Optimal Robust Controller Design for Heartbeat System

Dr. Amjad J. Humaidi
Control and Systems Control Engineering Department, University of Technology /Baghdad
Email: aaaacontrol2010@yahoo.com
Kadhim Yakoob Yousif
Control and Systems Control Engineering Department, University of Technology /Baghdad
Email: kadhemyy1984@yahoo.com
Ruaa Muayad AL_Wardie
Control and Systems Control Engineering Department, University of Technology /Baghdad
Email: rm_technology@yahoo.com

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ABSTRACT

In the present work, an optimal control approach is suggested to design a robust controller for the heart system. The work objective is to generate artificial or synthetic optimal Electrocardiogram (ECG) signals using the nonlinear optimal robust control. This includes the design of a state feedback controller to stabilize the ECG under uncertainty. The approach key is to reflect the uncertainty bound in the cost function, and the solution to the optimal control problem is a solution to the robust control problem.

The robust optimal control strategy is applied for nonlinear heart system; different structures of uncertainties have been sought. Unfortunately, for the present case study only unmatched structure of uncertainty has been found in both state and input matrices.

Keywords: Optimal Control, Robust Control, Heartbeat System

INTRODUCTION

The human heart is a complex and yet robust system. One of the most important signals that relates to human heart operation is the electrocardiogram (ECG) signal. It is a time-varying signal representing the electrical potential generated by the electrical activity in the cardiac tissue. A single cycle of the ECG reflects the contraction and relaxation of the heart, leading to the heart's pumping action. The ECG can be measured by recording the potential between two electrodes placed on the surface of the skin at some pre-determined points. Characteristic information extracted from the ECG signal can be used to indicate the state of cardiac health as well as a potential heart problem [1].

A cycle of the heartbeat consists of two states: diastole which is the relaxed state, and systole which is the contracted state. The cycle starts when the heart is in the diastolic state. The pacemaker that is located at the top of the right atrium-one of the upper chambers of the heart-triggers an electrochemical wave that spreads slowly over the atrium. This electrochemical wave causes the muscle fibers to contract and

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2412-0758/University of Technology-Iraq, Baghdad, Iraq

push the blood into the ventricles—the lower chambers of the heart. The electrochemical wave then spreads rapidly over the ventricles causing the whole ventricle to contract into the systolic state, and pumping the blood into the lung and the arteries. Immediately following the systolic state, the muscle fibers relax quickly and return the heart to the diastolic state to complete one cycle of the heartbeat [2].

In real life, the systems to be controlled have many uncertainties that can make the performance deviate from the nominal design. That a controller is said to be robust if it works even if the actual system deviates from its ideal model on which the controller design is based [3]. Needless to say, it is very important for a controller to be robust because, as modern day system becomes more and more complex, it is in fact impossible to find the exact model of a system. First, despite the great scientific progresses made so far, there are still phenomena that are poorly understood and hence impossible to model precisely. Second, some modern systems are so complex even if accurate models are available, they will be too complicated to use. Finally, some systems may be subjected to deterioration or other changes during their life-time. All these facts indicate the necessity of having robust controllers for such systems.

The approach to be presented in this work is quite different from other robust controllers. One does not attempt to solve a robust control problem directly. Rather, the robust control problem is translated into an optimal control problem. After the translation, one can erase the robust control problem from memory and concentrate on the optimal control problem. As long as the optimal control problem can be solved, then robust optimal control problem, can be guaranteed be solved too. In fact, the solution to the optimal control problem is a solution to the robust control problem if the matching condition is satisfied (otherwise, a computable sufficient condition needs to be checked, and part of the solution to the optimal control problem is a solution to the robust control problem).

For linear system, the optimal control problem often reduces to a linear quadratic regulator (LQR) problem, whose solution can always be obtained by solving an algebraic Riccati equation. For nonlinear systems, the robust control design is rather complex. The complexity is in terms of efficient computation. For nonlinear systems, one may not be able to easily compute the solution to the optimal control problem as analytical solutions, forcing us to use numerical solutions [4].

Mathematical Model of Heartbeat

Much effort has been invested into the development of mathematical models that describe the operation of the human heart. One of the crucial developments is by Zeeman [2], where he developed a mathematical model that captured three important qualities of cardiac characteristics: (i) stable equilibrium; (ii) threshold for triggering an action potential; and (iii) return to equilibrium. The resulting models are a second-order nonlinear differential equation representing the heartbeat system, and a third-order nonlinear differential equation that can be applied to the nerve impulse. Other interesting and related models were presented in [2, 3]. The third order nonlinear heartbeat model is given by [1, 2, 5]

$$\dot{x}_1 = -(x_1^3 + x_1 x_2 + x_3) \qquad \dots(1)$$
$$\dot{x}_2 = -2x_1 - 2x_2 \qquad \dots.(2)$$
$$\dot{x}_3 = -x_2 - 1 + u \qquad \dots.(3)$$

where $x_1(t)$ refers to the length of the muscle fiber, $x_2(t)$ represents tension in the muscle fiber, $x_3(t)$ is related to electrochemical activities, ε is a positive constant, and $u(t)$ represents cardiac pacemaker control signal which directs the heart into the diastolic and the systolic states.

Choosing the output as $y(t) = x_3(t)$. Differentiating the output with respect to t yields

$$\dot{y} = -x_2 - 1 + u \quad \ldots (4)$$

To proceed on the output tracking control design task, define the tracking error as $e = y - y_r$, where $y_r(t)$ is the reference input. It follows that

$$\dot{e} = \dot{x}_3 - \dot{y}_r = -x_2 - 1 + u \cong v - \dot{y}_r \quad \ldots (5)$$

Where: $v(t)$ is the transformed input. Let the tracking control law for the transformed input $v(t)$ be given by [1, 2, 5]

$$v = -Ke + \dot{y}_r = -K(x_3 - y_r) + \dot{y}_r \quad \ldots (6)$$

$$u = -K(x_3 - y_r) + \dot{y}_r + x_2 + 1 \quad \ldots (7)$$

$$\dot{x}_1 = -\frac{1}{\varepsilon}(x_1^3 + x_1 x_2 + x_3) \quad \ldots (8)$$

$$\dot{x}_2 = -2x_1 - 2x_2 \quad \ldots (9)$$

$$\dot{x}_3 = -x_2 - 1 + u \quad \ldots (10)$$

Solution of Robust Optimal Control to Unmatched Uncertainty of Nonlinear System:

Consider the following nonlinear system [4, 7, 8]:

$$\dot{x} = \boldsymbol{A}(x) + \boldsymbol{B}(x)\,u + \boldsymbol{C}(x)\boldsymbol{f}(x) \quad \ldots (11)$$

Where: $\boldsymbol{f}(x)$ models the uncertainty in the system dynamics, and $\boldsymbol{C}(x)$ can be any matrix. The reason for introducing $\boldsymbol{C}(x)$ is to make the definition of uncertainty $\boldsymbol{f}(x)$ more flexible. The following assumptions are made:

- ❒ $\boldsymbol{A}(0) = 0$ and $\boldsymbol{f}(0) = 0$, so that $x = 0$ is an equilibrium.
- ❒ The uncertainty $\boldsymbol{f}(x)$ is bounded.

The robust problem can be stated by finding a feedback control law $u = u_o(x)$ such that the closed-loop system

$$\dot{x} = \boldsymbol{A}(x) + \boldsymbol{B}(x)\,u_o(x) + \boldsymbol{C}(x)\boldsymbol{f}(x) \quad \ldots (12)$$

is globally asymptotically stable for all uncertainties $\boldsymbol{f}(x)$.The robust control problem can be solved indirectly by translating it into an optimal control problem. For the auxiliary system [4, 7, 8],

$$\dot{x} = \boldsymbol{A}(x) + \boldsymbol{B}(x)\,u + \propto (\boldsymbol{I} - \boldsymbol{B}(x)\boldsymbol{B}(x)^+)\,\boldsymbol{C}(x)\,v \quad \ldots (13)$$

Where: $\boldsymbol{B}(x)^+$ is pseudo inverse and is given by

$$\boldsymbol{B}(x)^+ = \left(\boldsymbol{B}(x)^T\boldsymbol{B}(x)\right)^{-1}\boldsymbol{B}(x)^T$$

The objective is to find a feedback control law ($u_o(x)$,$v_o(x)$) that minimizes the following cost functional

$$J = \int_0^\infty (\boldsymbol{f}_{max}(x)^2 + \rho^2\boldsymbol{g}_{max}(x)^2 + \beta^2\,\|x\|^2 + \|u\|^2 + \rho^2\|v\|^2)\,dt \quad \ldots (14)$$

Where: $\alpha \geq 0$, $\rho \geq 0$ and $\beta \geq 0$ are design parameters. $\boldsymbol{f}_{max}(x)$ and $\boldsymbol{g}_{max}(x)$ are nonnegative functions such that [6,7]

$$\|\boldsymbol{B}(x)^{+}\,\boldsymbol{C}(x)\boldsymbol{f}(x)\| \leq \boldsymbol{f}_{max}(x) \qquad \ldots (15)$$

$$\|\alpha^{-1}\,\boldsymbol{f}(x)\| \leq \boldsymbol{g}_{max}(x) \qquad \ldots(16)$$

If one chooses α, ρ and β such that the solution to optimal control problem, denoted by $(u_o(x), v_o(x))$, exists, and the following condition is satisfied [7]:

$$2\,\rho^2\,\|v_o(x)\|^2 \leq \beta'^2\,\|x\|^2 \qquad \forall x \in R^n \qquad \ldots(17)$$

For some β' such that $|\beta'|<|\beta|$, then $u_o(x)$, the u-component of the solution to optimal control problem is a solution to the robust control problem.

Design of Robust Optimal Controller to Heart System

One can easily formulate the nonlinear system of Eq.s (8)-(10) into matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & -\frac{1}{\varepsilon}\\ -2 & -2 & 0\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}[u(t)] + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} f(x) \qquad \ldots.(18)$$

The nonlinearity of the above system can be isolated outside the state matrix and mimics the following expression;

$$\dot{x} = \boldsymbol{A}(x) + \boldsymbol{B}(x)\,u + \boldsymbol{C}(x)\boldsymbol{f}(x)$$

Then the input matrix becomes $\boldsymbol{B} = [0 \quad 0 \quad 1]^T$. The design parameters can be selected as; $\propto = 10^3$, $\beta = 10^4$ and $\rho = 10^6$. Letting $\varepsilon \in (0.1, 0.5)$ and its nominal value is set at $\varepsilon_o = 0.2$, then

$$\boldsymbol{A}(\varepsilon) - \boldsymbol{A}(\varepsilon_o) = \begin{bmatrix}0 & 0 & -\frac{1}{\varepsilon}\\ -2 & -2 & 0\\ 0 & -1 & 0\end{bmatrix} - \begin{bmatrix}0 & 0 & -\frac{1}{\varepsilon_o}\\ -2 & -2 & 0\\ 0 & -1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The above is equivalent to the following expression [7];

$$\boldsymbol{A}(\varepsilon) - \boldsymbol{A}(\varepsilon_o) = \boldsymbol{BB}^{+}\left(\boldsymbol{A}(\varepsilon) - \boldsymbol{A}(\varepsilon_o)\right) + (\boldsymbol{I} - \boldsymbol{BB}^{+})\left(\boldsymbol{A}(\varepsilon) - \boldsymbol{A}(\varepsilon_o)\right)$$

Where: the matched component is given by

$$\boldsymbol{BB}^{+}\left(\boldsymbol{A}(\varepsilon) - \boldsymbol{A}(\varepsilon_o)\right)$$

And the unmatched component is given by

$$(\boldsymbol{I} - \boldsymbol{BB}^{+})\left(\boldsymbol{A}(\varepsilon) - \boldsymbol{A}(\varepsilon_o)\right)$$

Since,

$$\boldsymbol{B}^{+} = (\boldsymbol{B}^T\boldsymbol{B})^{-1} \qquad \ldots(19)$$

The above pseudo inverse can be further expressed in terms of heartbeat model as;

$$\boldsymbol{B}^{+} = \left([0 \quad 0 \quad 1]\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\right)^{-1} [0 \quad 0 \quad 1] = [0 \quad 0 \quad 1]$$

The matched component can be determined by the following:

$$\boldsymbol{BB}^{+}\left(\boldsymbol{A}(\varepsilon) - \boldsymbol{A}(\varepsilon_o)\right) = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}[0 \quad 0 \quad 1]\begin{bmatrix}0 & 0 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

It is clear that it is not satisfied, in other words, all uncertainties are unmatched. As such, the unmatched component can be given by;

$$(\boldsymbol{I} - \boldsymbol{BB}^{+})\left(\boldsymbol{A}(\varepsilon) - \boldsymbol{A}(\varepsilon_o)\right) = \left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}0 & 0 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $f(x)$ is assumed to be $f(x) = -\frac{1}{\varepsilon}(x_1^3 + x_1 x_2)$, then based on condition of Eq.(15), the following condition has to be satisfied:

$$\|\boldsymbol{B}^{+}\,\boldsymbol{C}(x)\,\boldsymbol{f}(x)\| \le \boldsymbol{f}_{max}(x) \qquad ..(20)$$

$$\left\| \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{f}(x) \right\| = \boldsymbol{f}_{max}(x) = 0$$

Moreover, the condition of Eq.(16) has to be satisfied;

$$\|\alpha^{-1}\boldsymbol{f}(x)\| \le \boldsymbol{g}_{max}(x) \qquad \ldots(21)$$

Numerically Eq.(21) can be written as;

$$\left\| -\frac{0.001}{\varepsilon}(x_1^3 + x_1 x_2) \right\| \le \boldsymbol{g}_{max}(x) \le \sqrt{\frac{a^2+b^2}{3}} \times \sqrt{x_1^2 + x_2^2}$$

If one let a = 5 and b = 5, then one will have

$$\|-0.005 \times (x_1^3 + x_1 x_2)\| \le \sqrt{\frac{(0.05)^2+(0.05)^2}{3}} \times \sqrt{x_1^2 + x_2^2}$$

or,

$$\|-0.005 \times (x_1^3 + x_1 x_2)\| \le 0.04 \times \sqrt{x_1^2 + x_2^2} = \boldsymbol{g}_{max}(x)$$

Therefore, the objective is to find a feedback control law $(u_o(x), v_o(x))$ that minimizes the following cost functional

$$\begin{aligned} &\boldsymbol{f}_{max}(x)^2 + \rho^2 \boldsymbol{g}_{max}(x)^2 + \beta^2\|x\|^2 + \|u_0(x)\|^2 + \rho^2\|v_0(x)\|^2 \\ &\quad = (10^6)^2\left(0.04\sqrt{x_1^2 + x_2^2}\right)^2 + 10^8\,(x_1^2 + x_2^2 + x_3^2) + u^2 + (10^6)^2 v^2 \\ &\quad = 17 \times 10^8 x_1^2 + 17 \times 10^8\, x_2^2 + \times\, 10^8\, x_3^2 + u^2 + 10^{12} v^2 \end{aligned}$$

Such that

$$\boldsymbol{Q} = \begin{bmatrix} 17 \times 10^8 & 0 & 0 \\ 0 & 17 \times 10^8 & 0 \\ 0 & 0 & 10^8 \end{bmatrix}, \boldsymbol{R} = \begin{bmatrix} 1 & 0 \\ 0 & 10^{12} \end{bmatrix}$$

Hence, the corresponding optimal control problem is as follows; for the auxiliary system

$$\dot{x} = \boldsymbol{A}(x) + \boldsymbol{B}(x)u + \propto (\boldsymbol{I} - \boldsymbol{B}(x)\boldsymbol{B}(x)^{+})\boldsymbol{C}(x)v \qquad ...(2٢)$$

Where:

$$\begin{aligned} &\propto (\boldsymbol{I} - \boldsymbol{B}(x)\boldsymbol{B}(x)^{+})\boldsymbol{C}(x)v \\ &\quad = 1000 \times \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} v \qquad \ldots(2٣) \end{aligned}$$

Substituting Eq.(23) into Eq.(22), we get

$$\boldsymbol{A} = \begin{bmatrix} -0.028 & -110 & 0 \\ 0 & -0.025 & 0.00013 \\ 0 & 0 & -0.0926 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & 1000 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\boldsymbol{Q} = \begin{bmatrix} 17 \times 10^8 & 0 & 0 \\ 0 & 17 \times 10^8 & 0 \\ 0 & 0 & 17 \times 10^8 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} 1 & 0 \\ 0 & 10^{12} \end{bmatrix}$$

MATLAB package has been used to solve this LQR problem. The solution to the algebraic Riccati equation

$$\boldsymbol{SA} + \boldsymbol{A}^T\boldsymbol{S} + \boldsymbol{Q} - \boldsymbol{SBR}^{-1}\boldsymbol{B}^T\boldsymbol{S} = 0 \qquad \ldots(2٤)$$

where, $\boldsymbol{S}$ is given by

$$\boldsymbol{S} = 1.0e{+}08 \times \begin{bmatrix} 0.3754 & -0.1466 & -0.0002 \\ -0.1466 & 3.5798 & 0.0001 \\ -0.0002 & 0.0001 & 0.0001 \end{bmatrix}$$

The corresponding control $u = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{S}x$ is given by

$$u_0 = 1.0\mathrm{e} + 004\,[1.8685 \quad -0.7293 \quad -1.0009]x$$
$$v_0 = 1.0\mathrm{e} + 004\,[-0.0000 \quad 0.0000 \quad 0.0000]x$$

Let us check if the sufficient condition is satisfied;

$$2\rho^2\|v_0(x)\|^2 \leq \beta^2\|x\|^2 \qquad \forall x \in \qquad \ldots(2٥)$$

The above condition can be formulated as

$$\beta^2\|x\|^2 - 2\rho^2\|v_0(x)\|^2 \geq 0$$

or,

$$\beta^2\|x\|^2 - 2\rho^2\|v_0(x)\|^2 = 10^8\|x\|^2 - 2 \times (10^{12})\|v_0(x)\|^2$$
$$= x^T \begin{bmatrix} 10^8 & 0 & 0 \\ 0 & 10^8 & 0 \\ 0 & 0 & 10^8 \end{bmatrix} x \geq 0$$

Therefore, the condition is satisfied.

Simulation Results

The model together with the suggested controller is simulated using MATLAB package. The optimal control strategy has been coded and listed in m-file format. Two scenarios have been considered; one for the case with robust optimal controller and the other without robust optimal controller. The latter case considers the situation of classical optimal control without taking into account the inclusion of uncertainty within performance index structure.

Figure (1) shows a single cycle of ECG signal. Figure (2) shows different behaviors of heart system states; muscle fiber length, electrochemical activity and muscle fiber tension. The control signal in this system is represented by pacemaker signal. It is evident from the figure that there is almost coincidence between actual and prescribed ECG signal in the case of robust optimal controller.

The above scenario has been repeated if robustness has not been considered in optimal controller. It is clear from Figure (3) that the actual electrochemical signal does not coincide with typical ECG signal and a crucial situation would appear.

CONCLUSIONS

In the present work, a robust optimal controller is presented for nonlinear system of heartbeat under uncertainty. The results showed that the suggested controller could enable the electrochemical signal to keep track typical ECG signal, while there is a lack of tracking when the robustness is discarded in the optimal controller.

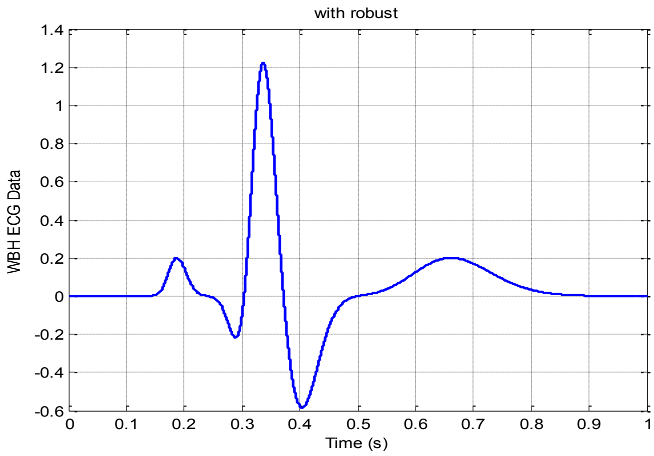

Figure (1) A single cycle of an ECG signal

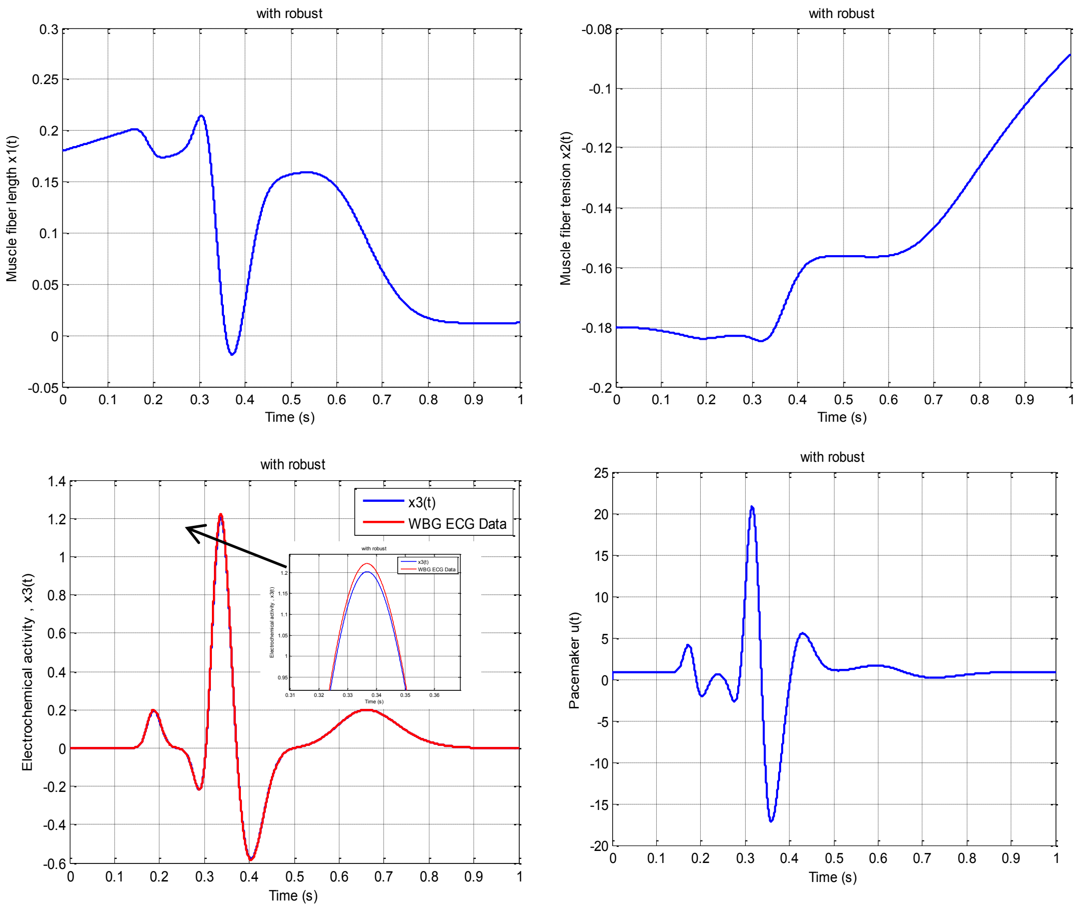

Figure (2) State behavior for heartbeat system with robust optimal controller

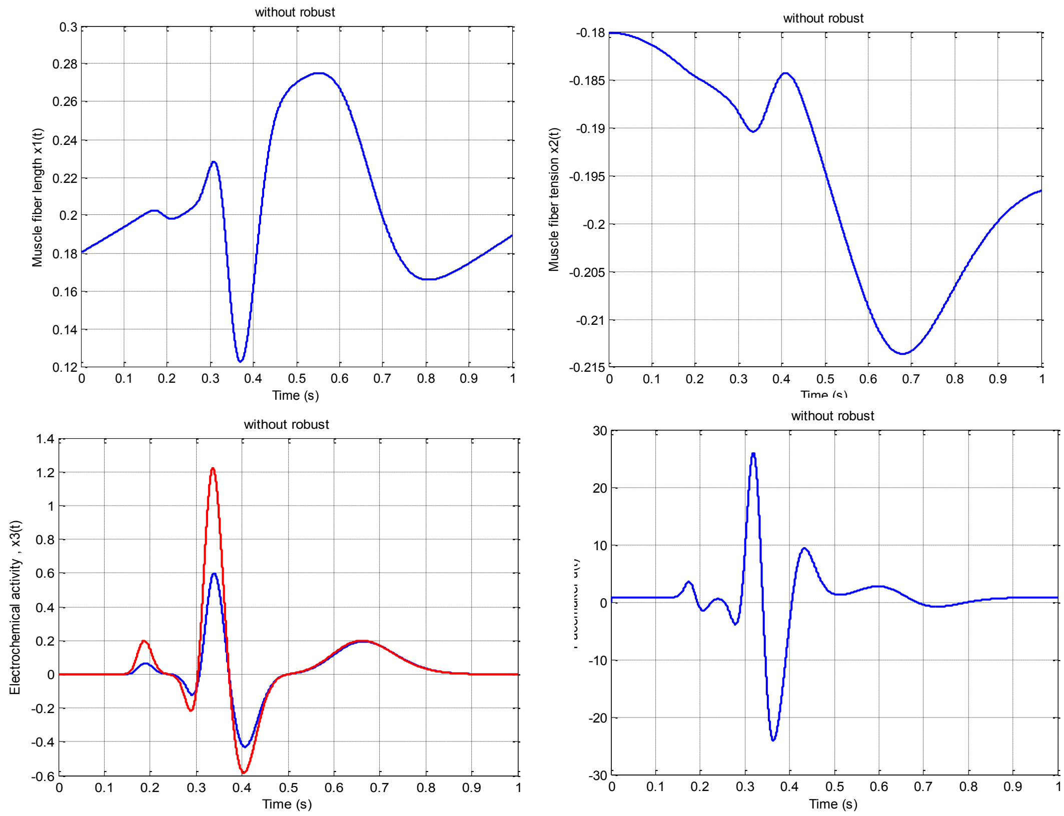

Figure (3) State behavior for heartbeat system without robust optimal controller

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