On $3-$ *Monotone Approximation by Piecewise Positive Functions*

Eman Samir Bhaya
University of Babylon, College of Education, Department of Mathematics

Malik Saad Al-Muhja
University of Al-Muthana, College of Sciences, Department of Mathematics and Computer Application

Abstract.
In 2005 Halgwrd [3], introduced a paper for $f \in C[-1,1]$ *with* $1 < p < \infty$ *, be a convex function, we are interested in estimating the degree of 3-monotone approximation for the function* f *, which are copositive on* $[-1,1]$*. We obtained that* f *and* g *are piecewise positive in* $[-1,1]$ *in terms of the Ditzian-Totik modulus of smoothness .*

1. Introduction and auxiliary results.

Let $Y_s = \{a < y_1 < y_2 < ... < y_s < b\}$, $s \geq 0$. We denote by $\Delta^0(Y_s)$, the set of all functions f , such that $(-1)^{s-k} f(x) \geq 0$, for $x \in [y_j, y_{j+1}]$, $0 \leq k \leq s$. Functions f and g , that belong to the same class $\Delta^0(Y_s)$ are said to be *copositive* on $[a,b]$. *Copositive approximation* is the approximation of a function f , from $\Delta^0(Y_s)$, class by polynomials that are copositive with f . Also , let $E_n^0(f,k)_P = \inf_{p_n \in \Pi_n \cap \Delta^0(Y_s)} \| f - p_n \|_P$ be the *degree of copositive polynomial approximation* of f .

We denote $J_j(n,\varepsilon) = [y_j - \Delta_n(y_j)n^{\varepsilon}, y_j + \Delta_n(y_j)n^{\varepsilon}] \cap [a,b]$, $0 \leq j \leq s+1$, and denote $O_n(Y_s,\varepsilon) = \bigcup_{j=1}^{s} J_j(n,\varepsilon)$, and $O_n^*(Y_s,\varepsilon) = \bigcup_{j=0}^{s+1} J_j(n,\varepsilon)$. [2]

Functions f and g are called *weakly almost copositive* on I , with respect to Y_S if they are copositive on $I \setminus O_n^*(Y_s,\varepsilon)$, where $\varepsilon > 0$. We define a function class $(\varepsilon - alm\Delta)_n^0(Y_s) = \{f : (-1)^{s-k} f(x) \geq 0, \ \ for \ x \in I \setminus O_n^*(Y_s,\varepsilon)\}$, the set of all weakly almost nonnegative functions on I , if $\varepsilon > 0$.

The *degree of weakly almost copositive polynomial approximation* of f in $L_P[a,b] \cap \Delta^0(Y_s)$, by means $p \in \Pi_n \cap (\varepsilon - alm\Delta)_n^0(Y_s)$ is $E_n^0(f, \varepsilon - almY_s)_P$ $= \inf\{\| f - p\|_P : p \in \Pi_n \cap (\varepsilon - alm\Delta)_n^0(Y_s)\}$.

These results can be summarized in the following theorem (see [5] and [8]) .

Theorem A.
There are functions f_1 *and* f_2 *in* $C^1[-1,1]$, *with* $r \geq 1$, *sign changes such that*

$$\limsup_{n\to\infty} \frac{E_n^0(f_1,r)}{\omega_4(f_1,n^{-1},[-1,1])} = \infty \ \ and \ \ \limsup_{n\to\infty} \frac{E_n^0(f_2,r)_P}{\omega_2(f_2,n^{-1},[-1,1])_P} = \infty, \ 1 < p < \infty,$$

where $E_n^0(f,r)_P$ *is the degree of the best copositive* L_P *(* C *if* $p = \infty$ *), approximation to* f *, by polynomials from* Π_n *.*

Recently , Y. Hu , D. Leviatan and X. M. Yu [6], showed that theorem A can be considerably improved , thus together with theorem A, revealing an interesting and unexpected difference between the cases $p=\infty$, and $1<p<\infty$, for copositive polynomial approximation . Their result is stated as follows .

Theorem B.
Let $f\in C[-1,1]$, *change sign* r, *times at* $-1<y_1<...<y_r<1$, *and let* $\delta=\min\limits_{0\le i\le r}|y_{i+1}-y_i|$, *where* $y_\circ=-1$ *and* $y_{r+1}=1$. *Then there exists a constant* $C=C(r,\delta)$, *but otherwise independent of* f *and* n, *such that for each* $n\ge 4\delta^{-1}$, *there is a polynomial* $p_n\in\Pi_{Cn}$, *copositive with* f ,
satisfying

$$\|f-p_n\|_{L_\infty[-1,1]}\le C\omega_2\left(f,n^{-1},[-1,1]\right). \tag{1.1}$$

In [2] Bhaya , E. and other , showed that in the second result ω_2 in (1.1) can not replaced by $\omega_3\left(f,b-a,[a,b]\right)_P$, for $0<p<1$, i.e., she proved .

Theorem C.
Given any $A>0$, $n\in\tilde{N}$, $a<0$, $0<b$, $0<p<1$ *and* $0<\varepsilon<2$, *there exists* f *in* $L_P[a,b]\cap\Delta^0(Y_s)$, *such that*

$$E_n^0\left(f,\varepsilon-almY_s\right)_P>\omega_3\left(f,b-a,[a,b]\right)_P. \tag{1.2}$$

The second result in [2], shows that τ-modulus of any order $k>0$ can be used for $0<p<1$.

Theorem D.
Let f *in* $L_P[a,b]\cap\Delta^0(Y_s)$, $0<p<1$, *and* k *be a positive integer . Then there exists a polynomial* p_{k-1} *in* $\Pi_{k-1}\cap(\varepsilon-alm\Delta)_n^0(Y_s)$, *satisfying* $\|f-P_n\|_P\le c(p)\tau_k\left(f,b-a,[a,b]\right)_P$.

2. The main results

We will modify this polynomial near the points of sign change obtaining a smooth piecewise polynomial approximation f_n, with controlled first and third derivatives . We will consider σ_i that its convexity at $\{y_i,y_i',y_i''\}$ with f .

Theorem 2.1
Let f *in* $L_P[a,b]\cap\Delta^0(Y_s)$. *Then for each* $n\ge 4\delta^{-1}$, *there exists a function* f_n *in* $\Delta^3[-1,1]\cap\left(S-\Delta^0(Y_s)\right)$, *copositive with* f *in* $Y=\bigcup\limits_{i=1}^{k}\rho_i$, *such that*

$$\|f-f_n\|_{L_P[-1,1]}\le C(k)\omega_3^\phi\left(f,n^{-1},[-1,1]\right)_P, \tag{2.2}$$

$$\left\|\phi(x)^3f_n^{(3)}(x)\right\|_{L_P[-1,1]}\le C(k)n^3\omega_3^\phi\left(f,n^{-1},[-1,1]\right)_P, \tag{2.3}$$

and

$$\|\Delta_n(x)f_n'(x)\|_{L_P[-1,1]}\ge C\omega_3^\phi\left(f,n^{-1},[-1,1]\right)_P,\ \textit{for}\ x\in Y, \tag{2.4}$$

where $\left(S-\Delta^0(Y_s)\right)$ *is the set of all piecewise positive .*

Proof. Let $n \geq 4\delta^{-1}$, and index $1 \leq i \leq k$, be fixed . For $x \in \mathrm{I}_i^*$, we set σ_i to be the polynomial of degree ≤ 2, which vanishes at y_i,

$$\sigma_i(x) = \frac{x - y_i}{y_i'' - y_i'}\left\{\frac{x - y_i'}{y_i'' - y_i}\sigma_i(y_i'') + \frac{x - y_i''}{y_i - y_i'}\sigma_i(y_i')\right\} \text{ [4]},$$

where $\sigma_i(y_i')$ and $\sigma_i(y_i'')$ are chosen so that

$$|\sigma_i(y_i')| = \begin{cases} c\omega_3^\phi(f, n^{-1}, [-1,1])_P \operatorname{sgn}(f(y_i')) & ; if\ |f(y_i')| \leq c\omega_3^\phi(f, n^{-1}, [-1,1])_P, \\ f(y_i') & ; \qquad o.w \end{cases}$$

and

$$|\sigma_i(y_i'')| = \begin{cases} c\omega_3^\phi(f, n^{-1}, [-1,1])_P \operatorname{sgn}(f(y_i'')) & ; if\ |f(y_i'')| \leq c\omega_3^\phi(f, n^{-1}, [-1,1])_P, \\ f(y_i'') & ; \qquad o.w. \end{cases}$$

If $f(y_i') = 0$, then $\operatorname{sgn}(f(y_i'))$, equals the sign f on (y_{i-1}, y_i) . Since $\sigma_i \in \Pi_2$, and $\sigma_i(y_i')$ and $\sigma_i(y_i'')$, have opposite signs , then the only zero of σ_i in I_i^* is y_i .

Hence , σ_i is copositive with f in I_i^* . Also , the first derivative of σ_i,

$$\sigma_i'(x) = \frac{2x - y_i - y_i'}{(y_i'' - y_i')(y_i'' - y_i)}\sigma_i(y_i'') + \frac{2x - y_i - y_i''}{(y_i'' - y_i')(y_i - y_i')}\sigma_i(y_i')$$

is a linear function , and

$$\sigma_i'\left(\frac{y_i + y_i'}{2}\right) = \frac{-\sigma_i(y_i')}{(y_i - y_i')}, \text{ and } \sigma_i'\left(\frac{y_i + y_i''}{2}\right) = \frac{\sigma_i(y_i'')}{(y_i'' - y_i)}$$

are of the same sign , which implies that σ_i', does not change sign in ρ_i, and for any $x \in \rho_i$.

$$\|\sigma_i'(x)\|_{L_P[-1,1]} \geq 2^{\frac{1}{P}} \min\left\{\left|\sigma_i'\left(\frac{y_i + y_i'}{2}\right)\right|, \left|\sigma_i'\left(\frac{y_i + y_i''}{2}\right)\right|\right\} = 2^{\frac{1}{P}} \min\left\{\frac{|\sigma_i(y_i')|}{|y_i - y_i'|}, \frac{|\sigma_i(y_i'')|}{|y_i'' - y_i|}\right\}$$

$$\geq 2^{\frac{1}{P}} \frac{1}{c\Delta_n(x)} \min\{|\sigma_i(y_i')|, |\sigma_i(y_i'')|\} \geq 2^{\frac{1}{P}} \frac{1}{\Delta_n(x)} \min\{\omega_3^\phi(f, n^{-1}, [-1,1])_P, \omega_3^\phi(f, n^{-1}, [-1,1])_P\}$$

$$\geq 2^{\frac{1}{P}} \frac{1}{\Delta_n(x)} \omega_3^\phi(f, n^{-1}, [-1,1])_P. \qquad (2.5)$$

From [3], we have

$$\|f - \sigma_i\|_{L_P[-1,1]} \leq C\omega_3^\phi(f, n^{-1}, [-1,1])_P. \qquad (2.6)$$

It is well known (see proof of Lemma 8 in [7]), that there exists a polynomial $Q_n(x)$, of degree $\leq n$, which is a polynomial of best approximation to f in $[-1,1]$, and satisfying

$$\|f - Q_n\|_{L_P[-1,1]} \leq C\omega_3^\phi(f, n^{-1}, [-1,1])_P, \qquad (2.7)$$

and

$$\left\|\phi(x)^3 Q_n^{(3)}(x)\right\|_{L_P[-1,1]} \leq Cn^3\omega_3^\phi(f, n^{-1}, [-1,1])_P. \qquad (2.8)$$

Now , we define the piecewise polynomial function $S(x) \in C[-1,1]$, as follows

$$S(x) = \begin{cases} 1 & ; if\ x \notin \bigcup_{i=1}^k \mathrm{I}_i^*, \\ 0 & ; if\ x \in \bigcup_{i=1}^k \rho_i, \\ c & ; if\ x \in \left[y_i', \frac{y_i + y_i'}{2}\right], 1 \leq i \leq k. \end{cases}$$

Finally , the function

$$f_n(x)=\begin{cases}\left|Q_n(x)-\sigma_i(x)\right| S(x)+\sigma_i(x) & ;if\ \ x\in \mathrm{I}_i^*,\\ Q_n(x) \quad ; & o.w\end{cases}$$

is copositive with f in $\Upsilon=\bigcup_{i=1}^{k}\rho_i$, and indeed f_n, coincides with σ_i in ρ_i, and , let C be an absolute constant such that

$\|f-f_n\|_{L_P[-1,1]}\le C\|f-\sigma_i\|_{L_P[-1,1]}\le C\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P$.

From (2.5), then

$\|\Delta_n(x)f_n'(x)\|_{L_P[-1,1]}\ge Ch_{j(i)}\|f_n'(x)\|_{L_P[-1,1]}\ge C\Delta_n(x)\frac{1}{\Delta_n(x)}\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P=C\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P$.

Now , to prove the remaining (2.3), for $x\in\left[y_i',\frac{y_i+y_i'}{2}\right]$, $1\le i\le k$ (for $x\notin\bigcup_{i=1}^{k}\mathrm{I}_i^*$, from (2.8), we have (2.3) is valid , and for $x\in\Upsilon$ it is trivial), from [4], look at

$$\left|\phi(x)^3 f_n^{(3)}(x)\right|\le Cn^3\left|\mathrm{I}_i^*\right|^3\sum_{v=0}^{3}\left|Q_n^{(v)}-\sigma_i^{(v)}\right|\left|S^{(3-v)}\right|, \text{ such that}$$

$\phi(x)\approx n\Delta_n(x)\approx n\left|\mathrm{I}_i^*\right|$, for $x\in\mathrm{I}_i^*$, then

$$\begin{aligned}\left\|\phi(x)^3 f_n^{(3)}(x)\right\|_{L_P[-1,1]}&\le\left\|\phi(x)^3 f_n^{(3)}(x)\right\|_{L_\infty[-1,1]}\\ &\le Cn^3\left|\mathrm{I}_i^*\right|^3\sum_{v=0}^{3}\left\|Q_n^{(v)}-\sigma_i^{(v)}\right\|_{L_\infty[-1,1]}\left\|S^{(3-v)}\right\|_{L_\infty[-1,1]}\\ &\le C(p,v,n,2)n^3\left|\mathrm{I}_i^*\right|^{3-k-1}\sum_{v=0}^{3}\left\|Q_n-\sigma_i\right\|_{L_P[-1,1]}\|S\|_{L_P[-1,1]}\\ &\le C(p,v,n,2)n^3\left(\|Q_n-f\|_{L_P[-1,1]}+\|f-\sigma_i\|_{L_P[-1,1]}\right).\end{aligned}$$

Now , from (2.6) and (2.7), we get

$$\left\|\phi(x)^3 f_n^{(3)}(x)\right\|_{L_P[-1,1]}\le C(p,v,n,2)n^3\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P .$$

□

Also , let us introduce the following auxiliary proposition .

Proposition 2.9

If $\hat f$ *in* $C^3[-1,1]\cap L_P[-1,1]$ *is such that* $\left|(1-x^2)^{3/2}\hat f^{(3)}(x)\right|\le M$, $x\in[-1,1]$, $-1<y_1<...<y_k<1$, *and* $\delta=\min_{1\le i\le k+1}\left|y_{i+1}-y_i\right|$, *then for every* $n\ge C$, *there exists a polynomial* $p_n\in\Pi_n$, *such that*

$$\left\|\hat f-p_n\right\|_{L_P[-1,1]}\le C\omega_3^{\phi}\left(\hat f,n^{-1},[-1,1]\right)_P, \qquad (2.10)$$

and

$$\left\|\Delta_n(x)\left(\hat f'-p_n'\right)\right\|_{L_P[-1,1]}\le C\frac{2}{n}\omega_2^{\phi}\left(\hat f',n^{-1},[-1,1]\right)_P \qquad (2.11)$$

where the constant C*, depends only on* k *and* p .

Proof. Note that (2,10) is trivial (see [1] theorem 3.2.1) . In (2.11) is valid since $x\in[-1,1]$, then from [1], we get

$$\begin{aligned}\left\|\Delta_n(x)\left(\hat f'-p_n'\right)\right\|_{L_P[-1,1]}&\le\frac{2}{n}\left\|\hat f'-p_n'\right\|_{L_P[-1,1]}\\ &\le C\frac{2}{n}\omega_2^{\phi}\left(\hat f',n^{-1},[-1,1]\right)_P .\end{aligned}$$

□

Now , let us introduce the following theorem as a main result

Theorem 2.12

Let f *in* $L_P[a,b]\cap\Delta^0(Y_s)$, *change sign* $k\ge 1$, *times at* $-1<y_1<...<y_k<1$, *and let* $\delta=\min_{0\le i\le k}|y_{i+1}-y_i|$, *where* $y_\circ=-1$ *and* $y_{k+1}=1$. *Then there exists a constant* C, *such that for each* $n>C$, *there is a function* g *in* $L_P[a,b]\cap\Delta^0(Y_s)$, *copositive with* f , *and satisfying*

$$\|f-g\|_{L_P[-1,1]}\le C\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P \qquad (2.13)$$

where the constant C, *depends only on* k .

Proof. If $n\ge 4\delta^{-1}$, there exists $p_N\in\Pi_N$, let $g=p_N+2^k\|f-p_N\|_{L_P[-1,1]}\eta\prod_{i=1}^{k}T_N(y_i,x)$ in $L_P[a,b]\cap\Delta^0(Y_s)$, where N is sufficiently large ($N=\left(\left(18\sqrt{C}\right)+1\right)n$ will do)[4], and $\eta=\pm1$ is such that $\operatorname{sgn}(f(x))=\eta\prod_{i=1}^{k}\operatorname{sgn}(x-y_i)$.

Also , let f_n in $\Delta^3[-1,1]\cap\left(S-\Delta^0(Y_s)\right)$ be a function which was described in theorem 2.1 . (3.9) can be written as

$$\left\|\left(1-x^2\right)^{3/2}f_n^{(3)}(x)\right\|_{L_P[-1,1]}\le C(k)n^3\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P \ .$$

It follows from proposition 2.9, that there exists a polynomial $p_N\in\Pi_N$, best approximation to f_n and satisfies (2.7), such that

$$\begin{aligned}\|f_n-p_N\|_{L_P[-1,1]}&\le\|f_n-f\|_{L_P[-1,1]}+\|f-p_N\|_{L_P[-1,1]}\\&\le C\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P \qquad (2.14)\end{aligned}$$

and

$$\|\Delta_n(x)(f_n'-p_N')\|_{L_P[-1,1]}\le C\frac{2}{n}\omega_2^{\phi}\left(f_n',n^{-1},[-1,1]\right)_P \ . \qquad (2.15)$$

Together with (2.4), this implies $\operatorname{sgn}(p_N(x))=\operatorname{sgn}(f_n(x))$, $x\in \Upsilon=\bigcup_{i=1}^{k}\rho_i$.

In turn , it follows that p_N is copositive with f in $\Upsilon=\bigcup_{i=1}^{k}\rho_i$, and also by (2.2), (2.10) and (2.14), we get

$$\begin{aligned}\|f-g\|_{L_P[-1,1]}&=\|f-f_n+f_n-g\|_{L_P[-1,1]}\\&\le\|f-f_n\|_{L_P[-1,1]}+\|f_n-g\|_{L_P[-1,1]}\\&\le\|f-f_n\|_{L_P[-1,1]}+\left\|f_n-p_N-2^k\|f-p_N\|_{L_P[-1,1]}\eta\prod_{i=1}^{k}T_N(y_i,x)\right\|_{L_P[-1,1]}\\&\le\|f-f_n\|_{L_P[-1,1]}+\|f_n-p_N\|_{L_P[-1,1]}+C\|f-p_N\|_{L_P[-1,1]}\\&\le C\omega_3^{\phi}\left(f,n^{-1},[-1,1]\right)_P \ .\end{aligned}$$

□

References

[1] Al-Muhja , Malik S. , On k -monotone approximation in L_p spaces , M. Sc. Thesis , University of Kufa , 2009 .

[2] Bhaya , E. and A. H. **, *Weak Copositive Approximation and Whitney Theorem in*** L_P**,** $0<p<1$ **,** J. J. App. , Sci. , Vol8 , No.2 , 51-57 , 2006 .

[3] Halgwrd M. **, *On the Shape Preserving Approximation* ,** Mc. Sc. Thesis , Babylon University , 2005 .
[4] Kopotun , Kirill A. **, *On Copositive Approximation by Algebraic Polynomials* ,** AMS , 1991 .
[5] S. P. Zhou **, *A Counterexample in Copositive Approximation* ,** Israel J. Math. , 78 ; 75-83 , 1992 .
[6] Y. Hu , D. Leviatan and X. M. Yu **, *Copositive Polynomial Approximation in*** $C[0,1]$ **,** J. Anal. , 1 ; 85-90 , 1993 .
[7] Yingkang Hu **, *On Equivalence of Moduli of Smoothness of Polynomial in*** L_P **,** $0 < p \leq \infty$ **,** J. Approx. Theory , 136 ; 182-197 , 2005 .
[8] Zhou , S. P. **, *On Copositive Approximation* ,** Approx. Theory Appl. , 104-110 , 1993 .