $\tilde{W}$ is invariant with respect to the involution J: $\tilde{W}(J(\xi)) = \tilde{W}(\xi)$. Thus, boundary singularities are identified (in a natural way) with the invariant (with respect to the involution J) ones. This identification is called the *Arnol'd passage*.

Let $a = (a_1, a_2, \ldots, a_n)$ be a critical point of the function W. Then $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$ where $\tilde{a}_2 = \sqrt{a_2}$ and $\tilde{a}_1 = a_1, \ldots, \tilde{a}_n = a_n$, is a critical point of the function $\tilde{W}$. The point $\tilde{a}_- = (\tilde{a}_1, -\tilde{a}_2, \ldots, \tilde{a}_n)$ (obtained from $\tilde{a}$ by the involution J) is critical as well. Thus, if a is a critical point of the function W and a □ $\mathbb{R}^n_+ \backslash \partial\mathbb{R}^n_+$, then a pair of critical points of the function $\tilde{W}$ corresponds to a. If a □ $\partial\mathbb{R}^n_+$ $(a_2 = 0)$ is a critical point of the function W, then a single critical point of the function $\tilde{W}$ corresponds to a.

If a critical point is "usual," then spreadings of bifurcating extremals (bif-spreadings) are represented by the row $(l_0, l_1, \ldots, l_n)$, where l_i is the number of critical points of the Morse index i. If we are dealing with a boundary critical point, then bif-spreadings are represented by the following matrix of order $2\times(n+1)$:

$$\begin{pmatrix} \tilde{l}_0 & \tilde{l}_1 & \ldots & \tilde{l}_n \\ l_0 & l_1 & \ldots & l_n \end{pmatrix}$$

Here $\tilde{l}_i$ is the number of boundary critical points of index i, while l_i is the number of usual (situated inside $\mathbb{R}^n_+$) critical points of index i.

5. Hyperbolic umbilic Singularities

Consider the following function germ of type hyperbolic umbilic

$$W(x, y) = \frac{x^3}{3} + xy^2.$$

This function has multiplicity 4 and codimension 3. Since, the ideal $I_W = (W_x, W_y) = (x^2 + y^2, x\,y)$ is generated by the functions $x^2 + y^2$and $x\,y$, so a basis for m_2/I_W can be to be taken $\{x,\ y,\ x^2 - y^2\}$. Hence, the deformation of the function W is given by

$$\widehat{W}(x, y, \lambda_1, \lambda_2, \lambda_3) = \frac{x^3}{3} + xy^2 + \lambda_1(x^2 - y^2) + \lambda_2 x + \lambda_3 y$$

In this section we considered the function $\widehat{W}$ when its symmetric with respect to the variable y, that is;

$$\widehat{W}(x, y, \lambda_1, \lambda_2) = \frac{x^3}{3} + xy^2 + \lambda_1(x^2 - y^2) + \lambda_2 x \qquad (5.1)$$

The function $\widehat{W}$ has four critical points. In the following Lemma we will the conditions of the existence of the critical points of the function $\widehat{W}$.

Lemma 5.1 The function $\widehat{W}$ has,

1- four real nondegenerate critical points if $\lambda_2 < -3\lambda_1^2$,

2- two real nondegenerate critical points if $-3\lambda_1^2 < \lambda_2 < \lambda_1^2$,

3- zero real nondegenerate critical points if $\lambda_2 > \lambda_1^2$.

Proof Since, the function $\widehat{W}$ has the following critical points,

$$P_1 = \left(-\lambda_1 + \sqrt{\lambda_1^2 - \lambda_2}, 0\right),$$

$$P_2 = \left(-\lambda_1 - \sqrt{\lambda_1^2 - \lambda_2}, 0\right),$$

$$P_3 = \left(\lambda_1, \sqrt{-(3\lambda_1^2 + \lambda_2)}\right),$$

and

$$P_4 = \left(\lambda_1, -\sqrt{-(3\lambda_1^2 + \lambda_2)}\right).$$

The points P_1 and P_2 are real when $\lambda_2 < \lambda_1^2$ and the points P_3, P_4 are real when $\lambda_2 < -3\lambda_1^2$, so the four points are real and nondegenerate if $\lambda_2 < -3\lambda_1^2$ .The points P_3 and P_4 are complex when $\lambda_2 > -3\lambda_1^2$, hence the only real critical points are P_1 and P_2 in the region $-3\lambda_1^2 < \lambda_2 < \lambda_1^2$.

Finally, the points P_1 and P_2 are complex if $\lambda_2 > \lambda_1^2$, hence there is no real critical points in the region $\lambda_2 > \lambda_1^2$. □

Lemma 5.2 The caustic of the function $\widehat{W}$ is given by the parameters equation

$$\lambda_2^2 + 3\lambda_2\lambda_1^2 - \lambda_2\lambda_1^2 - 3\lambda_1^4 = 0$$

Proof Since, the critical points of the function $\widehat{W}$ are degenerate on the cones defined by the equation $x^2 - y^2 - \lambda_1^2 = 0$, so the points P_1 and

P_2 are degenerate on the curve γ_1 defined by the equation $\lambda_2 = \lambda_1^2$. The points P_3 and P_4 are degenerate on curve γ_2 defined by the equation $\lambda_2 = -3\lambda_1^2$. Hence, the four points are degenerate on the curve defined by the equation $(\lambda_2 - \lambda_1^2)\,(\lambda_2 + 3\lambda_1^2) = 0$. □

The Geometric description of the caustic is given by the following graph

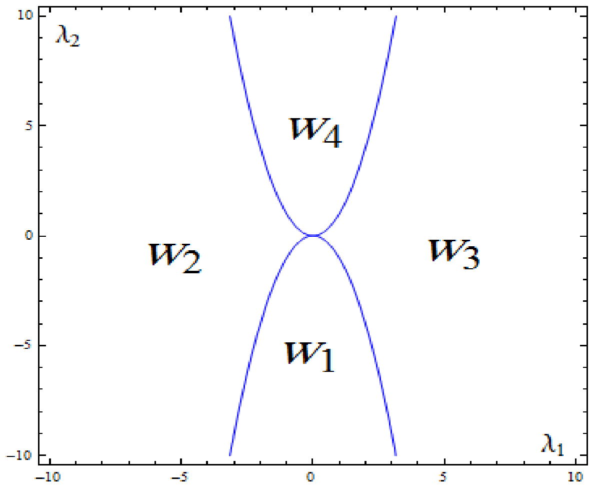

Fig. 1 describe the caustic of function (5.1)

The complement of the caustic is the union of four connected open subsets W_1 □ W_2 □ W_3 □ W_4 , every region has a fixed number of critical points such that if λ_1, λ_2 □ W_1 then the function $\widehat{W}$ has four nondegenerate critical points, if λ_1, λ_2 □ $(W_2$ □ $W_3)$ then the function $\widehat{W}$ has two nondegenerate critical points and if λ_1, λ_2 □ W_4 then the function $\widehat{W}$ has zero critical points. The spreading of the critical points is summarize in the following table ,

Critical point	Type	Conditions
$\boldsymbol{P_1}$	minimum	$\lambda_2 < -3\lambda_1^2, \lambda_1 \geq 0 \; or - 3\lambda_1^2 < \lambda_2 < \lambda_1^2, \lambda_1 < 0$
	saddle	$-3\lambda_1^2 < \lambda_2 < \lambda_1^2,\; \lambda 1 > 0$

	not exist	$\lambda_2 > \lambda_1^2$
P_2	maximum	$-3\lambda_1^2 < \lambda_2 < \lambda_1^2, \lambda_1 > 0 \ or \lambda_2 < -3\lambda_1^2, \lambda_1 \leq 0$
	saddle	$-3\lambda_1^2 < \lambda_2 < \lambda_1^2, \lambda_1 < 0$
	not exist	$\lambda_2 > \lambda_1^2$
P_3	saddle	$\lambda_2 < -3\lambda_1^2$
	not exist	$\lambda_2 > -3\lambda_1^2$
P_4	saddle	$\lambda_2 < -3\lambda_1^2$
	not exist	$\lambda_2 > -3\lambda_1^2$

Table (5.1)

6. Singularities of the function of codimension fifteen.

In this section we considered the function of degree five defined as follows

$$\widetilde{W}(x_1, x_2, \lambda_1, \lambda_2) = \frac{x_1^5}{5} + x_1x_2^4 + x_1^3x_2^2 + \frac{x_1^3}{3} + x_1x_2^2 + \lambda_1x_1^2 + \lambda_2x_2^2 \ (6.1)$$

Function (6.1) has multiplicity 16 and then it has codimension 15. The main purpose is to find geometrical description (bifurcation diagram) of the caustic of function (6.1) and then determine the spreading of the critical points of this function. To avoid some difficulties in the study of function (6.1) we assume the following $x_1 = x$ and $x_2^2 = y$, so the study of function (6.1) is equivalent to the study of the function

$$\widetilde{W}(x, y, \lambda_1, \lambda_2) = \frac{x^5}{5} + xy^2 + x^3y + \frac{x^3}{3} + xy + \lambda_1x^2 + \lambda_2y, \quad y \geq 0. \quad (6.2)$$

Since, the germ of function (6.2) is

$$W_0(x, y) = \frac{x^5}{5} + xy^2$$

So from section (4) we have

$$I = (W_x, yW_y) = (x^4, xy^2).$$

Accordingly, the multiplicity of function (6.2) is $\bar{\mu} = 10$ where $\mu = 6$ and $\mu_0 = 4$, so it has at the origin codimension 9, hence by theorem (2.2) the number of critical points of function (6.2) is 10, four points lies on the boundary $y = 0$ and six points lies in the interior, so the caustic of function (6.2) is the union of three sets,

$$\Sigma = \Sigma_1^{int} \cup \Sigma_1^{ext} \cup \Sigma_0$$

where Σ_1^{int} and Σ_1^{ext} are the subsets (components) of the caustic corresponding to the degeneration of boundary singularities along the boundary and along the normal, respectively, while Σ_0 is the component corresponding to the degeneration of interior (nonboundary) critical points.

6.1 Degeneration along the boundary.

To determine the set Σ_1^{int} we consider boundary critical points of function (6.2) such that the second-order partial derivatives of this function with respect to x vanishes at these points, i.e, the following relations are valid:

$$\frac{\partial \widetilde{W}}{\partial x}(x, 0, \lambda_1, \lambda_2) = \frac{\partial^2 \widetilde{W}}{\partial x^2}(x, 0, \lambda_1, \lambda_2) = 0$$

or

$$x^4 + x^2 + 2\lambda_1 x = 4x^3 + 2x + 2\lambda_1 = 0$$

From these relations, we easily obtain that the set Σ_1^{int} is defined by the equation $\lambda_1 = 0$.

To determine the set Σ_1^{ext} we consider boundary critical points of function (6.2) such that the first-order partial derivatives of this function with respect to y vanishes at these points, i.e, the following relations are valid:

$$\frac{\partial \widetilde{W}}{\partial x}(x, 0, \lambda_1, \lambda_2) = \frac{\partial \widetilde{W}}{\partial y}(x, 0, \lambda_1, \lambda_2) = 0$$

or

$$x^4 + x^2 + 2\lambda_1 x = x^3 + x + \lambda_2 = 0$$

From these relations, it is easy to see that the set Σ_1^{ext} is given by the equation

$$\lambda_2(\lambda_2 - 2\lambda_1) = 0.$$

Thus, the union of the sets Σ_1^{int} and Σ_1^{ext} is the union of the coordinate cross $\lambda_1\lambda_2 = 0$ and the line $\lambda_2 - 2\lambda_1 = 0$.

6.2 Degeneration of interior (nonboundary).

To determine the set Σ_0 we consider boundary critical points of function (6.2) which are defined by the following relations :

$$\frac{\partial \widetilde{W}}{\partial x}(x, y, \lambda_1, \lambda_2) = \frac{\partial \widetilde{W}}{\partial y}(x, y, \lambda_1, \lambda_2) = 0, \quad y > 0$$

or

$$x^4 + y^2 + 3x^2 y + x^2 + y + 2\lambda_1 x = 2x\,y + x^3 + x + \lambda_2 = 0. \qquad (6.3)$$

The determinate of the Hessian matrix of function (6.2) is given by the equation

$x^4 + 2x^2 - 4x\lambda_1 + (2y + 1)^2 = 0.$ (6.4)

It is easy to obtain from (6.3) and (6.4) the following equation

$3x^4 + 4x^2 + 6(\lambda_2 - 2\lambda_1)x + 1 = 0.$ (6.5)

Theoretically it is not easy to find the parameters equation of the set Σ_0, so we used program (Mathematic 6.0) to find geometric description of the set Σ_0 . By solving the following system

$$3x^4 + 4x^2 + 6(\lambda_2 - 2\lambda_1)x + 1 = 0,$$
$$2\,x\,y + x^3 + x + \lambda_2 = 0.$$

we obtain the values of x and y. Substitute the results in the equation (6.4) we see that the caustic ($\Sigma = \Sigma_1^{int} \cup \Sigma_1^{ext} \cup \Sigma_0$) of function (6.2) is given by the following graph,

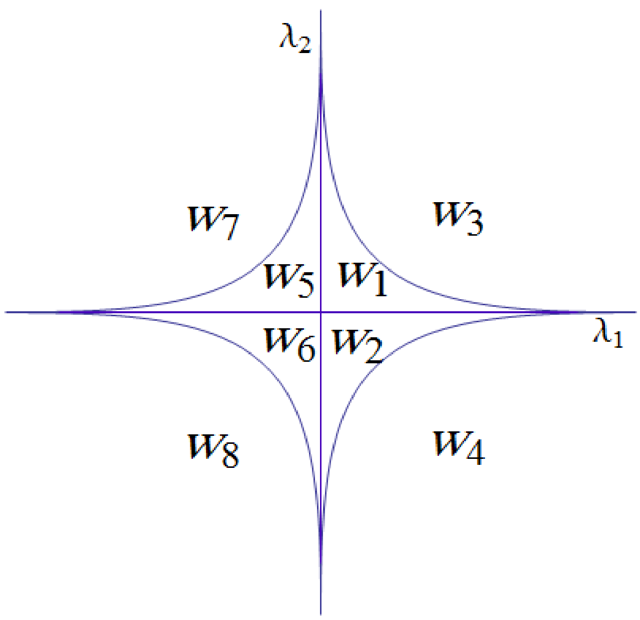

Fig. 2 describe the caustic of function (6.2).

The caustic of function (6.2) decomposes the plane of parameters into eight regions W_i, $i = 1,2,3,4,5,6,7,8$; every region contains a fixed number of critical points such that, if $\lambda_1, \lambda_2 \in (W_1 \cup W_3 \cup W_6 \cup W_8)$ then we have three critical points (boundary minimum, boundary maximum and interior saddle). If $\lambda_1, \lambda_2 \in (W_2 \cup W_4)$ then function (6.2) has two critical points (boundary saddle and boundary maximum). Finally, if $\lambda_1, \lambda_2 \in (W_5 \cup W_7)$ then the function has two critical points (boundary minimum and boundary saddle).

The matrices of bif-spreadings are as follows:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

7. Applications

In the bifurcation analysis of extremals the following two questions need to be answered (1) describe geometrical structure of Caustic (bifurcation diagram of the function) (2) determine the bifurcation spreading of the critical points in the complements of Caustic. The study of these problems in our work based on the local method of Lyapunov-Schmidt and singularities of smooth maps [8] .

To illustrate the results obtained in sections (5) and (6) we gave two examples, the first is to find the bifurcation solutions of Duffing equation of type,

$$\alpha\, \ddot{u} + \beta\, u + u^2 = \psi, \qquad (7.1)$$

$$u(0) = u(1) = 0.$$

where α, β are the parameters of the problem, $u = u(t)$, $t \in [0,1]$ and ψ is symmetric function with respect to the involution $I: \psi(t) \mapsto \psi(1-t)$. The second example is to find the bifurcation solutions of Duffing equation of type,

$$\alpha\, \ddot{u} + \beta\, u + u^2 + u^4 = 0, \ (7.2)$$

$$u(0) = u(1) = 0.$$

Suppose that $f_1, f_2 : E \to M$ are nonlinear Fredholm operators of index zero from Banach space E to Banach space M, where $E = C^2([0,1], \mathbb{R})$ is the space of all continuous functions that have derivative of order at most two, $M = C^0([0,1], \mathbb{R})$ is the space of all continuous function and f_1, f_2 are defined by the operator equations,

$f_1(u,\lambda) = \alpha\,\ddot{u} + \beta\,u + u^2,$

$$f_2(u,\lambda) = \alpha\,\ddot{u} + \beta\,u + u^2 + u^4$$

where

$u = u(t),\ t \,\square\, [0,1],\ \lambda = (\alpha\,,\beta),$

and $\cdot = \frac{d}{dt}.$

Every solution of the equations in (7.1) and (7.2) is a solution of the operator equations ,

$f_1(u,\lambda) = \psi, \qquad \psi \,\square\, M.$ (7.3)

and

$f_2(u,\lambda) = 0$ (7.4)

respectively. Since, the operators f_1 and f_2 have variational property, then there exist functionals V_1 and V_2 such that,

$f_1 = grad_H V_1$ and $f_2 = grad_H V_2.$

where,

$$V_1(u,\lambda,\psi) = \int_0^1 \left(\alpha\,\frac{(\dot{u})^2}{2} + \beta\,\frac{u^2}{2} + \frac{u^3}{3} - u\,\psi\right)dt,$$

$$V_2(u,\lambda) = \int_0^1 \left(\alpha\,\frac{(\dot{u})^2}{2} + \beta\,\frac{u^2}{2} + \frac{u^3}{3} + \frac{u^5}{5}\right)dt.$$

In this case every solution of equation (7.3) is a critical point of the functional V_1 and every solution of equation (7.4) is a critical point of the functional V_2.

In the following theorem we showed that the study of bifurcation of extremals of the functionals V_1 and V_2 is reduced to the study of bifurcation of extremals of the functions (5.1) and (6.1) respectively

.**Theorem 7.1** The normal forms of the key functions $\breve{W}_1$ and $\breve{W}_2$ corresponding to the functionals V_1 and V_2 respectively are given by,

$$\breve{W}_1(\hat{x},\rho) = \frac{x^3}{3} + xy^2 + \lambda_1(x^2 - y^2) + \lambda_2 x, \quad \hat{x} = (x,y), \quad \rho = (\lambda_1,\lambda_2).$$

$$\breve{W}_2(\hat{y},\rho_1) = \frac{x_1^5}{5} + x_1x_2^4 + x_1^3x_2^2 + \frac{x_1^3}{3} + x_1x_2^2 + \tilde{\lambda}_1\,x_1^2 + \tilde{\lambda}_2 x_2^2, \qquad \hat{y} = (x_1,x_2), \quad \rho_1 = (\tilde{\lambda}_1\,,\tilde{\lambda}_2).$$

Proof Since, the Fréchet derivative of the nonlinear operators f_1 and f_2 at the point $(0,\lambda)$ is

$$Ah = df_1(0,\lambda)h =$$

$$df_2(0,\lambda)h = \alpha\frac{d^2h}{dt^2} + \beta h, \qquad h \,\square\, E.$$

hence the linearized equation corresponding to the equations $f_1(u,\lambda) = 0$ and $f_2(u,\lambda) = 0$ is given by,

$$A\,h = 0, \quad h \in E,$$
$$h(0) = h(1) = 0.$$

The solution of the linearized equation which satisfies the initial conditions is given by

$$e_p(t) = c_p \sin(p\pi t), \quad p = 1,2,\ldots$$

and the characteristic equation corresponding to this solution is

$$\beta - \pi^2 p^2 \alpha = 0,$$

this equation gives in $\alpha\beta$ − plane characteristic lines ℓ_p . The characteristic lines ℓ_p consist the points (α,β) for which the linearized equation has non-zero solutions. The point of intersection of the characteristic lines in $\alpha\beta$ − plane is bifurcation point. So the bifurcation

$E = N \oplus N^{\perp}$,

$N^{\perp} = \{v \in E: \int_0^1 v\, e_k\, dt = 0,\ k = 1,2\}$.

Similarly, the space M can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$M = N \oplus \tilde{N}^{\perp}$,

$\tilde{N}^{\perp} = \{\omega \in M: \int_0^1 \omega e_k\, dt = 0,\ k = 1,2\}$.

There exist two projections $P: E \to N$ and $I - P: E \to N^{\perp}$ such

point for the equations (7.3) and (7.4) is $(\alpha,\beta) = (0,0)$.Localized parameters α,β as following ,

$\alpha = 0 + \delta_1$, $\beta = 0 + \delta_2$, δ_1,δ_2 are small parameters, lead to bifurcation along the modes,

$$e_1(t) = c_1 \sin(\pi t),$$
$$e_2(t) = c_2 \sin(2\pi t).$$

Since, $\|e_1\| = \|e_2\| = 1$ then we have $c_1 = c_2 = \sqrt{2}$.

Let $N=Ker(A) = span\,\{e_1, e_2\}$, then the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

that $Pu = \bar{u}$ and $(I - P)u = v$,

(I is the identity operator). Hence every vector $u \in E$ can be written in the form ,

$$u = \bar{u} + v, \bar{u} = y_1 e_1 + y_2 e_2 \in N,$$
$$v \in N^{\perp}, \quad y_i = \langle u, e_i \rangle.$$

Since $\psi \in M$ implies that

$\psi = \psi_1 + \psi_2, \psi_1 = \hat{q}_1 e_1 + \hat{q}_2 e_2 \in N$, $\psi_2 \in \tilde{N}^{\perp}$. Also, the symmetry of the function $\psi(t)$ with respect to the involution $I: \psi(t) \mapsto \psi(1-t)$ implies that $\hat{q}_2 = 0$. Thus, by the implicit function theorem , there exists a smooth map $\Theta: N \rightarrow N^{\perp}$, such that

$$\widetilde{W}_1(z,\gamma,\psi) = V_1(\Theta(z,\gamma,\psi),\gamma,\psi),$$

and

$$\widetilde{W}_2(z,\gamma) = V_2(\Theta(z,\gamma,\psi),\gamma), \qquad z = (y_1,y_2), \qquad \gamma = (\delta_1,\delta_2).$$

and then the key functions $\widetilde{W}_1$ and $\widetilde{W}_2$ can be written in the form ,

$$\widetilde{W}_1(z,\gamma) = V_1(y_1e_1 + y_2e_2 + \Theta(y_1e_1 + y_2e_2,\gamma),\gamma)$$

$$=W_1(z,\gamma) + o(|z|^3) + O(|z|^3)\,O(\gamma) ,$$

$$\widetilde{W}_2(z,\gamma) = V_2(y_1e_1 + y_2e_2 + \Theta(y_1e_1 + y_2e_2,\gamma),\gamma)$$

$$=W_2(z,\gamma) + o(|z|^3) + O(|z|^3)\,O(\gamma) ,$$

where,

$$W_1(z,\gamma) = -q_1y_1 + k_1y_1^2 + \frac{8\sqrt{2}y_1^3}{9\pi} + k_2y_2^2 + \frac{32\sqrt{2}y_1y_2^2}{15\pi},$$

$W_2(z,\gamma) = (6720$

$+$

The geometrical form of bifurcations of critical points and the first asymptotic of branches of bifurcating for the functions $\widetilde{W}_1$ and $\widetilde{W}_2$ are completely determined by their principal parts W_1 and W_2 respectively. By changing the parameters and the variables in the function W_1 as follows

$$k_1 = \rho_1 + \lambda_1,$$
$$k_2 = \rho_1 - \lambda_1,$$
$$\lambda_2 = -q_1,$$
$$y_1 = x ,$$

and

$$y_2 = y$$

we have the function W_1 is equivalent to the function $\widetilde{W}_1$. The functions W_1 and $\widetilde{W}_1$ have the same germ

$$W_0(x,y) = \frac{x^3}{3} + xy^2,$$

so they are contact equivalent. Hence the Caustic of the function W_1 is coincide with the Caustic of the function $\widetilde{W}_1$. Also, by changing variables in the function W_2 as follows:

$y_1 = x_1,\ y_2 = x_2$ we have that the functions W_2 and $\widetilde{W}_2$ are contact equivalent, since they have the same germ

$$G_0(x_1, x_2) = \frac{x_1^5}{5} + x_1 x_2^4 + x_1^3 x_2^2$$

The functions $\breve{W}_1$ and $\breve{W}_2$ has all the topological and analytical properties of functionals V_1 and V_2, so the study of bifurcation analysis of the equations (7.3) and (7.4) is equivalent to the study of bifurcation analysis of the functions $\breve{W}_1$ and $\breve{W}_2$. This shows that the study of bifurcation of extremals of the functionals V_1 and V_2 is reduced to the study of bifurcation of extremals of the functions (5.1) and (6.1) respectively. □

The point $a \in E$ is a solution of the equation $f(x, \lambda) = 0$ if and only if,

$$a = \sum_{i=1}^{n} \bar{\xi}_i e_i + \Phi(\bar{\xi}, \bar{\lambda}),$$

where $\bar{\xi}$ is a critical point of the key function *W*. From this, we note that the set of solutions of the equation $f(x, \lambda) = 0$ is coincide with the set of the critical points of the functional $V(x, \lambda)$ and the set of the critical points of the functional *V* is coincide with the set of the critical points of the key function *W*. Hence, the bifurcation analysis of problems (7.1) and (7.2) has been studied in section (5) and (6) respectively

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