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Ritz Variational Method for Buckling Analysis of Euler-Bernoulli Beams Resting on Two-Parameter Foundations

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ABSTRACT

The analysis of the least compressive load that cause buckling failures of Euler-Bernoulli beams resting on two-parameter elastic foundations (EBBo2PFs) is vital for safety. This article presents Ritz variational method (RVM) for the stability solutions of EBBo2PFs under in-plane compressive loads. The Ritz total potential energy functional, $\prod$, was derived for the problem as the sum of the strain energies of the thin beam, the two-parameter lumped parameter elastic foundation (LPEF) and the work potential due to the in-plane compressive load. Ritz functional $\prod$ was found to depend upon the buckling function $w(x)$ and its derivatives $\left(w'(x), w''(x)\right)$ with respect to the longitudinal coordinate. The principle of minimization of $\prod$ was implemented for each considered boundary condition to find the $w(x)$ corresponding to minimum $\prod$. Three cases of boundary conditions investigated were: clamped at both ends, clamped at one end and free at the other, simply supported at both ends. For each case, $w(x)$ was found in terms of unknown generalized buckling parameters c_i, and buckling shape functions $\varphi_i(x)$ satisfying the boundary conditions. Thus $\prod$ was expressed in terms of the parameters c_i. The Ritz functional was subsequently minimized with respect to the parameters yielding an algebraic eigenvalue problem. The condition for nontrivial solutions of homogeneous algebraic equations was used to find the characteristic buckling equations that were solved to find the eigenvalues. The eigenvalues were used to find the buckling loads and the critical buckling load. It was found that a one-parameter RVM solution for the EBBo2PF with both ends clamped, and with one clamped and one free end gave similar critical buckling load solutions to those presented in the literature. It was also found that an n-parameter RVM solution for the EBBo2PFs with both ends simply supported yielded exact buckling load solutions because exact sinusoidal buckling shape functions were used.

1. Introduction

1.1. Background

The subject matter of beams on elastic foundations (BoEFs) has been extensively applied to the study of foundation beams. In such studies the governing equations of beam theory are modified by the incorporation of

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the effect of the reaction forces from the supporting foundation. Beam on elastic foundation studies have been influenced by studies on beams, and studies on foundations; and are focused on how the interaction effects of the foundation affect the beam behaviour in bending, buckling and vibration.

The earliest beam theory was proposed by Euler and also by Bernoulli, and is commonly referred to as the Euler-Bernoulli beam theory (EBBT) or the classical beam theory (CBT). EBBT was derived using the Navier hypothesis that plane cross-sections which are initially normal to the middle plane of the longitudinal axis of the beam before deformation would remain plane and normal to the middle plane of the longitudinal axis after deformation; and the middle plane of the longitudinal axis is unstretched and free of strains. Hence the middle plane is a neutral plane in pure bending. (Ike, 2018a; Ike, 2018b; Ike, 2023a). Consequently, the EBBT disregards shear deformation effects and can only apply to slender/thin beams for which the ratio of thickness, h, to the span, l, is less than or equal to 0.05. In beams with ratios of thickness to span greater than 0.05, and for composite and laminated beams, shear deformation effects have been found to be important factors governing their behaviour in bending, stability or vibration.

Beams with $h/l > 0.05$ are called moderately thick or thick beams depending on the actual value of h/l. If $0.05 \leq h/l \leq 0.10$ the beam is called moderately thick, and thick when $h/l > 0.10$.

Moderately thick beams and thick beams are formulated by consideration of the effects of shear deformation in order to truly reflect the actual behaviour of such beams.

Several shear deformable beam theories have been proposed and implemented by researchers in a bid to overcome the limitations of the CBT. Shear deformation beam theories have been derived by Timoshenko, Levinson (1981), Dahake and Ghugal (2013), and Sayyad and Ghugal (2011), to name only some contributors.

Despite the limitations of EBBT, it has been widely used because of the prevalence of thin beams in practical structural applications. This work is focused on thin beams and thus uses EBBT.

Elastic foundations have been described analytically via continuously distributed parameter and lumped/discrete parameter idealizations. Continuously distributed parameter idealizations utilize the well-established elasticity theory framework to determine the mathematical relations for the reaction forces from the soil on the beam structures. The resulting mathematical formulations have been found to be extremely complex, and have not found extensive usage. On the other hand, lumped parameter idealizations utilize one, two or a definite number of soil foundation parameters to derive the soil reaction forces on the beam. They are commonly utilized majorly as a result of the simple nature of the resulting equations, leading to simple governing equations for the beam or elastic foundation problem.

Lumped parameter elastic foundation (LPEF) models have been proposed by several researchers. LPEF models include:

(i) Winkler model, also called a one-parameter LPEF model (Ike, 2018a; Ike, 2018b; Ike, 2023).
(ii) Pasternak, Vlasov, Hetenyi, and Filonenko-Borodich models, also called two-parameter LPEF models (Ike, 2023b; Ike, 2023c; Ike et al, 2023a).
(iii) Kerr (1985), a three-parameter LPEF model.

The Winkler's one-parameter LPEF model is illustrated in Figure 1, and it assumes that the soil behaves as a bed of vertical, independent, non-interacting, closely spaced, linearly elastic springs that obey Hooke's law. Consequently, the soil reaction at any point on the beam is directly proportional to the beam vertical deflection at the concerned point; and the constant of proportionality is the Winkler constant, k, that is used as the one parameter to define the soil reaction in the Winkler model. The resulting soil reaction equation is a simple equation, which also yields another simple equation when incorporated into the thin beam equation.

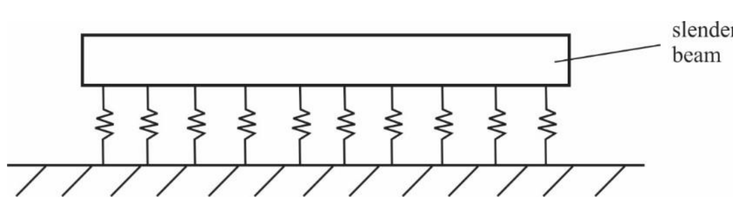

Figure 1: Thin beam on Winkler foundation and illustration of Winkler one-parameter lumped parameter elastic foundation (LPEF) model as a bed of non-interacting, vertical, linear elastic springs

As such, the Winkler foundation disregards the shear interactions of the vertical springs and yields discontinuity issues in deflections and/orslopes especially when the loading is a point load. Other foundation models were proposed to address the shortcomings of the Winkler one-parameter LPEF model. The two-parameter LPEF models were proposed variously by Pasternak, Vlasov, Hetenyi and Filonenko-Borodich as shown graphically in Figure 2.

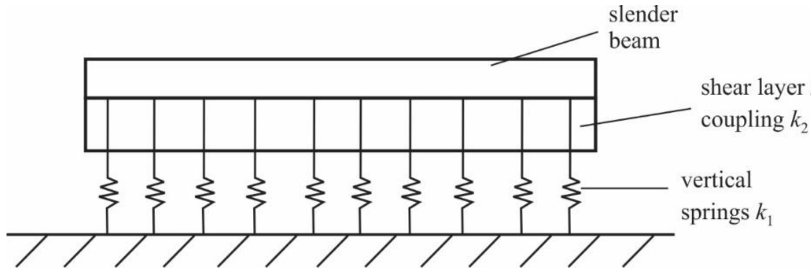

Figure 2: Thin beam on two-parameter LPEF with illustration of the two parameter LPEF as a bed of non-interacting, vertical, linear elastic springs with a shear coupling introduced at the interface of the vertical springs and the beam to model the shear interaction of the vertical springs.

As suggested by the name, two-parameter LPEF models utilize two parameters to determine the reaction of the soil on the thin beam. The first foundation parameter is analogous to the Winkler parameter k_1. The second foundation parameter k_2 accounts for the shear coupling effect of the vertical springs. The reaction pressure $r(x)$ is consequently expressed using the two parameters, resulting in another simple expression for $r(x)$.

Researchers who worked on continuously distributed foundation models include: Vlasov and Leontiev (1966), Jones and Xenophontos (1977), Vallabhan and Das (1988, 1991), Akhazhanov et al (2020, 2022, 2023a, 2023b), Huang et al (2019), Akhmediev et al (2023), and Zhang et al (2020).

1.2. Literature review

BoEFs which are under in-plane compressive forces can undergo failure by buckling, even when they have not attained their material strengths. This usually occurs when the compressive force reaches a certain critical threshold value, usually called the critical buckling load. It is thus significant for the analysis and design of EBBo2PFs under in-plane compressive loads to perform a buckling load analysis aimed at determining the least load that could cause buckling failures.

Literature review shows that the study of stability of beams on elastic foundations have been done using the following methods:

(i) Theory of elasticity methods
(ii) Finite element methods (FEMs)
(iii) Differential transform methods (DTMs)
(iv) Variational iteration methods (VIMs)
(v) Exact methods
(vi) Recursive differentiation methods (RDMs)
(vii) Point collocation methods (PCMs)
(viii) Finite sine transformation method (FSTM)
(ix) Generalized integral transform method (GITM)
(x) Fourier series methods
(xi) Stodola-Vianello iteration methods (SVIMs)

The methods of the mathematical theory of elasticity were used for BoEF problems by Gholami and Alizadeh (2022), Anyaegbunam (2014), and Thanh and Linh (2021), but failed to investigate stability problems.

FEMs were employed for BoEFs investigations by Mama et al (2020), Worku and Habte (2022), Alzubaidi et al (2023), Wieckowski and Swaitkiewicz (2021). Teodoru and Musat (2008) further investigated the use of FEM for EBBo2PF. Gulkan and Alemdar (1999) used the exact sinusoidal shape functions in the FEM to derive expressions for the stiffness matrices, nodal forces, and geometric matrices for a EBBo2PF with simply supported ends. They obtained general solutions that were comparable to those in the literature for simply supported boundary condition because exact sinusoidal series shape functions that satisfy all boundary conditions were used in the formulations. Soltani (2020) used FEM for solving the GDES of BoEF and found stability solutions for

EBBoWF for simple boundary conditions.

Olotu et al (2021) used DTMs to obtain numerical solutions for free transverse dynamic analysis of non-uniform beams rested on variable one-parameter LPEF. In their study, the Winkler coefficients varied along the longitudinal beam directions. The DTM adopted to the governing differential equation of motion which was variable in the coefficients, transformed the problem from a differential equation to an algebraic equation. They used algebraic matrix solvers in MAPLE computer codes to obtain accurate solutions for the beam vibration problems for fixed ends and simply supported ends. Their work however failed to investigate stability problems of EBBo2PFs.

Aslami and Akimov (2016) also worked on analytical solutions of flexural vibrations of EBBo2PFs with simply supported ends for continuously distributed parameter thin beams under vibration. Their work failed to investigate stability of EBBo2PFs.

Exact mathematical methods for solving the buckling problems of EBBo2PFs have been applied by Hetenyi (1946), Timoshenko and Gere (1985) and Wang et al (2005) for a variety of end supports for the stability problem. The exact solutions were derived by seeking closed form analytical solutions to the governing differential equations of stability (GDES) for the EBBo2PF such that the GDES are satisfied over the domain and the boundary conditions are simultaneously satisfied. This derivation of exact solutions requires rigorous mathematical techniques for solving ODEs and PDEs and exact solutions are unavailable for several cases of non-homogenous beam materials, non-prismatic beam cross-sections, variable foundations and complex boundaries. This explains the necessity for numerical methods that could achieve approximate, yet accurate solutions.

Hassan (2008) used the exact methods for solving ordinary differential equations to obtain solutions for buckling of EBBoEF for different types of supported ends. Aristizabal-Ochoa (2013) investigated the stability problems of EBBoEFs for various cases of end supports using approximate methods for solving ordinary differential equations.

Anghel and Mares (2019) obtained accurate critical buckling load solutions for EBBoEFs via collocation methods.

Atay and Coskun (2009) applied the VIM for accurate stability analysis of EBBoEF for cases of beams with prismatic and non-prismatic cross-sections.

Akgoz et al (2016) investigated the bending analysis of EBBoEF via the method of singular convolution but failed to consider buckling studies.

Hariz et al (2022) studied the stability problem of Timeshenko beam on two-parameter LPEFs. Yue (2021) used an iterative method for solving the thick Bo2PF where the beam is idealized as refined beam model.

Ike (2018a) applied the FSTM to obtain exact natural transverse vibration frequencies of prismatic cross-section EBBoWF, but did not consider buckling investigation of the EBBoWF. Ike (2022) has applied the GITM to obtain exact eigensolutions to the transversely vibrating EBBoWF, but did not consider buckling.

Ike (2018b) applied point collocation method (PCM) to the flexural analysis of EBBoWF, and obtained acceptable results; but did not consider buckling. Ike et al (2018) solved Euler buckling problems using Picard's iteration method and found accurate buckling load solutions to the eigenvalue problem.

Ikwueze et al (2018) applied least squares weighted residual method to find critical buckling load of Euler columns with fixed-pinned ends. Ofondu et al (2018) used the Stodola-Vianello iteration method (SVIM) to find acceptable approximate solutions to Euler column buckling analysis for clamped-pinned boundaries.

Ike et al (2023b) and Ike (2023d) applied the SVIM and polynomial displacement basis functions for the eigensolutions of EBBoWF where the beam has clamped-clamped and simple end supports respectively.

In another work, Ike (2023e) used the SVIM and exact trigonometric basis functions to solve the eigenvalue problems of EBBoWF with Dirichlet boundary conditions.

Ike (2023b) used SVM and exact shape functions for the exact eigen solution of EBBo2PFs. Ike et al (2023a) and Ike (2023c) have further used the SVIM for EBBo2PFs based on polynomial basis functions for clamped-clamped and simply supported boundaries, respectively.

Ike (2024) used the Fourier series method (FSM) to obtain exact stability solutions for EBBo2PFs with simply supported ends. The work used a Newtonian equilibrium technique to formulate the GDES in a first principles, rigorous method, and the FSM was adopted for the solution due to the ease of the Fourier series to undergo differentiation and integration, because of the inherent orthogonality properties.

Taha (2014) used a recursive differentiation method (RDM) for the approximate solutions of boundary value problems (BVPs) and specifically illustrated the application of RDM to EBBo2PFs under simply supported ends.

Taha and Hadima (2015) also presented RDM for buckling analysis of non-uniform BoEFs. Naidu and Rao (1995) presented stability solutions for EBBo2PFs for various end support conditions and values of the foundation parameters. Rao and Raju (2002) presented closed form solutions for the buckling analysis of EBBo2PFs for various foundation parameters and end support conditions.

Aristizabal-Ochoa (2013) has also investigated the stability of EBBoEF under various support condition cases.

In this work, the Ritz variational method is adopted to obtain buckling solutions for EBBo2PFs. The Ritz vartiational functional $\prod$ is derived for the thin beam resting on two-parameter LPEF by summing the strain energy of the beam, and the elastic foundation and the work potential of the in-plane compressive force. The principle of minimum potential energy is applied to minimize the Ritz functional $\prod$.

1.3. Novelty of the study

The novelty of the study is the first principles, systematic derivation of the Ritz functional for the EBBo2PF problem under in-plane compression; and the systematic application of the principle of minimum total energy to obtain the eigen equation of the system.

2. Theoretical framework of the studied EBBo2PF

The EBBo2PF studied is shown in Figure 3 for axial compressive loading by force *P*.

Figure 3: Euler-Bernoulli beam rested on two-parameter elastic foundation

The beam has a span *l* and the ends are supported according to the support conditions investigated in this paper.

2.1. Fundamental assumptions

The following are assumed:

(i) The thin beam material is linearly elastic, homogeneous and isotropic.
(ii) The thin beam is resting on a linearly elastic, homogeneous, isotropic two-parameter foundation.
(iii) The transverse displacements are considered to be very small with respect to the beam thickness.
(iv) The axial strains are so small and neglected.
(v) Normal strains in the transverse directions are so small and are insignificant.
(vi) Transverse shear stresses are infinitesimally small, and are neglected.
(vii) Middle planes to the beam cross-sections are plane and orthogonal to the longitudinal axis of the beam before and after deformations.

2.2. Displacement field

The displacement field components about the *x, y, z* Cartesian coordinate directions are:

$$u(x,y,z) = -z\frac{\partial w}{\partial x}$$
$$v(x,y,z) = 0 \qquad (1)$$
$$w(x,y,z) = w(x)$$

where $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ are the displacement field components about the *x, y, z* Cartesian coordinate directions, respectively.

2.3. Strain field

The strain field is found using the strain displacement equations of small displacement elasticity theory. Thus,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \qquad (2)$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

Since $w(x)$ does not vary with z

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \qquad (3)$$
$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} + \frac{\partial}{\partial z}\left(-z\frac{\partial w}{\partial x}\right) = 0$$

$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$ are normal strains in the *x*, *y*, *z* coordinate directions, $\gamma_{xy}, \gamma_{yz}, \gamma_{xz}$ are shear strains.

2.3.1 Stress fields

The stresses are found from the strain fields using the stress-strain relations. Thus,

$$\sigma_{xx} = E\varepsilon_{xx} = -Ez\frac{\partial^2 w}{\partial x^2} \qquad (4)$$
$$\sigma_{yy} = E\varepsilon_{yy} = 0$$
$$\sigma_{zz} = E\varepsilon_{zz} = 0$$
$$\tau_{xy} = G\gamma_{xy} = 0$$
$$\tau_{yz} = G\gamma_{yz} = 0$$
$$\tau_{xz} = G\gamma_{xz} = 0$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are normal stresses in the *x*, *y*, *z* directions, $\tau_{xy}, \tau_{yz}, \tau_{xz}$ are shear stresses, *E* is the Young's modulus of elasticity, and *G* is the shear modulus.

2.4. Strain Energy of Euler Bernoulli Beam (*SE$_b$*)

The strain energy (SE_b) of an Euler-Bernoulli beam is given by the triple integral over the beam domain as:

$$SE_b = \frac{1}{2}\int_{-b/2}^{b/2}\int_{0}^{l}\int_{-h/2}^{h/2}\left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right)dxdydz \qquad (5)$$

where $0 \le x \le l; -b/2 \le y \le b/2; -h/2 \le z \le h/2,$ *h* is the depth (thickness) of the beam cross-section, *b* is the width of the beam, *l* is the length of the beam.

Simplifying, the non-vanishing expression for SE_b is

$$SE_b = \frac{1}{2}\int_{0}^{l}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}\sigma_{xx}\varepsilon_{xx}\ dxdydz \qquad (6)$$

$$SE_b = \frac{1}{2}\int_{0}^{l}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}\left(-Ez\frac{\partial^2 w}{\partial x^2}\right)\left(-z\frac{\partial^2 w}{\partial x^2}\right)dxdydz \qquad (7)$$

$$SE_b = \frac{1}{2}\int_{0}^{l}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}Ez^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2\ dxdydz \qquad (8)$$

$$SE_b = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \qquad (9)$$

I is the moment of inertia of the beam cross section, where

$$I = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} z^2 dydz = \frac{bh^3}{12} \qquad (10)$$

2.5. Strain energy of the two-parameter elastic foundation (SE_f)

The strain energy of the two-parameter elastic foundation is:

$$SE_f = \frac{1}{2}\int_{-b/2}^{b/2}\int_0^l r_{s_1} w(x)dxdy + \frac{1}{2}\int_{-b/2}^{b/2}\int_0^l r_{s_2} w'(x)dxdy \qquad (11)$$

where r_{s1} and r_{s2} are the reactive pressures from the elastic foundation.
For two-parameter foundations, the reactive pressures r_{s1} and r_{s2} are:

$$r_{s_1} = k_1 w(x)$$
$$r_{s_2} = k_2 w'(x) = k_2\frac{dw}{dx} \qquad (12)$$

where k_1 and k_2 are the two parameters of the foundation.
Hence,

$$SE_f = \frac{1}{2}b\int_0^l k_1\left(w(x)\right)^2 dx + \frac{1}{2}b\int_0^l k_2\left(w'(x)\right)^2 dx \qquad (13)$$

2.6. Work potential of the axial load, P

The work potential W_p of the applied load P on the beam is given by Ike (2024) as:

$$W_P = \int_0^l P\frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx = \frac{1}{2}\int_0^l P\left(w'(x)\right)^2 dx \qquad (14)$$

2.7. Ritz total potential energy functional $(\prod)$

The total potential energy functional $(\prod)$ is:

$$\prod = SE_b + SE_f - W_p \qquad (15)$$

$$\prod = \frac{1}{2}\int_0^l EI\left(w''(x)\right)^2 dx + \frac{1}{2}\int_0^l k_1 b\left(w(x)\right)^2 dx + \frac{1}{2}\int_0^l k_2 b\left(w'(x)\right)^2 dx - \frac{1}{2}\int_0^l P\left(w'(x)\right)^2 dx \qquad (16)$$

Hence,

$$\prod = \frac{1}{2}\int_0^l \left\{EI(w'')^2 + k_1 b\left(w(x)\right)^2 + k_2 b\left(w'(x)\right)^2 - P\left(w'(x)\right)^2\right\} dx \qquad (17)$$

Let

$$k_1 b = \bar{k}_1$$
$$k_2 b = \bar{k}_2 \qquad (18)$$

Then

$$\prod = \frac{1}{2}\int_0^l \left\{EI\left(w''(x)\right)^2 + \bar{k}_1\left(w(x)\right)^2 + \bar{k}_2\left(w'(x)\right)^2 - P\left(w'(x)\right)^2\right\} dx \qquad (19)$$

Alternatively, for prismatic, homogeneous beams, *EI* is a constant which can be factored out to give Π as follows:

$$\Pi = \frac{EI}{2}\int_0^l \left\{ (w''(x))^2 + \frac{\bar{k}_1}{EI}(w(x))^2 + \frac{\bar{k}_2}{EI}(w'(x))^2 - \frac{P}{EI}(w'(x))^2 \right\} dx \tag{20}$$

Let

$$\frac{\bar{k}_1}{EI} = \alpha_1$$
$$\frac{\bar{k}_2}{EI} = \alpha_2 \tag{21}$$
$$\frac{P}{EI} = \beta$$

Then

$$\Pi = \frac{EI}{2}\int_0^l \left((w''(x))^2 + \alpha_1 (w(x))^2 + \alpha_2 (w'(x))^2 - \beta (w'(x))^2 \right) dx$$
$$= \frac{EI}{2}\int_0^l (w''(x))^2 + \alpha_1 (w(x))^2 + (\alpha_2 - \beta)(w'(x))^2 \, dx \tag{22}$$

3. Methodology

The Ritz variational methodology for solving the EBBo2PF buckling problem is illustrated for one-parameter buckling shape function, two-parameter buckling shape function, three-parameter buckling shape function and an *n*-parameter buckling shape function. The problem can be solved for any number of parameters used for buckling shape configuration. However, with increase in number of parameters, the solution accuracy is expected to increase. However, a one-parameter shape function that satisfies all boundary conditions can be used to achieve accurate results.

3.1. One-parameter buckling shape function

A one-parameter buckling shape function is given by:

$$w(x) = c_1 \varphi_1(x) \tag{23}$$

where c_1 is the generalized parameter of the shape function, $\varphi_1(x)$ is the buckling shape function which is constructed or chosen to satisfy both the displacement and force boundary conditions of the problem.

Then the Ritz functional for the EBBo2PF is:

$$\Pi = \frac{EI}{2}\int_0^l \left\{ (c_1 \varphi_1''(x))^2 + \alpha_1 (c_1 \varphi_1(x))^2 + (\alpha_2 - \beta)(c_1 \varphi_1'(x))^2 \right\} dx \tag{24}$$

Simplifying Equation (24) gives:

$$\Pi = \frac{EI}{2} c_1^2 \left\{ \int_0^l (\varphi_1''(x))^2 \, dx + \alpha_1 \int_0^l (\varphi_1(x))^2 \, dx + (\alpha_2 - \beta) \int_0^l (\varphi_1'(x))^2 \, dx \right\} \tag{25}$$

Let $I_1 = \int_0^l (\varphi_1''(x))^2 \, dx$

$$I_2 = \int_0^l (\varphi_1(x))^2 \, dx \tag{26}$$

$$I_3 = \int_0^l \left(\varphi_1'(x)\right)^2 dx$$

Then,

$$\prod = \frac{EI}{2} c_1^2 \left(I_1 + \alpha_1 I_2 + (\alpha_2 - \beta) I_3\right) = \prod(c_1) \quad (27)$$

Extrema of $\prod$ correspond to a zero of derivative $\prod$ with respect to c_1:

$$\frac{\partial \prod}{\partial c_1} = 0 \quad (28)$$

Hence,

$$EIc_1 \left(I_1 + \alpha_1 I_2 + (\alpha_2 - \beta) I_3\right) = 0 \quad (29)$$

Dividing by EI,

$$c_1 \left(I_1 + \alpha_1 I_2 + (\alpha_2 - \beta) I_3\right) = 0 \quad (30)$$

For nontrivial solutions, $c_1 \neq 0,$ the characteristic buckling equations is:

$$I_1 + \alpha_1 I_2 + (\alpha_2 - \beta) I_3 = 0 \quad (31)$$

Solving,

$$(\beta - \alpha_2) I_3 = I_1 + \alpha_1 I_2 \quad (32)$$

Dividing by I_3 gives:

$$\beta - \alpha_2 = \frac{I_1 + \alpha_1 I_2}{I_3} \quad (33)$$

Solving for β,

$$\beta = \alpha_2 + \frac{I_1 + \alpha_1 I_2}{I_3} = \frac{P}{EI} \quad (34)$$

Then,

$$P_{cr} = EI\left(\alpha_2 + \frac{I_1 + \alpha_1 I_2}{I_3}\right) \quad (35)$$

In standard form,

$$P = \frac{EI}{l^2}\left(\alpha_2 + \frac{I_1 + \alpha_1 I_2}{I_3}\right) l^2 \quad (36)$$

$$P = \frac{EI}{l^2} K_{cr}\left(\alpha_1 \ \alpha_2, l^2\right) \quad (37)$$

where

$$K_{cr} = \left(\alpha_2 + \frac{I_1 + \alpha_1 I_2}{I_3}\right) l^2 \quad (38)$$

3.2. Two-parameter buckling shape function

Here, $w(x) = c_1\varphi_1(x) + c_2\varphi_2(x)$ (39)

where $\varphi_1(x)$ and $\varphi_2(x)$ are the buckling shape functions which satisfy the boundary conditions, and c_1 and c_2 are the generalized parameters of $w(x)$.

The Ritz functional corresponding to this two-parameter buckling shape function is:

$$\prod = \frac{EI}{2} \int_0^l \left(\left(c_1\varphi_1''(x) + c_2\varphi_2''(x)\right)^2 + \alpha_1 \left(c_1\varphi_1(x) + c_2\varphi_2(x)\right)^2 + (\alpha_2 - \beta)\left(c_1\varphi_1'(x) + c_2\varphi_2'(x)\right)^2 \right) dx \quad (40)$$

Expanding,

$$\Pi = \frac{EI}{2}\int_0^l \left(c_1^2 \left(\varphi_1''(x)\right)^2 + 2c_1c_2\varphi_1''(x)\varphi_2''(x) + c_2^2 \left(\varphi_2''(x)\right)^2 \right) + \alpha_1 \left(c_1^2\varphi_1^2(x) + 2c_1c_2\varphi(x)\varphi_2(x) + c_2^2\varphi_2^2(x) \right)$$
$$+ (\alpha_2 - \beta)\left(c_1^2 \left(\varphi_1'(x)\right)^2 + 2c_1c_2\varphi_1'(x)\varphi_2'(x) + c_2^2 \left(\varphi_2'(x)\right)^2 \right)\Big) dx \tag{41}$$

The Ritz functional can be simplified as:

$$\Pi = \frac{EI}{2}\Bigg\{ c_1^2 \int_0^l \left(\varphi_1''(x)\right)^2 + \alpha_1 \left(\varphi_1(x)\right)^2 + (\alpha_2 - \beta)\left(\varphi_1'(x)^2\right) dx$$
$$+ 2c_1c_2 \int_0^l \left(\varphi_1''(x)\varphi_2''(x) + \alpha_1\varphi_1(x)\varphi_2(x) + (\alpha_2 - \beta)\left(\varphi_1'(x)\varphi_2'(x)\right)\right) dx$$
$$+ c_2^2 \int_0^l \left(\left(\varphi_2''(x)\right)^2 + \alpha_1 \left(\varphi_2(x)\right)^2 + (\alpha_2 - \beta)\left(\varphi_2'(x)\right)^2 \right) dx \Bigg\} = f(c_1, c_2) \tag{42}$$

For extremizing $\Pi(c_1, c_2)$ with respect to c_1 and c_2,

$$\frac{\partial \Pi}{\partial c_1} = 0$$
$$\frac{\partial \Pi}{\partial c_2} = 0 \tag{43}$$

Hence,

$$\frac{\partial \Pi}{\partial c_1} = \frac{EI}{2}\Bigg\{ 2c_1 \int_0^l \left(\left(\varphi_1''(x)\right)^2 + \alpha_1 \left(\varphi_1(x)\right)^2 + (\alpha_2 - \beta)\left(\varphi_1'(x)\right)^2 \right) dx$$
$$+ 2c_2 \int_0^l \left(\varphi_1''(x)\varphi_2''(x) + \alpha_1\varphi_1(x)\varphi_2(x) + (\alpha_2 - \beta)\varphi_1'(x)\varphi_2'(x) \right) dx \Bigg\} = 0 \tag{44}$$

$$\frac{\partial \Pi}{\partial c_2} = \frac{EI}{2}\Bigg\{ 2c_1 \int_0^l \left(\varphi_1''(x)\varphi_2''(x) + \alpha_1\varphi_1(x)\varphi_2(x) + (\alpha_2 - \beta)\left(\varphi_1'(x)\varphi_2'(x)\right)\right) dx$$
$$+ 2c_2 \int_0^l \left(\left(\varphi_2''(x)\right)^2 + \alpha_1 \left(\varphi_2(x)\right)^2 + (\alpha_2 - \beta)\left(\varphi_2'(x)\right)^2 \right) dx \Bigg\} \tag{45}$$

Simplifying,

$$EI\Bigg(c_1 \int_0^l \left(\left(\varphi_1''(x)\right)^2 + \alpha_1 \left(\varphi_1(x)\right)^2 + (\alpha_2 - \beta)\left(\varphi_1'(x)\right)^2 \right) dx$$
$$+ c_2 \int_0^l \left(\varphi_1''(x)\varphi_2''(x) + \alpha_1\varphi_1(x)\varphi_2(x) + (\alpha_2 - \beta)\varphi_1'(x)\varphi_2'(x) \right) dx \Bigg) = 0 \tag{46}$$

$$EI\Bigg(c_1 \int_0^l \left(\varphi_1''(x)\varphi_2''(x) + \alpha_1\varphi_1(x)\varphi_2(x) + (\alpha_2 - \beta)\varphi_1'(x)\varphi_2'(x) \right) dx$$
$$+ c_2 \int_0^l \left(\left(\varphi_2''(x)\right)^2 + \alpha_1 \left(\varphi_2(x)\right)^2 + (\alpha_2 - \beta)\left(\varphi_2'(x)\right)^2 \right) dx \Bigg) = 0 \tag{47}$$

But $EI \neq 0,$ hence we have:

$$c_1k_{11} + c_2k_{12} = 0$$
$$c_1k_{21} + c_2k_{22} = 0 \tag{48}$$

where,

$$k_{11} = \int_0^l \left(\left(\varphi_1''(x)\right)^2 + \alpha_1 \left(\varphi_1(x)\right)^2 + (\alpha_2 - \beta)\left(\varphi_1'(x)\right)^2 \right) dx \quad (49)$$

$$k_{12} = k_{21} = \int_0^l \left(\varphi_1''(x)\varphi_2''(x) + \alpha_1 \varphi_1(x)\varphi_2(x) + (\alpha_2 - \beta)\left(\varphi_1'(x)\varphi_2'(x)\right)^2 \right) dx \quad (50)$$

$$k_{22} = \int_0^l \left(\left(\varphi_2''(x)\right)^2 + \alpha_1 \left(\varphi_2(x)\right)^2 + (\alpha_2 - \beta)\left(\varphi_2'(x)\right)^2 \right) dx \quad (51)$$

In matrix form,

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (52)$$

For nontrivial solutions, the characteristic buckling equation is:

$$\begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix} = 0 \quad (53)$$

Expanding,

$$k_{11}k_{22} - k_{12}k_{21} = 0 \quad (54)$$

3.3. n-parameter buckling shape function

For Ritz solution using an *n*-parameter buckling shape function,

$$w(x) = \sum_{i=1}^{n} c_i \varphi_i(x) \quad (55)$$

Then

$$\prod = \frac{EI}{2} \int_0^l \left\{ \left(\sum_{i=1}^{n} c_i \varphi_i''(x) \right)^2 + \alpha_1 \left(\sum_{i=1}^{n} c_i \varphi_i(x) \right)^2 + (\alpha_2 - \beta) \left(\sum_{i=1}^{n} c_i \varphi_i'(x) \right)^2 \right\} dx = \prod(c_1, c_2, \ldots, c_n) \quad (56)$$

For extrema,

$$\frac{\partial \prod}{\partial c_1} = 0, \quad \frac{\partial \prod}{\partial c_2} = 0, \quad \vdots \quad \frac{\partial \prod}{\partial c_n} = 0 \quad (57)$$

Or, $\frac{\partial \prod}{\partial c_i} = 0$

where $i = 1, 2, \ldots, n$.

The conditions for extremum yield a system of n equations as follows:

$$\begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \\ 0 \end{pmatrix} \quad (58)$$

For nontrivial solutions,

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \neq 0$$

The characteristic buckling equation is

$$\begin{vmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{vmatrix} = 0 \qquad (59)$$

where k_{ij} are the elements of the buckling matrix.

4. Results

The results are presented for different boundary conditions of the EBBo2PF problem.

4.1. Results for EBBo2PF with clamped ends

The boundary conditions for EBBo2PF with both ends ($x = 0$, and $x = l$) clamped are:

$$\begin{aligned} w(0) &= w(l) = 0 \\ w'(0) &= w'(l) = 0 \end{aligned} \qquad (60)$$

Using trigonometric shape functions, one particular $\varphi(x)$ that satisfies the clamped boundary conditions is:

$$\varphi_1(x) = 1 - \cos\left(2\pi x/l\right) \qquad (61)$$

A one-parameter deflection function, would therefore be:

$$w(x) = c_1\varphi_1(x) = c_1\left(1 - \cos\left(2\pi x/l\right)\right) \qquad (62)$$

where c_1 is a, yet, undetermined deflection parameter.

Then, I_1, I_2 and I_3 are evaluated as:

$$I_1 = \int_0^l (\varphi_1''(x))^2\, dx = \int_0^l \left(\left(\frac{2\pi}{l}\right)^2 \cos\left(\frac{2\pi x}{l}\right)\right)^2 dx = \left(\frac{2\pi}{l}\right)^4 \int_0^l \cos^2\left(\frac{2\pi x}{l}\right) dx = \frac{16\pi^4}{l^4}\frac{l}{2} = \frac{8\pi^4}{l^3} \qquad (63)$$

$$I_2 = \int_0^l (\varphi_1(x))^2\, dx = \int_0^l \left(1 - \cos\left(\frac{2\pi x}{l}\right)\right)^2 dx = \int_0^l \left(1 - 2\cos\left(\frac{2\pi x}{l}\right) + \cos^2\left(\frac{2\pi x}{l}\right)\right) dx = \frac{3l}{2} \qquad (64)$$

$$I_3 = \int_0^l (\varphi_1'(x))^2\, dx = \int_0^l \left(\frac{2\pi}{l}\sin\left(\frac{2\pi x}{l}\right)\right)^2 dx = \int_0^l \left(\left(\frac{2\pi}{l}\right)^2 \sin^2\left(\frac{2\pi x}{l}\right)\right) dx$$

$$I_3 = \frac{4\pi^2}{l^2}\int_0^l \sin^2\left(\frac{2\pi x}{l}\right) dx = \frac{4\pi^2}{l^2}\frac{l}{2} = \frac{2\pi^2}{l} \qquad (65)$$

Substituting the above obtained particular vales of I_1, I_2, and I_3 into Equation (35) or Equation (36) the following is obtained:

$$P_{cr} = EI\left(\alpha_2 + \frac{4\pi^2}{l^2} + \frac{3l^2\alpha_1}{4\pi^2}\right) \qquad (66)$$

Or, in the standard form

$$P_{cr} = \frac{EI}{l^2}\left(\alpha_2 + \frac{4\pi^2}{l^2} + \frac{3l^2\alpha_1}{4\pi^2}\right)l^2 \quad (67)$$

The least value of P that causes buckling is the critical buckling load P_{cr} expressed as:

$$P_{cr} = \frac{EI}{l^2}\left(4\pi^2 + \alpha_2 l^2 + \frac{3\alpha_1 l^4}{4\pi^2}\right) \quad (68)$$

Equation (106) is expressed in terms of critical buckling load coefficient $K(\alpha_1, \alpha_2)$ as:

$$P_{cr} = \frac{EI}{l^2}K(\alpha_1, \alpha_2) \quad (69)$$

where,

$$K(\alpha_1, \alpha_2) = \frac{3}{4\pi^2}\alpha_1 l^4 + 4\pi^2 + \frac{\alpha_2 l^2}{\pi^2}\pi^2 \quad (70)$$

Let $\bar{\alpha}_2 = \frac{\alpha_2 l^2}{\pi^2} = \frac{k_2 l^2}{\pi^2 EI}$ (71)

Then,

$$K(\alpha_1, \alpha_2) = \frac{3}{4\pi^2}\alpha_1 l^4 + 4\pi^2 + \bar{\alpha}_2\pi^2$$

Further simplification gives:

$$K(\alpha_1, \alpha_2) = \frac{3}{4\pi^2}\alpha_1 l^4 + (4 + \bar{\alpha}_2)\pi^2 \quad (72)$$

The buckling load parameters $K(\alpha_1, \alpha_2)$ are calculated for values of $\alpha_1 l^4 = 0, 1, 100$ and for values of $\bar{\alpha}_2 = 0, 0.5, 1.0$ and 2.5 and presented in Table 1. Table 1 also shows $K(\alpha_1, \alpha_2)$ obtained by Rao and Raju (2002) and by Naidu and Rao (1995) using the Finite Element Method.

Table 1: Buckling load parameters of EBBo2PF with clamped ends at *x* = 0 and *x* = *l*

$\bar{\alpha}_2 = \alpha_2 l^2 / \pi^2$	Method / Reference	$\alpha_1 l^4$		
		0	1	100
0	Present study	39.4784176	39.5544089	47.07750638
	Rao and Raju (2002)	39.478	39.554	47.077
	FEM (Naidu and Rao, 1995)	39.479	39.555	47.077
0.5	Present study	44.4132198	44.48921069	52.01230858
	Rao and Raju (2002)	44.413	44.489	52.012
	FEM (Naidu and Rao, 1995)	44.414	44.490	51.542
1	Present study	49.34802201	49.02401289	56.94711078
	Rao and Raju (2002)	49.348	49.424	56.9471
	FEM (Naidu and Rao, 1995)	49.349	49.425	56.877
2.5	Present study	64.15242861	64.22841949	71.75151738
	Rao and Raju (2002)	64.152	64.228	71.751
	FEM (Naidu and Rao, 1995)	64.153	64.229	71.681

4.2. Results for EBBo2PF with clamped free ends

The boundary conditions are:

$$w(0)=0 \qquad w(l)=0$$
$$w'(0)=0 \qquad w'(l)=0 \tag{73}$$

A shape function that satisfies the geometric conditions, but does not satisfy the force boundary condition $w'''(l)=0$ is:

$$\varphi(x)=1-\cos\left(\frac{\pi x}{2l}\right) \tag{74}$$

Hence, $w(x)=c_1\left(1-\cos\left(\frac{\pi x}{2l}\right)\right)$ (75)

$$\frac{\partial \Pi}{\partial c_1}=0$$

$$I_1+\alpha_1 I_2+(\alpha_2-\beta)I_3=0$$

$$I_1=\int_0^l (\varphi_1''(x))^2\,dx=\int_0^l\left(\left(\frac{\pi}{2l}\right)^2\cos\left(\frac{\pi x}{2l}\right)\right)^2 dx \tag{76}$$

$$I_1=\frac{\pi^4}{16l^4}\int_0^l \cos^2\left(\frac{\pi x}{2l}\right)dx=\frac{\pi^4}{16l^4}\cdot\frac{l}{2}=\frac{\pi^4}{32l^3} \tag{77}$$

$$I_2=\int_0^l(\varphi_1(x))^2\,dx=\int_0^l\left(1-\cos\left(\frac{\pi x}{2l}\right)\right)^2 dx=\frac{(3\pi-8)l}{2\pi} \tag{78}$$

$$I_3=\int_0^l(\varphi_1'(x))^2\,dx=\int_0^l\left(\frac{\pi}{2l}\sin\left(\frac{\pi x}{2l}\right)\right)^2 dx$$

$$I_3=\left(\frac{\pi}{2l}\right)^2\int_0^l\sin^2\left(\frac{\pi x}{2l}\right)dx=\frac{\pi^2}{4l^2}\left(\frac{l}{2}\right)=\frac{\pi^2}{8l} \tag{79}$$

Hence,

$$\frac{\pi^4}{32l^3}+\alpha_1\frac{(3\pi-8)l}{2\pi}+(\alpha_2-\beta)\frac{\pi^2}{8l}=0 \tag{80}$$

Simplifying,

$$(\beta-\alpha_2)\frac{\pi^2}{8l}=\frac{\pi^4}{32l^3}+\alpha_1 l\left(\frac{3\pi-8}{2\pi}\right) \tag{81}$$

Further simplifying,

$$\beta-\alpha_2=\frac{8l}{\pi^2}\left(\frac{\pi^4}{32l^3}+\frac{\alpha_1 l(3\pi-8)}{2\pi}\right) \tag{82}$$

Hence,

$$\beta-\alpha_2=\frac{\pi^2}{4l^2}+\frac{8\alpha_1 l^2(3\pi-8)}{2\pi^3} \tag{83}$$

Then,

$$\beta-\alpha_2=\frac{\pi^2}{4l^2}+\frac{4\alpha_1 l^2(3\pi-8)}{\pi^3} \tag{84}$$

Making β the subject, we have:

$$\beta = \frac{P}{EI} = \alpha_2 + \frac{\pi^2}{4l^2} + \frac{4\alpha_1 l^2(3\pi - 8)}{\pi^3} \tag{85}$$

Expressed in terms of P, gives:

$$P = EI\left(\alpha_2 + \frac{\pi^2}{4l^2} + \frac{4\alpha_1 l^2(3\pi - 8)}{\pi^3}\right) \tag{86}$$

The critical value of P is P_{cr} which is:

$$P_{cr} = \frac{EI}{l^2}\left(\alpha_2 l^2 + \frac{\pi^2}{4} + \frac{4\alpha_1 l^4(3\pi - 8)}{\pi^3}\right) = \frac{EI}{l^2}K(\alpha_1, \alpha_2) \tag{87}$$

where:

$$K(\alpha_1, \alpha_2) = \alpha_2 l^2 + \frac{\pi^2}{4} + \frac{4(3\pi - 8)\alpha_1 l^4}{\pi^3} = \frac{\alpha_2 l^2}{\pi^2}\pi^2 + \frac{\pi^2}{4} + \frac{4(3\pi - 8)}{\pi^3}\alpha_1 l^4 \tag{88}$$

Alternatively,

$$K(\alpha_1, \alpha_2) = \pi^2\left(\bar{\alpha}_2 + 0.25\right) + \frac{4(3\pi - 8)}{\pi^3}\alpha_1 l^4$$

where,

$$\bar{\alpha}_2 = \frac{\alpha_2 l^2}{\pi^2} \tag{89}$$

$$K(\alpha_1 = 0, \bar{\alpha}_2 = 0) = 0.25\pi^2 = 2.4674011$$

$$K(\alpha_1 = 1, \bar{\alpha}_2 = 1) = 12.5208106$$

$K(\alpha_1, \alpha_2)$ are calculated for $\alpha_1 l^4 = 0, 1, 100$ and $\bar{\alpha}_2 = 0, 0.5, 1, 2.5$ and presented in Table 2 together with previous results by Naidu and Rao (1995) using the FEM, and Rao and Raju (2002).

Table 2: Buckling load coefficients of EBBo2PF clamped at *x* = 0 and free at *x* = *l*

$\bar{\alpha}_2 = \alpha_2 l^2 / \pi^2$	**Method / Reference**	$\alpha_1 l^4$		
		0	1	100
0	Present study	2.4674011	2.651206202	20.84791128
	Rao and Raju (2002)	2.4674	2.6512	20.848
	FEM (Naidu and Rao, 1995)	2.467	2.652	20.848
0.5	Present study	7.402203301	7.586008403	25.78271349
	Rao and Raju (2002)	7.4022	7.5860	25.783
	FEM (Naidu and Rao, 1995)	7.402	7.591	25.79
1	Present study	12.3370055	12.5208106	30.71751569
	Rao and Raju (2002)	12.337	12.521	30.717
	FEM (Naidu and Rao, 1995)	12.337	12.521	30.718
2.5	Present study	27.1414121	27.3252172	45.52192229
	Rao and Raju (2002)	27.141	27.325	45.522
	FEM (Naidu and Rao, 1995)	27.142	27.325	45.522

4.3. Results for EBBo2PF with simply supported ends

The boundary conditions for EBBo2PF with both ends ($x = 0$ and $x = l$) simply supported are:

$$w(0) = w(l) = 0$$
$$M(0) = M(l) = 0 \tag{90}$$

where $M(x)$ is the bending moment at x.

Hence using the bending moment-deflection equation, the force boundary conditions are expressed using deflections as:

$$w''(0) = w''(l) = 0 \quad (91)$$

Suitable functions, $\varphi(x)$, can be found as:

$$\varphi_n(x) = \sin\left(\frac{n\pi x}{l}\right), \qquad n = 1, 2, 3, \ldots \quad (92)$$

Then, for an analytical solution, $w(x)$ is given in the infinite sine series:

$$w(x) = \sum_{i=1}^{\infty} c_i \sin\left(\frac{n\pi x}{l}\right) \quad (93)$$

$i = 1, 2, 3, \ldots, \infty$; c_i are the generalized displacement parameters.

For a truncated series solution,

$$w(x) = \sum_{i=1}^{n} c_i \sin\left(\frac{i\pi x}{l}\right) \quad (94)$$

Then, for n-parameter Ritz buckling solutions, the Ritz functional $\prod$ is:

$$\prod = \frac{EI}{2}\left\{\int_0^l \left(\sum_{i=1}^{n}\left(c_i\varphi_1''(x)\right)^2 + \alpha_1\left(\sum_{i=1}^{n} c_i\varphi_1(x)\right)^2 + (\alpha_2 - \beta)\left(\sum_{i=1}^{n} c_i\varphi_i'(x)\right)^2\right)dx\right\} = \prod(c_1, c_2, \ldots, c_n) \quad (95)$$

Simplification of Equation (95) gives:

$$\prod = \frac{EI}{2}\left\{\int_0^l \left\{\sum_{i=1}^{n}\left(-\left(\frac{i\pi}{l}\right)^2 c_1 \sin\left(\frac{i\pi x}{l}\right)\right)^2 + \alpha_1\sum_{i=1}^{n}\left(c_i \sin\left(\frac{i\pi x}{l}\right)\right)^2 + (\alpha_2 - \beta)\left(\sum_{i=1}^{n}\left(\frac{i\pi}{l}\right)c_1 \cos\left(\frac{i\pi x}{l}\right)\right)^2\right\}dx\right\} \quad (96)$$

Hence,

$$\prod = \frac{EI}{2}\left\{c_i c_j \int_0^l \left\{\left(\frac{i\pi}{l}\right)^2\left(\frac{j\pi}{l}\right)^2 \sin\left(\frac{i\pi x}{l}\right)\sin\left(\frac{j\pi x}{l}\right) + \alpha_1 \sin\left(\frac{i\pi x}{l}\right)\sin\left(\frac{j\pi x}{l}\right) + (\alpha_2 - \beta)\left(\frac{i\pi}{l}\right)\left(\frac{j\pi}{l}\right)\cos\left(\frac{i\pi x}{l}\right)\cos\left(\frac{j\pi x}{l}\right)\right\}dx\right\} \quad (97)$$

($i = 1, 2, 3, \ldots, n$; $j = 1, 2, 3, \ldots, n$)

Thus,

$$\prod = \frac{EI}{2}\left\{c_i c_j \left(\frac{i\pi}{l}\right)^2\left(\frac{j\pi}{l}\right)^2 I_1 + \alpha_1 I_1 + (\alpha_2 - \beta)\frac{i\pi}{l}\frac{j\pi}{l} I_2\right\} \quad (98)$$

where

$$I_1 = \int_0^l \sin\left(\frac{i\pi x}{l}\right)\sin\left(\frac{j\pi x}{l}\right)dx \quad (99a)$$

$$I_2 = \int_0^l \cos\left(\frac{i\pi x}{l}\right)\cos\left(\frac{j\pi x}{l}\right)dx \quad (99b)$$

Using the orthogonality properties of the trigonometric functions, the integrals I_1 and I_2 are easily evaluated.

$$I_1 = \int_0^l \sin\left(\frac{i\pi x}{l}\right)\sin\left(\frac{j\pi x}{l}\right)dx = \begin{cases} 0 & i \neq j \\ l/2 & i = j \end{cases} \quad (100a)$$

$$I_2 = \int_0^l \cos\left(\frac{i\pi x}{l}\right)\cos\left(\frac{j\pi x}{l}\right)dx = \begin{cases} 0 & i \neq j \\ l/2 & i = j \end{cases} \tag{100b}$$

Hence for nontrivial solutions, $i = j$ and $I_1 = I_2 = l/2$

Then,

$$\Pi = \frac{EIl}{4}\left\{c_i^2\left(\left(\frac{i\pi}{l}\right)^4 + \alpha_1 + (\alpha_2 - \beta)\left(\frac{i\pi}{l}\right)^2\right)\right\} \qquad (i = 1, 2, 3, \ldots, n) \tag{101}$$

From the principle of minimization of total potential energy, the functional Π is minimum with respect to c_i when

$$\frac{\partial \Pi}{\partial c_i} = 0 \tag{102}$$

$(i = 1, 2, 3, \ldots, n)$

Hence,

$$\frac{\partial \Pi}{\partial c_i} = \frac{EIl}{4}\left\{2c_i\left(\left(\frac{i\pi}{l}\right)^4 + \alpha_1 + (\alpha_2 - \beta)\left(\frac{i\pi}{l}\right)^2\right)\right\} = 0 \tag{103}$$

This is an algebraic homogeneous eigenvalue equation.

For nontrivial solutions, $c_i \neq 0$, the characteristic buckling equation is:

$$\left(\frac{i\pi}{l}\right)^4 + \alpha_1 + (\alpha_2 - \beta)\left(\frac{i\pi}{l}\right)^2 = 0 \tag{104}$$

Solving for β, gives:

$$(\beta - \alpha_2)\left(\frac{i\pi}{l}\right)^2 = \left(\frac{i\pi}{l}\right)^4 + \alpha_1 \tag{105}$$

Simplifying,

$$\beta - \alpha_2 = \left(\frac{l}{i\pi}\right)^2\left(\left(\frac{i\pi}{l}\right)^4 + \alpha_1\right) = \left(\frac{i\pi}{l}\right)^2 + \alpha_1\left(\frac{l}{i\pi}\right)^2 \tag{106}$$

Making β the subject gives:

$$\beta = \alpha_2 + \alpha_1\left(\frac{l}{i\pi}\right)^2 + \left(\frac{i\pi}{l}\right)^2 = \frac{P}{EI} \tag{107}$$

Hence,

$$P = EI\left(\alpha_2 + \alpha_1\left(\frac{l}{i\pi}\right)^2 + \left(\frac{i\pi}{l}\right)^2\right) \tag{108}$$

Expressing in the standard form,

$$P = \frac{EI}{l^2}\left(\alpha_2 + \alpha_1\left(\frac{l}{i\pi}\right)^2 + \left(\frac{i\pi}{l}\right)^2\right)l^2 = \frac{EI}{l^2}\left(\alpha_2 l^2 + \frac{\alpha_1 l^4}{(i\pi)^2} + (i\pi)^2\right) \tag{109}$$

Thus,

$$P = \frac{EI}{l^2}K_{b_i}(\alpha_1, \alpha_2) \tag{110}$$

where K_{bi} is the ith buckling load coefficient

$$K_{b_i}(\alpha_1, \alpha_2) = \alpha_2 l^2 + \frac{\alpha_1 l^4}{(i\pi)^2} + (i\pi)^2 \tag{111}$$

The least value of P occurs when $i = 1$ and $K_{b(i=1)}$ is called the critical buckling load parameter, K_{bcr}

$$K_{b(i=1)} = \left(\alpha_2 l^2 + \frac{\alpha_1 l^4}{\pi^2} + \pi^2\right) = K_{b_{cr}} = \lambda^2 \quad (112)$$

where λ is a buckling load parameter related to $K_{b_{cr}}$

$$P_{cr} = \frac{EI}{l^2} K_{b(i=1)} = \frac{EI}{l^2}\left(\alpha_2 l^2 + \frac{\alpha_1 l^4}{\pi^2} + \pi^2\right)$$

$$P_{cr} = \frac{EI}{l^2}\lambda^2 \quad (113)$$

$$P_{cr} = \frac{EI}{l^2}\left(\frac{\alpha_2 l^2}{\pi^2} + \frac{\alpha_1 l^4}{\pi^4} + 1\right)\pi^2 = \frac{EI}{l^2}\lambda^2$$

$$P_{cr} = \frac{EI}{l^2}(\bar{\alpha}_2 + \bar{\alpha}_1 + 1)\pi^2 = \frac{EI}{l^2}\lambda^2$$

Values of $K_{b_{cr}} = \lambda^2$ are calculated for $\alpha_1 l^4 = 0$, and $\alpha_2 \left(l/\pi\right)^2 = 0$, 1, and 2.5, and for $\alpha_1 l^4 = 100$ and $\alpha_2 \left(l/\pi\right)^2 = 0$, 1, 2.5; and presented as λ in Table 3, along with previous results presented by Ike (2023b, 2024), Taha (2014), Anghel and Mares (2019) and Ike (2023c).

Table 3
Critical buckling load coefficient $\lambda = \sqrt{K_{cr}}$ of EBBo2PF with simply supported ends (x = 0, and x = l)

$\alpha_1 l^4$	$\bar{\alpha}_2 = \alpha_2 \left(l/\pi\right)^2 = 0$				
	Taha (2014)	Anghel and Mares (2019)	Ike (2023b, 2024)	Ike (2023c)	Present
0	3.1415	3.1413	3.141593	3.143621	3.141593
100	4.4723	4.4721	4.472329	4.473579	4.472329
	$\bar{\alpha}_2 = \alpha_2 \left(l/\pi\right)^2 = 1$				
	Taha (2014)	Anghel and Mares (2019)	Ike (2023b, 2024)	Ike (2023c)	Present
0	4.4428	4.4427	4.44283	4.444317	4.44283
100	5.4654	5.4653	5.465467	5.466505	5.465467
	$\bar{\alpha}_2 = \alpha_2 \left(l/\pi\right)^2 = 2.5$				
	Taha (2014)	Anghel and Mares (2019)	Ike (2023b, 2024)	Ike (2023c)	Present
0	5.8774	5.8772	5.877382	5.878466	5.877382
100	6.6840	6.6838	6.683991	6.68484	6.683991

5. Discussion

5.1. General discussion

This article has presented Ritz variational method (RVM) for buckling solutions of EBBo2PFs. The Ritz total potential energy functional $\prod$ was derived for the EBBo2PF under in-plane compressive force. This functional, $\prod$, was found as the sum of the strain energies of EBB, the two-parameter LPEF, and the work potential due to the in-plane compressive load. The displacement field used was that of the EBBT and the strain field components were found using small displacement linear elasticity theory. Stress fields were found from the strain field using one-dimensional constitutive relations.

The Ritz functional, $\prod$, was constructed as a function of x, $w(x)$, $w'(x)$ and $w''(x)$. The principle of minimization of $\prod$ was used to find $w(x)$ corresponding to minimum $\prod$. The boundary conditions considered were:

(i) EBBo2PFs with clamped ends at $x = 0$, and $x = l$
(ii) EBBo2PFs clamped at $x = 0$, and free at $x = l$
(iii) EBBo2PFs with simple supports at $x = 0$, and $x = l$.

For each of the boundary conditions considered, $w(x)$ was determined in terms of generalized unknown buckling parameters c_i, and buckling shape functions $\varphi_i(x)$ constructed to satisfy the boundary conditions. Thus $\prod$ become expressed in terms of the generalized unknown buckling parameters c_i as $\prod(c_1, c_2, \ldots, c_n)$ for an n-parameter Ritz formulation.

The criterion for the calculus of minimization of the Ritz functional, $\prod$, was then used to determine the eigenvalue equation for the problem as an algebraic equation in terms of the unknown parameters. The condition for nontrivial solutions of the eigenvalue problem was used to determine the characteristic buckling equation.

5.2. Discussion on EBBo2PF with clamped ends

A one-parameter Ritz formulation was used for EBBo2PFs with clamped ends. The buckling shape function was expressed in trigonometric basis functions as Equation (61) and the buckling deflection function expressed as Equation (62). By substitution in the Ritz functional and minimization, the characteristic buckling equation was found as Equation (66). Solving Equation (66) yielded the critical buckling load P_{cr} as Equation (68). The buckling load coefficients $K(\alpha_1, \alpha_2)$ for the EBBo2PFs with clamped ends were calculated for $\alpha_1 l^4 = 0, 1,$ and 100 for $\bar{\alpha}_2 = 0, 0.5, 1.0,$ and 2.5 and shown in Table 1 along with previous solutions by Rao and Raju (2002), and by Naidu and Rao (1995) using the FEM. Table 1 illustrates that the present results are closely similar to previous results by Naidu and Rao (1995) via the FEM and the results by Rao and Raju (2002).

5.3. Discussion on results for EBBo2PFs with clamped-free ends

In this case, a one-parameter buckling shape function shown in Equation (74) was used to express the buckling function as Equation (75). The algebraic eigenvalue problem obtained by minimizing $\prod$ with respect to c_i gave Equation (80) which was solved to obtain the critical buckling load P_{cr} expression given by Equation (77). The critical buckling load coefficients $K(\alpha_1, \alpha_2)$ were calculated for various values of $\alpha_1 l^4 = 0, 1, 100$ and for $\bar{\alpha}_2 = 0, 0.5, 1$ and 2.5, and presented in Table 2; along with previous results from the literature by the FEM method and by Rao and Raju (2002). Table 2 shows that the present Ritz results are similar to the previous results by Naidu and Rao (1995) using the FEM and by Rao and Raju (2002).

5.4. Discussion on results for EBBo2PF with simply supported ends

In this case n-parameter buckling function is constructed as Equation (93) using sinusoidal basis functions in Equation (92) which satisfy the boundary conditions of simple supports at the beam ends. Substitution of Equation (94) into the Ritz functional gave $\prod$ as $\prod(c_1, c_2, \ldots, c_n)$ which is shown explicitly as Equation (101). The eigenvalue equation is obtained for minimization of $\prod$ as Equation (103). The conditions for nontrivial solutions yield the characteristic buckling equation as Equation (104). The eigenvalue is found as the buckling load expression given for the ith buckling load as Equation (109). The buckling load coefficient for the ith buckling mode is given by Equation (111). The least buckling load is found for the first buckling mode when $i = 1$ and thus the critical buckling load P_{cr} for this case is found as Equation (113). The critical buckling load coefficient given by Equation (112) is evaluated for various values of $\alpha_1 l^4 = 0, 1, 100$ and for $\bar{\alpha}_2 = 0, 1,$ and 2.5; and presented in Table 3, along with previous results by Taha (2014), Anghel and Mares (2019), Ike (2023b, 2023c, 2024). Table 3 illustrates that the present RVM results are identical with results obtained by Ike (2023b, 2024). The identical results obtained in this work and the previous results by Ike (2023b, 2024) was because the exact buckling shape functions were used in this work (for the case of simply supported boundaries) and those research works that applied the SVIM and the Fourier series method. Similarly, the present RVM results were similar to previous results by Ike (2023c), which used the polynomial basis functions in the SVIM for the eigensolution of the problem. Table 3 further shows that the present RVM results are similar to previous results by Taha (2014), Anghel and Mares (2019).

6. Conclusion

This article has studied the Ritz variational method for the buckling load solutions of EBBo2PFs under in-plane compressive load, *P*. The study was done for three cases of boundary conditions; namely:

(i) EBBo2PFs with clamped ends at $x = 0$, and $x = l$
(ii) EBBo2PFs clamped at $x = 0$, and free at $x = l$
(iii) EBBo2PFs with simple supports at $x = 0$, and $x = l$.

In conclusion,

(i) The results for critical buckling load for EBBo2PF with clamped ends are closely similar to previous results that used the FEM and results by Rao and Raju (2002).
(ii) The present RVM results for critical buckling load are similar to previous results in the literature that used the FEM and results by Rao and Raju (2002).
(iii) The present RVM critical buckling load solutions for EBBo2PFs with simple end supports are identical with previous solutions that used the exact sinusoidal buckling shape functions in the SVIM and the Fourier series method (FSM). The present RVM results for critical buckling loads are similar to pervious results that used fourth degree polynomial shape functions in the SVIM, and other previous solutions by Taha (2014) and Anghel and Mares (2019) who applied collocation methods. The present RVM results for simply supported EBBo2PFs are exact because exact buckling shape functions were used to construct the solutions, and the total potential energy functional $\prod$ was minimized everywhere in the domain and all the boundary conditions were also satisfied.
(iv) Expectedly, the critical buckling load solutions obtained in the present RVM study for EBBo2PFs reduced to the critical buckling load solutions for EBBoWFs when the second foundation parameter α_2 became equal to zero.

Notation

x, y, z	three dimensional Cartesian coordinates
z	transverse coordinate
x	longitudinal coordinate
y	coordinate determining the beam width
u	displacement in longitudinal (axial) x direction
v	displacement in the y direction
w	displacement in the z direction
ε_{xx}	normal strain in x direction
ε_{yy}	normal strain in y direction
ε_{zz}	normal strain in in the transverse z direction
γ_{xy}, γ_{yz}	shear strains
γ_{xz}	transverse shear strain
E	Young's modulus of elasticity
G	shear modulus
σ_{xx}	normal stress in x direction
σ_{yy}	normal stress in y direction
σ_{zz}	normal stress in z direction
τ_{xy}, τ_{yz}	shear stresses
τ_{xz}	transverse shear stress
h	depth (thickness) of beam
b	breath of beam
l	span (length) of beam

SE_b	strain energy of thin beam in bending
I	moment of inertia of beam cross-section
SE_f	strain energy of the two-parameter elastic foundation
r_{s1}	reaction pressure from the two-parameter foundation corresponding to the first Winkler-parameter
r_{s2}	reaction pressure from the two-parameter foundation corresponding to the second parameter of the foundation
k_1	Winkler foundation parameter or first parameter of the two-parameter elastic foundation
k_2	second foundation parameter of the two-parameter elastic foundation
$\bar{k}_1$	foundation parameter defined in terms of k_1 and b
$\bar{k}_2$	foundation parameter defined in terms of k_2 and b
p	axial compressive force
W_p	work potential of the applied load
Π	total potential energy functional
α_1	parameter defined in terms of $\bar{k}_1$, and EI
α_2	parameter defined in terms of $\bar{k}_2$, and EI
β	compressive load parameter defined in terms of P and the beam properties EI
$F(x, w(x), w'(x), w''(x))$	integrand in the total potential energy functional
Γ	integral
$\varphi_i(x)$	i buckling shape function
$M(x)$	bending moment distribution
c_i	i generalized parameter of the buckling function
I_1	integral defined in terms of $\varphi_i''(x)$
I_2	integral defined in terms of $\varphi_i'(x)$
I_3	integral defined in terms of $\varphi_i'(x)$
P_{cr}	critical buckling load
K_{bcr}	critical buckling load coefficient
K_{ij}	elements of the buckling matrix
$\bar{\alpha}_1$	parameter defined in terms of α_1 and l
$\bar{\alpha}_2$	parameter defined in terms of α_2, l and π
λ	buckling load parameter defined in terms of $\sqrt{K_{bcr}}$
$\lvert \; \rvert$	determinant
cos	cosine function
sin	sine function
$\frac{\partial}{\partial x}$	partial differential operator with respect to x
$\frac{\partial}{\partial c_i}$	partial differential operator with respect to c_i
$\int_0^l (\;)dx$	integration with respect to x between the limits $x = 0$, and $x = l$
EBBo2PF	Euler-Bernoulli beam on two-parameter elastic foundation
EBB	Euler-Bernoulli beam
EBBT	Euler Bernoulli beam theory
CBT	classical beam theory
TBT	Timoshenko beam theory
EBBoWF	Euler-Bernoulli beam on Winkler foundation
EBBoEF	Euler-Bernoulli beam on elastic foundation
VIM	variational iteration method
GDES	governing differential equation(s) of stability
GITM	generalized integral transform method
SVIM	Stodola-Vianello iteration method
BoEF	beam on elastic foundation
DTM	differential transform method
RDM	recursive differentiation method
PCM	point collocation method
FSM	Fourier series method

ODEs	ordinary differential equations
PDEs	partial differential equations
BVPs	boundary value problems
LPEF	lumped parameter elastic foundation
RVM	Ritz variational method
FEM	Finite element methods

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