Iraqi journal of civil engineering (IJCE)
P-ISSN: 1992-7428 ; E-ISSN: 2706-686X

Vol. 18 , No. 2 (2024) , pp. 73 ~ 87
DOI: 10.37650/ijce.2024.180206

Iraqi Journal of Civil Engineering

Journal homepage: https://ijce.uoanbar.edu.iq/

Stodola-Vianelo Iteration Method for Solving Transverse Harmonic Natural Vibration Problems of Euler-Bernoulli Beams on Winkler Foundations

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PAPER INFO

Paper history (8pt)
Received: 10 /04/2024
Revised: 07 /10/2024
Accepted: 18 /11/2024

Keywords:
Stodola-Vianello iteration method
Euler-Bernoulli beam on Winkler foundation
Natural vibration frequency
Beam-foundation parameter
Eigenequation
Convergence criterion

ABSTRACT

Free transverse vibration frequency analysis of Euler-Bernoulli beams on Winkler foundation (EBBoWF) is a significant part of their analysis for averting failures by resonance. Resonant failure of EBBoWF occurs when the loading frequency exciting the vibration coincides with the least natural frequency. This study aims at using the Stodola-Vianello iteration method (SVIM) for the natural transverse vibration analysis of EBBoWF. Generally, the problem is governed by a non-homogenous partial differential equation (PDE) for forced vibrations, but simplifies to a homogeneous PDE for free vibrations where excitation forces are absent. For harmonic vibrations, and harmonic displacement response $u(x, t)$, the equations are decoupled in terms of the independent spatial and time variables, resulting in a fourth order ordinary differential equation (ODE) in $\bar{U}(x)$, the displacement modal function for $u(x, t)$. The study's focus is on homogenous, prismatic, isotropic thin beams leading to ODEs with constant parameters. SVIM was used to express the ODE as Stodola-Vianello iteration equations with four constants of integration, determinable via the boundary conditions. Specific application of SVIM to the EBBoWF with simple end supports used exact sinusoidal shape functions and boundary conditions to determine the integration constants. Convergence criterion at the nth iteration was used to find the eigenequation which was solved for the eigenvalues. The natural transverse vibration frequencies ω_n at the nth modes were found in terms of frequency parameters $\bar{\lambda}_n$. Values of $\bar{\lambda}_n$ calculated for the first five modes n = 1, 2, 3, 4, 5, and for values of $\alpha_1^4 l^4 = 1, 10, 100, 1000, 10000$ showed that the present SVIM gave exact results compared to other previous results. The exact solutions were obtained because exact shape functions were used in the SVIM equations resulting in satisfaction of the governing equations at the domain and the boundaries.

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1. Introduction

The eigenfrequency analysis of Euler-Bernoulli beams resting on Winkler foundations (EBBoWFs) under transverse free vibrations is important for determining the natural frequencies and in order for design to avert resonance failures. Resonant failures of dynamic systems occur when the excitation frequencies coincide with the structures natural vibration frequencies. The determination of the structures' natural vibration systems thus become vital in design against resonance, as they are the excitation frequencies at which the systems amplitude of vibration become infinitely large (Al-Anbaki and Pavic, 2017).

Studies on beams resting on elastic foundations have always proceeded with theoretical models of beams and the supporting soil which is generally described as elastic foundation.

A typical simply supported beam with rectangular cross-section is shown in Figure 1. The breadth, thickness and span are identified on Figure 1.

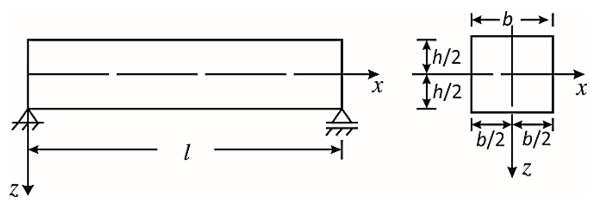

Fig. 1 Typical simply supported beam with rectangular cross-section

Beams have been modelled theoretically based on their ratios of thickness (h) to span (l) as thin, moderately thick or thick. Beams with $h/l \leq 0.05$ are called thin or slender beams. When $0.05 < h/l < 0.10$, the beam is called moderately thick. When $h/l \geq 0.10$, the beam is called a thick beam.

Thin beam problem is the one most commonly encountered and is the subject of this work. Thin beam theory, commonly called Euler-Bernoulli beam theory (EBBT), was first formulated by Euler and independently by Bernoulli. The EBBT formulation relied on the Euler-Bernoulli-Navier orthogonality hypothesis which is that the plane cross-sections that are orthogonal to the middle plane of the longitudinal axis of the beam would remain plane and orthogonal to the middle plane after deformation (Ike, 2018a, 2018b, 2021).

It is further assumed that the middle surface of the beam's longitudinal axis is free of stretching and strains, and is a neutral surface in bending deformation under dynamic or static applied loads.

The implication of the EBBT orthogonality hypothesis is that the shear deformations that are responsible for distortions of the plane cross-section are disregarded or absent. This renders the EBBT limited to the analysis of thin beams for which shear deformation effects do not alter significantly the bending, vibration and buckling behaviours.

Efforts to construct beam theories that adequately consider shear deformation effects have resulted in Timoshenko beam theory (TBT), shear deformation beam theories (SDBTs), and refined beam theories (ReBTs) (Sayyad and Avhad, 2019; Ike, 2022; Geetha et al., 2023; Sohani and Eipakchi, 2021).

TBT is a first order shear deformation beam theory (FSDBT) that yields a constant shear stress across the thickness, and thus violates the shear stress free boundary conditions at the beam surfaces ($z = \pm 0.5h$) (Simsek, 2016; Ike, 2019). TBT needs shear correction factors in order to predict accurate stress fields for the thick beam problem. Other SDBTs and ReBTs do not need shear correction factors and result in transverse shear variation profiles that comply with the shear stress free boundary conditions at the beam top and bottom surfaces (Razouki et al., 2020; Nguyen et al., 2022; Ghumare and Sayyad, 2017; Sayyad and Ghugal, 2017; Emadi et al., 2023; Ike 2022).

1.1. Foundation Models

Models for elastic foundations have been proposed by Winkler, Pasternak, Hetenyi, Vlasov, and Kerr as one-, two- and three parameter lumped models (Boudaa et al., 2021; Al-Azzawi and Daud, 2020, Akhazhanov et al., 2023). The lumped parameters can be constant or variable parameters (Soltani and Asgarian, 2019; Motaghian et al., 2018; Al-Azzawi and Daud, 2020). For elastic foundations with variable parameters, the lumped parameter values present a variation with the beams longitudinal coordinate variable, resulting in more complicated soil reactions, and yielding more complex Euler-Bernoulli beam on elastic foundation (EBBoEF) equations with variable coefficients. Such problems present more difficult solution algorithms.

Winkler foundation model is a one-parameter model that assumes that the soil reaction at any point on the beam is directly proportional to the beam's deflection at that point. This yields a simple equation for the soil reaction where the proportionality constant is the Winkler modulus, which can vary with the longitudinal coordinate of the beam, for variable Winkler foundations (Ike, 2018a, 2018b; Mutman and Coskun, 2013; Kacer et al., 2011; Ike, 2023a, 2023b, 2023c, Ike et al., 2023a). A typical figure of a beam resting on Winkler elastic foundation is shown in Figure 2.

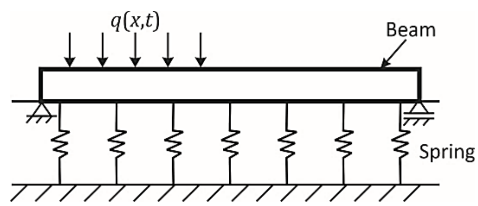

Fig. 2 Model of beam resting on a Winkler elastic foundation

Pasternak, Vlasov, Hetenyi and Filonenko-Borodich foundations are two-parameter foundations, while Kerr foundation is a three-parameter lumped idealization (Dutta et al., 2021; Ike 2023a, Ike et al., 2023b). The two-parameter-foundations overcome the lapses of lack of continuity between the vertical springs of the one-parameter model by introducing a second shear coupling parameter to ensure the continuity of the deformation between springs. The resulting soil reaction is also simple and results in equally simple beam on two-parameter foundation equations over the domain. A typical beam resting on a two-parameter foundation is shown in shown in Figure 3.

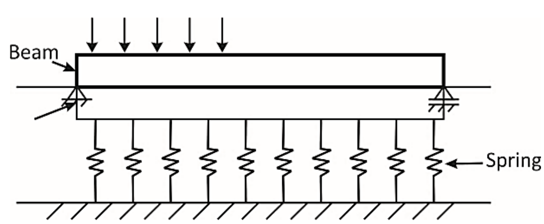

Fig. 3 Model of beam resting on a two-parameter elastic foundation

Another elastic foundation model which is derived using the theory of elasticity is the continuum elastic foundation model. As implied by the name, the soil is modeled using a three-dimensional (3D) continuum model of the theory of elasticity. The 3D continuum model can be reduced to two-dimensional (2D) continuum models for beams. However, in each case, the elastic continuum model yields complicated soil reaction equations which also result to more complicated thin beam on elastic foundation equations.

1.2. Literature Review

Mutman and Coskun (2013) investigated the natural vibrations of nonhomogeneous EBBoWF using the Homotopy perturbation method (HPM) and obtained accurate natural frequencies for various end support cases.

Balkaya et al. (2009) studied the natural frequencies analysis of thin beam using the differential tansform method, and found accurate natural frequencies for various boundary conditions studied.

Chen (2000) obtained accurate natural vibration frequencies of prismatic EBBoEF using differential quadratic element method (DQEM).

Kacer et al. (2011) used the differential transform method (DTM) for the natural vibration frequency solutions of EBBoWF for the case of variable Winkler foundation parameter.

Ike (2018b) used the Fourier sine transform method (FSM) to obtain exact free transverse vibration frequencies for EBBoWF with simple supports.

Tazabekova et al. (2018) used He's Variational iteration method (VIM) to study natural transverse vibration problems of EBBoWFs. Their study considered a variety of end supports; and the VIM results for clamped ends gave natural frequencies and mode shapes with fast convergence. They derived an efficient algorithm based on the He's VIM which could readily be adapted for more complex elastic foundations.

Adair et al. (2018) also investigated vibration of beams on elastic foundation using VIM; and obtained frequency parameters that compared well with other results from the literature.

Rahbar-Ranji and Shalibaztabar (2017) studied the natural vibration frequency analyses of beams on Pasternak foundation using Legendre polynomial and Rayleigh-Ritz method. Natural frequency parameters for EBBoWF were obtained from their study when the second foundation parameter vanished. Bezerra, Soares and Hoefel (2017) used the finite element method (FEM) for the free vibration analysis of Euler-Bernoulli beam on Pasternak foundation. Solhani and Eipakchi (2020) derived the governing equations of a vibrating beam with moderately large deflection and arbitrary cross-section using the first order shear deformation theory. Their beam was considered homogeneous, isotropic and it was subjected to axial loads. The von-Karman equations were used for the kinematic and Hooke's law for the constitutive equations. Ofondu et al. (2018) used the Stodola-Vianello iteration method (SVIM) for the critical buckling load analysis of Euler columns but did not study EBBoWFs. Ike et al. (2023a, 2023b) used SVIM for the critical buckling load solutions of EBBoWF and Euler-Bernoulli beam on two-parameter foundations (EBBo2PFs) respectively for the cases of clamped ends. They found satisfactory solutions with buckling shape functions that were not exact. They however did not study free vibration problems of EBBoWF. Ike (2023a) used SVIM to find exact buckling solutions for EBBoWF with simple end supports, but did not consider free vibration analysis. The work used exact buckling shape functions.

Ike (2023b) used exact shape functions to find the buckling loads of Euler-Bernoulli beam on Pasternak foundations (EBBoPFs) but did not consider free vibration studies of Euler-Bernoulli beams on elastic foundations (EBBoEFs). Ike (2023c) used SVIM and polynomial shape function to determine critical buckling load solution for EBBoWF, but did not study free vibration analysis of EBBoWF. Ike (2023d) studied SVIM implementation for EBBo2PFs via polynomial basis functions, and obtained satisfactorily accurate solutions for cases with simple supports, but did not consider free vibration studies. Ike (2023e) studied free vibration of EBBoWF using generalized integral transform method. In a recent paper, Ike (2024) implemented the SVIM for the "free torsional vibration analysis of monosymmetric box-beam bridges" with simple supports and obtained exact solutions via exact shape functions. This paper presents SVIM in a novel way for thee free vibration analysis of EBBoWF. The work is presented in a novel rigorous first principles manner and exact shape functions are used to obtain the SVIM equations.

2. Governing differential equation of vibratory motion for Euler-Bernoulli beam on Winkler foundation

The governing partial differential equation (GPDE) of transverse vibratory motion for an Euler-Bernoulli beam resting on a Winkler foundation (EBBoWF) is given by the nonhomogeneous partial differential equation (PDE) when there is a forcing function $q(x, t)$.

$$EI\frac{\partial^4 u(x,t)}{\partial x^4} + ku(x,t) + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} = q(x,t) \quad (1)$$

where $0 \le x \le l,\ t > 0$

In Equation (1), $u(x, t)$ is the transverse deflection, k is the Winkler foundation constant, x is the longitudinal axis of the beam, l is the length of the beam, t is time, A is the cross-sectional area of the beam, ρ is the mass density of the beam material, E is the Young's modulus of elasticity of the beam material, I is the moment of inertia of the beam, $q(x, t)$ is the applied transverse dynamic load.

For natural vibrations, there is no applied excitation force, and

$$q(x,t) = 0 \quad (2)$$

The GPDE simplifies to the following homogeneous form of Equation (1):

$$EI\frac{\partial^4 u}{\partial x^4}+ku+\rho A\frac{\partial^2 u}{\partial t^2}=0 \tag{3}$$

For harmonic vibrations, the response is expectedly harmonic, and $u(x, t)$ can be expressed as the harmonic function:

$$u(x,t)=\bar{U}(x)\cos\left(\omega_n t+\phi\right) \tag{4}$$

wherein ϕ is the phase angle, ω_n is the natural frequency and $\bar{U}(x)$ is the displacement modal function.

Then from Equation (4), the GPDE is expressed in the decoupled form:

$$EI\frac{\partial^4\bar{U}(x)}{\partial x^4}\cos(\omega_n t+\phi)+k\bar{U}(x)\cos(\omega_n+\phi)+\rho A\frac{\partial^2\bar{U}(x)}{\partial t^2}\cos(\omega_n t+\phi)=0 \tag{5}$$

Simplifying,

$$\left(EI\frac{d^4\bar{U}(x)}{dx^4}+k\bar{U}(x)-\rho A\omega_n^2\bar{U}(x)\right)\cos(\omega_n t+\phi)=0 \tag{6}$$

Hence, rearranging Equation (6) gives the homogeneous ordinary differential equation (ODE) in $\bar{U}(x)$:

$$\frac{d^4\bar{U}(x)}{dx^4}+\left(\frac{k-\rho A\omega_n^2}{EI}\right)\bar{U}(x)=0 \tag{7}$$

Or, $$\frac{d^4\bar{U}(x)}{dx^4}+\left(\frac{k}{EI}-\frac{\rho A\omega_n^2}{EI}\right)\bar{U}(x)=0 \tag{8}$$

Introducing nondimensional parameters, yields:

$$\frac{d^4\bar{U}(x)}{dx^4}+\left(\alpha_1^4-\Omega_n^4\right)\bar{U}(x)=0 \tag{9}$$

where $\alpha_1^4=\dfrac{k}{EI}$

$$\Omega_n^4=\frac{\rho A\omega_n^2}{EI}=\frac{\bar{m}\omega_n^2}{EI} \tag{10}$$

$\bar{m}$ is the mass per unit length of the beam, $\bar{m}=\rho A$

n is the vibration mode number.

3. Stodola-Vianello iteration method (SVIM)

Equation (9) is expressed as:

$$\frac{d^4\bar{U}(x)}{dx^4}=(\Omega_n^2-\alpha_1^4)\bar{U}(x) \tag{11}$$

Integrating once,

$$\frac{d^3\bar{U}(x)}{dx^3}=\int_0^x(\Omega_n^4-\alpha_1^4)\bar{U}(x)dx+c_1 \tag{12}$$

where c_1 is an integration constant

$$\bar{U}'''(x)=(\Omega_n^4-\alpha_1^4)\int_0^x\bar{U}(x)dx+c_1 \tag{13}$$

Integrating,

$$\frac{d^2\bar{U}(x)}{dx^2} = (\Omega_n^4 - \alpha_1^4)\int_0^x\int_0^x \bar{U}(x)dxdx + c_1 x + c_2 \quad (14)$$

c_2 is the second integration constant

Integrating again,

$$\frac{d\bar{U}(x)}{dx} = (\Omega_n^4 - \alpha_1^4)\int_0^x\int_0^x\int_0^x \bar{U}(x)dxdxdx + \frac{c_1 x^2}{2} + c_2 x + c_3 \quad (15)$$

c_3 is the third constant of integration
Integrating again,

$$\bar{U}(x) = (\Omega_n^4 - \alpha_1^4)\int_0^x\int_0^x\int_0^x\int_0^x \bar{U}(x)dxdxdxdx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \quad (16)$$

c_4 is the fourth integration constant
The four integration constants are found by applying the boundary conditions. For simply supported ends $x = 0$, $x = l$,

$$\begin{aligned} &\bar{U}(x=0) = \bar{U}(x=l) = 0 \\ &\bar{U}''(x=0) = \bar{U}''(x=l) = 0 \end{aligned} \quad (17)$$

For Euler-Bernoulli beam (EBB) clamped at the ends $x = 0$, $x = l$, the boundary conditions are given by:

$$\begin{aligned} &\bar{U}(x=0) = \bar{U}(x=l) = 0 \\ &\bar{U}'(x=0) = \bar{U}'(x=l) = 0 \end{aligned} \quad (18)$$

For EBB clamped at $x = 0$, and simply supported at $x = l$, the boundary conditions are given by:

$$\begin{aligned} &\bar{U}(x=0) = \bar{U}(x=l) = 0 \\ &\bar{U}'(x=0) = 0 \\ &\bar{U}''(x=l) = 0 \end{aligned} \quad (19)$$

The SVIM iteration equation becomes

$$\bar{U}_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)\int_0^x\int_0^x\int_0^x\int_0^x \bar{U}_n(x)dxdxdxdx + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \quad (20)$$

$$\bar{U}'_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)\int_0^x\int_0^x\int_0^x \bar{U}_n(x)dxdxdx + \frac{c_1 x^2}{2} + c_2 x + c_3 \quad (21)$$

$$\bar{U}''_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)\int_0^x\int_0^x \bar{U}_n(x)dxdx + c_1 x + c_2 \quad (22)$$

$$\bar{U}'''_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)\int_0^x \bar{U}_n(x)dx + c_1 \quad (23)$$

4. Results

4.1. Euler-Bernoulli beam on Winkler foundation with simple supports at x = 0, and x = l

The EBBoWF with simple supports at the left and right hand ends is shown in Figure 4. For EBB with simple supports at $x = 0$, and $x = l$, modal shape function that satisfies boundary condition is:

$$\bar{U}_n(x) = a_n \sin\frac{n\pi x}{l} \quad (24)$$

where a_n is the amplitude of $\bar{U}_n(x)$ and $\sin\left(n\pi x/l\right)$ is the shape function that satisfies the simply supported conditions.

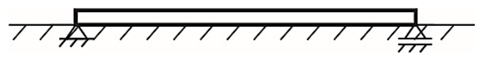

Fig. 4 Free vibrating simply supported Euler-Bernoulli beam on Winkler foundation

Then by the SVIM,

$$\bar{U}_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)\int_0^x\int_0^x\int_0^x\int_0^x a_n \sin\frac{n\pi x}{l}\,dxdxdxdx + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \quad (25)$$

$$\bar{U}''_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)\int_0^x\int_0^x a_n \sin\frac{n\pi x}{l}\,dxdx + c_1x + c_2 \quad (26)$$

Simplifying, gives:

$$\bar{U}''_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)a_n\left(-\left(\frac{l}{n\pi}\right)^2\right)\sin\frac{n\pi x}{l} + c_1x + c_2 \quad (27)$$

$$\bar{U}_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)a_n\left(\frac{l}{n\pi}\right)^4\sin\frac{n\pi x}{l} + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \quad (28)$$

Applying the boundary conditions,

$$\bar{U}''(x=0) = 0 = c_2 \quad (29)$$

$$\bar{U}''(x=l) = 0 = -(\Omega_n^4 - \alpha_1^4)a_n\left(\frac{l}{n\pi}\right)^2\sin n\pi + c_1l = 0 \quad (30)$$

$$c_1 = 0 \quad (31)$$

$$\bar{U}(x=0) = c_4 = 0 \quad (32)$$

$$\bar{U}(x=l) = (\Omega_n^4 - \alpha_1^4)a_n\left(\frac{l}{n\pi}\right)^4\sin n\pi + c_3l = 0 \quad (33)$$

$$c_3 = 0 \quad (34)$$

Thus,

$$\bar{U}_{n+1}(x) = (\Omega_n^4 - \alpha_1^4)a_n\left(\frac{l}{n\pi}\right)^4\sin\frac{n\pi x}{l} \quad (35)$$

At convergence of the iteration, after the *n*th iteration,

$$\bar{U}_{n+1}(x) = \bar{U}_n(x) \quad (36)$$

Then applying the convergence criterion yields:

$$\bar{U}_{n+1}(x) = a_n\sin\frac{n\pi x}{l} = (\Omega_n^4 - \alpha_1^4)a_n\left(\frac{l}{n\pi}\right)^4\sin\frac{n\pi x}{l} \quad (37)$$

Simplification of Equation (37) gives the characteristic eigenvalue equation as:

$$1 = (\Omega_n^4 - \alpha_1^4)\left(\frac{l}{n\pi}\right)^4 \quad (38)$$

Thus, further simplification of Equation (38) gives:

$$\Omega_n^4 - \alpha_1^4 = \left(\frac{n\pi}{l}\right)^4 \quad (39)$$

Expanding in terms of Ω^4 gives:

$$\Omega_n^4 = \alpha_1^4 + \left(\frac{n\pi}{l}\right)^4 = \frac{\bar{m}\omega_n^2}{EI} \quad (40)$$

Then, solving for ω_n^2 gives:

$$\omega_n^2 = \frac{EI}{\bar{m}}\left(\alpha_1^4 + \left(\frac{n\pi}{l}\right)^4\right) \quad (41)$$

Taking the square root of both sides of Equation (41) gives:

$$\omega_n = \sqrt{\alpha_1^4 + \left(\frac{n\pi}{l}\right)^4}\sqrt{\frac{EI}{\bar{m}}} = \sqrt{(\alpha_1^4 l^4 + (n\pi)^4}\,\frac{1}{l^2}\sqrt{\frac{EI}{\bar{m}}} \quad (42)$$

Expressing Equation (42) in terms of frequency parameter λ_n, gives:

$$\omega_n = \lambda_n^2\sqrt{\frac{EI}{\bar{m}}} = \lambda_n^2\sqrt{\frac{EI}{\rho A}} \quad (43)$$

where $\lambda_n^2 = \sqrt{\alpha_1^4 + \left(\frac{n\pi}{l}\right)^4} = \frac{1}{l^2}\sqrt{(\alpha_1 l)^4 + (n\pi)^4}$ (44)

$$\lambda_n^2 = \left(\frac{\bar{\lambda}_n}{l}\right)^2 \quad (45)$$

$$\bar{\lambda}_n^2 = \sqrt{(\alpha_1 l)^4 + (n\pi)^4} \quad (46)$$

When $\frac{k}{4EI} = \beta^4$ (46a)

where β^4 is another way to express the soil-structure parameter. Then,

$$\frac{k}{EI} = 4\beta^4 = \alpha_1^4 \quad (46b)$$

4.2. Results for transverse vibrations of simply supported Euler-Bernoulli Beam (EBB) without Winkler foundation

The. SVIM results can be used to obtain the natural transverse vibration frequencies of EBB without Winkler foundation by inputing $\alpha_1^4 = 0$ into Equation (42). Then,

$$\lambda_n^2 = \left(\frac{n\pi}{l}\right)^2 = \left(\frac{\bar{\lambda}_n}{l}\right)^2 \quad (47)$$

and $\omega_n = \left(\frac{n\pi}{l}\right)^2\sqrt{\frac{EI}{\rho A}}$ (48)

Equation (48) is identical with the exact solution for transverse vibration frequencies of EBB.

Thus, the relationship between α_1 and β is presented as Equation (46b). The values of $\bar{\lambda}_n$ ae calculated for values of $\alpha_1^4 l^4$ varying from $\alpha_1^4 l^4 = 0, 1, 10, 100, 1000,$ and 10,000 and presented in Table 1 together with previous values of $\bar{\lambda}_n$ computed by Rahbar Ranji and Shahbaztabar (2017), Ike (2023).

Furthermore, normalized free transverse vibration frequencies of simply supported EBBoWF for values of the following beam parameters $E = A = \rho = 1, \quad \alpha_1^4 l^4 = 1$ are determined and presented in Table 2 together for comparison purposes with previous results that used Homotopy perturbation method (HPM), Differential transform method (DTM), Differential quadrature element method (DQEM) and exact solution method.

Table 1 – Transverse natural frequency parameters for the first five vibration modes for the natural vibration of simply supported EBBoWF.

$\alpha_1^4 l^4$	$\bar{\lambda}_1$			$\bar{\lambda}_2$			$\bar{\lambda}_3$		
	Present study	**GITM Ike (2023)**	**Rahbar-Ranji and Shahbaztabar (2017)**	**Present study**	**GITM Ike (2023)**	**Rahbar-Ranji and Shahbaztabar (2017)**	**Present study**	**GITM Ike (2023)**	**Rahbar-Ranji and Shahbaztabar (2017)**
1	3.149624682	3.149624682	3.1496	6.284192925	6.284192925	6.28426	9.425076572	9.425076572	9.4251
10	3.219291184	3.219291184	3.2193 (3.228)*	6.293239752	6.293239752	6.2932 (6.293)*	9.427762796	9.427762796	9.4277 (9.427)*
100	3.74836425	3.74836425	3.7484 (3.748)*	6.381633292	6.381633292	6.3816 (6.382)*	9.454499603	9.454499603	9.4545 (9.454)*
1000	5.755620336	5.755620336	5.7556 (5.755)*	7.11210704	7.11210704	7.1121 (7.112)*	9.710176091	9.710176091	9.7102 (9.710)*
10,000	10.02426382	10.02426382	10.0243	10.36873551	10.36873551	10.3687	11.56520706	11.56520706	11.5652

$\alpha_1^4 l^4$	$\bar{\lambda}_4$			$\bar{\lambda}_5$		
	Present study	**GITM Ike (2023)**	**Rahbar-Ranji and Shahbaztabar (2017)**	**Present study**	**GITM Ike (2023)**	**Rahbar-Ranji and Shahbaztabar (2017)**
1	12.5664966	12.5664966	12.5665	15.70802772	15.70802772	15.7080
10	12.56763202	12.56763202	12.5676 (12.568)*	15.70860826	15.70860826	15.7086 (15.709)*
100	12.57894997	12.57894997	12.5790 (12.579)*	15.71440961	15.71440961	15.7144 (15.715)*
1000	12.69050177	12.69050177	12.6905 (12.690)*	15.77207279	15.77207279	15.7721
10,000	13.67163814	13.67163814	13.6716	16.31668659	16.31668659	16.3167

*__Note:__ the frequency parameter results that are enclosed in parenthesis and asterisked are taken from Zhou (1993).

Table 2 – Normalized natural transverse vibration frequency parameters of simply supported EBBoWF for $E = A = \rho = I = 1,\ \alpha_1^4 l^4 = 1,$ for first five modes of vibration.

Method / Reference	$\bar{\lambda}_1^2$	$\bar{\lambda}_2^2$	$\bar{\lambda}_3^2$	$\bar{\lambda}_4^2$	$\bar{\lambda}_5^2$
SVIM (Present)	9.92014	39.4911	88.8321	157.9168	246.7421
HPM (Coskun, 2003)	9.92014	39.4911	88.8321	157.9168	246.7421
DTM (Balkaya et al., 2009)	9.92014	39.4911	88.8321	–	–
DQEM (Chen, 2000)	9.92014	39.4911	89.4002	–	–
Exact solution (Ike, 2018, 2023)	9.92014	39.4911	88.8321	157.9168	246.7421

4.3. Numerical results

In order to further illustrate the effects of the Winkler foundation parameters on the natural transverse vibration frequencies of a simply supported thin beam on a Winkler foundation, some numerical examples are presented.

A beam of uniform cross-section with $E = 210GPa$, $\rho = 7850\text{kg/m}^3$, $h/l = 0.05$ and $l = 2\text{m}$ is considered.

Table 3 presents the comparison between the SVIM and FEM solutions for a simply supported EBBoWF. The second column presents the transverse frequency parameters for the present study without Winkler foundation, $(\alpha_1^4 l^4 = 0)$; the third column presents the Bezerra et al. (2017) solution without Winkler foundation; the fourth column presents the SVIM solution for Winkler foundation $\left(\frac{kl^4}{EI} = 5;\ \text{or}\ \alpha_1^4 l^4 = \frac{5}{4} = 1.25\right)$. The fifth and sixth columns present Bezerra et al. (2017) solutions using 30 elements and 70 elements respectively.

Table 3 – Comparison of transverse vibration frequency parameters

Vibration mode number	Without foundation		$\alpha_1^4 = 1.25$		
	Present	Bezerra et al. (2017)	Present	Bezerra et al. (2017)	
				FEM 30 elements	FEM 70 elements
1	9.87696	9.870	11.064	11.064	11.064
2	39.4784	39.478	39.794	39.794	39.794
3	88.8264	88.826	88.968	88.968	88.967
4	157.9137	157.914	157.996	157.996	157.993
5	246.7401	246.740	246.804	246.804	246.791

5. Discussion

The GPDE for transverse vibratory motion of EBBoWF represented by a fourth order non-homogeneous PDE for forced vibration cases, reduces to homogeneous PDE for natural vibrations where excitation forces are absent. For harmonic vibrations, and harmonic displacement response $u(x, t)$, the homogeneous PDE become decoupled in terms of the independent spatial and time variables; resulting in an ordinary differential equation (ODE) of fourth order in terms of $\bar{U}(x)$, the displacement modal function for $u(x, t)$. For homogeneous, prismatic, isotropic thin beams, the ODE has constant parameters.

SVIM was used to express the ODE in Stodola-iteration forms with fourth constants of integration that are determined using the end support conditions of deformation and forces. The SVIM equations were then used to solve the EBBoWF free vibration problem for the case of simple supports at $x = 0$, and $x = l$. Exact shape functions were implemented in the SVIM using sinusoidal functions that satisfy the simply supported boundary conditions at the beam ends. The criterion for convergence of the nth SVIM iteration scheme was utilized to find the eigenvalue equation, which was solved for nontrivial cases to get the eigenvalues. The natural transverse vibration frequencies ω_n were found in terms of frequency parameters $\bar{\lambda}_n^2$ for the nth vibration mode. $\bar{\lambda}_n^2$ was found to depend upon the beam foundation parameter α_1^4 and the vibrating mode, n. Values of λ_n were calculated for values of $\alpha_1^4 l^4 = 1, 10, 100, 1000, 10000$ and $n = 1, 2, 3, 4, 5$; and presented in Table 1, together with previously computed values by Rahbar-Ranji and Shahbaztabar (2017), Zhou (1993) and Ike (2023).

Table 1 illustrates that the present SVIM results for $\bar{\lambda}_n$ for all the first five modes of vibration are identical with the previous results presented using the Generalized integral transform method (GITM) by Ike (2023), Table 1 further establishes that the present SVIM results for $\bar{\lambda}_n$ are in close agreement with previous results by Zhou (1993) and Rahbar-Ranji and Shahbaztabar (2017).

Table 2 presents natural transverse vibration frequency parameters $\bar{\lambda}_n^2$ for simply supported EBBoWF for values of $E = A = \rho = I = 1,\ \alpha_1^4 l^4 = 1$ for the first five vibration modes.

Table 2 demonstrates that the parameters $\bar{\lambda}_n^2$ obtained in the present study are identical for all the first five modes of transverse vibration with the exact solution by Ike (2023) and the previous solutions using Homotopy perturbation method by Coskun (2003); Differential transform method (DTM) by Balkaya et al. (2009) and Differential quadrature element method (DQEM) by Chen (2000).

Table 3 shows that the present SVIM results for EBB without elastic foundation $(\alpha_1^4 l^4 = 0)$ are similar to results by Bezerra et al. (2017). Table 3 also shows that for $\alpha_1^4 l^4 = 1.25$, or $\frac{kl^4}{EI} = 5$, the present SVIM results are closely identical with finite element method results presented by Bezerra et al. (2017).

6. Conclusion

This study has presented SVIM for solving natural transverse harmonic vibration problems of homogeneous, isotropic, prismatic EBBoWFs. The governing PDE for the transverse forced vibrations of EBBoWF is a non-homogeneous PDE which has constant parameters for homogeneous, isotropic, prismatic thin beams, and variable parameters for non-homogeneous, non-isotropic, non-prismatic thin beams. For natural vibrations, there is no excitation force and the PDE reduces to homogeneous PDE with constant parameters for the case of prismatic, homogeneous, isotropic thin beam studied.

In conclusion:

(i) SVIM iteration formulation of the governing ODE for the decoupled homogeneous PDE was obtained using four successive integrations, and contains four constants of integration which are determinable using the four boundary conditions for the problem.

(ii) For simply supported EBBoWF studied, exact sinusoidal vibration shape functions were used in the SVIM equations and boundary conditions used to evaluate the four integration constants; thus leading to a full determination of the SVIM equation for the nth vibration mode.

(iii) The criterion for convergence of the SVIM equations at the nth iteration mode, which is the equality of the vibrating functions at the nth and $(n + 1)$th iterations, established the vibration function for the nth iteration.

(iv) The criterion for non-triviality of the solutions was used to derive the characteristic vibration equation at the nth vibration mode.

(v) The characteristic vibration equation gave exact eigenvalues for the simply supported EBBoWF for the *n*th vibration mode.
(vi) Exact eigenvalues were used to obtain the exact vibration frequencies for the first five modes of vibration.
(vii) The SVIM solution of the simply supported EBBoWF is exact because exact vibration shape functions were used in the formulation for the *n*th vibration mode.
(viii) The SVIM solutions were identical to the exact solutions obtained using the GITM by Ike (2023) and using HPM by Coskun (2003).
(ix) The effect of the Winkler foundation results is an increase in the natural frequencies of vibration of the EBBoWF for all the vibration modes.

Notations

EBBoWF	Euler-Bernoulli beam on Winkler foundation
SVIM	Stodola-Vianello iteration method
PDE	partial differential equation
ODE(s)	ordinary differential equation(s)
x	longitudinal (axial) coordinate
t	time
GITM	generalized integral transform method
DTM	differential transform method
HPM	homotopy perturbation method
DQEM	differential quadrature element method
h	thickness
l	span
EBBoEF	Euler-Bernoulli beam on elastic foundation
3D	three-dimensional
2D	two-dimensional
VIM	variational iteration method
FSM	finite sine transform
EBBo2PF	Euler-Bernoulli beam on two-parameter foundations
GPDE	governing partial differential equation
$u(x, t)$	transverse deflection
k	Winkler foundation constant
A	cross-sectional area of the beam
ρ	mass density of the beam material
E	Young’s modulus of elasticity of the beam material
I	moment of inertia of the beam
$q(x, t)$	applied transverse dynamic load
ϕ	phase angle
ω_n	natural frequency
$\bar{U}(x)$	displacement modal function
α_1	parameter defined in terms of k and EI
Ω_n	parameter defined in terms of ρA, ω_n and EI
$\bar{m}$	mass per unit length of beam
n	vibration mode number
c_1, c_2, c_3, c_4	constants of integration
λ_n	vibration frequency parameter defined in terms of α_1, n and l
EBBT	Euler-Bernoulli beam theory
TBT	Timoshenko beam theory

SDBTs	shear deformation beam theories
ReBTs	refined beam theories
EBB	Euler-Bernoulli beam
FEM	finite element method.

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