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Single Finite Sine Transform Method for Exact Bending Analysis of Simply Supported Kirchhoff Plate Under Parabolic Load

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ABSTRACT

Despite the importance of plates in structural analysis the flexural analysis of plates under parabolic load has not been extensively studied. This paper aims to present a single finite sine transform method for exact bending solutions of simply supported Kirchhoff plate under parabolic load. The governing equation of equilibrium is a fourth order non-homogeneous differential equation in terms of the deflection $v(x, y)$. The considered thin plate problem has Dirichlet boundary conditions at all the edges. This recommends the use of the finite sine integral transform method whose sinusoidal kernel function satisfies the boundary conditions. The sinusoidal function of x used for the sine transform kernel in this paper satisfies the Dirichlet boundary conditions along $x = 0$, $x = a$ edges. The transformation simplifies the problem from a partial differential equation (PDE) to an ordinary differential equation (ODE) in the transformed space. The general solution, obtained using methods for solving ODEs is found in terms of unknown constants of integration which are found by using the finite sine transform of Dirichlet boundary conditions along the $y = 0$, and $y = b$ edges. The solution in the physical domain space variables is then found by inversion as a rapidly convergent single series with infinite terms.

1. Introduction

Plates are structural members characterized by two in-plane dimensions (length and width respectively) and a transverse dimension called the thickness, *h*. They are commonly used in buildings, bridges, aircrafts, naval structures and retaining structures; and are subject to static or dynamic loads that are usually applied transverse to their surfaces. They have several advantages in structural applications, including their high flexural load carrying capacity (Timoshenko & Woinowsky-Krieger, 1959).

The plates resist transverse loads by bending. The middle surface (z = 0 plane), divides the plate into two halves, and does not stretch according to Kirchhoff's theory. The flexural behaviour of plates is determined by the thickness – least in-plane dimension ratio (h/a). Accordingly, thick plates have a/h < 8-10 while thin plates

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have 8-10 < $a/h \leq$ 80-100.

The flexural behaviour of thick plates is described by the theory of elasticity in three-dimensions (3D). Thick plate analysis uses the governing equations of equilibrium of 3D elasticity (Szilard, 2004). Simplified methods of formulation employ assumptions that reduce the rigours of mathematical analysis involved thus yielding first order shear deformation theories (FSDTs), higher order shear deformation theories, and refined plate theories (Rouzegar & Sharifpoor, 2015) which account for the effects of transverse shear deformations. First order shear deformation theories for plates were presented by Mindlin (1951), Ike et al. (2017a), Ike (2017a, 2018) using variational and equilibrium methods respectively.

The formulation for the flexural behaviour of thin plates ignores transverse shear deformations, and was first presented by Kirchhoff. The classical Kirchhoff-Love plate theory (KLPT) is a two-dimensional mathematical simplification of the 3D elasticity theory used to determine the stresses and deformations in thin plates subjected to transverse forces and bending moments. (Ike, 2015).

1.1. Literature review

Ghugal and Sayyad (2010, 2013) presented studies on stress analysis of thick laminated plates using trigonometric shear deformation theory (TSDT). The formulated TSDT satisfied the shear stress-free boundary conditions and accounted for transverse shear deformation effects. Nwoji et al. (2018a) solved the simply supported rectangular Mindlin plate bending problem subjected to bisinusoidal loading. Ghugal and Gajhbiye (2016) used the fifth order shear deformation theory to carry out accurate flexural analysis of thick isotropic plates. The theory accounted for transverse shear deformations and was variationally consistent.

Do et al. (2020) developed refined plate theory for static flexural analysis of functionally graded plates such that transverse shear deformation is accounted for. Nareem and Shimpi (2015) developed variationally consistent refined hyperbolic shear deformation plate theory suitable for static bending solutions. Ferreira and Roque (2011) used radial basis functions (RBFs) to analyze thick plates bending problems.

Onah et al. (2020) used stress function methods of elasticity theory to derive flexural solutions to thick circular plate problems under transverse loads. Onyeka et al. (2022a, 2022b, 2023a, 2023b, 2023c) used the energy minimization techniques to obtain satisfactorily accurate bending solutions to thick rectangular plates with clamped, clamped/simply supported and mixed boundary conditions. Onyeka and Okeke (2021) and Onyeka et al (2023a) presented polynomial shear deformation formulations for solving thick plate bending problems. Onyeka and Mama (2021) used direct variational methods (DVMs) of energy minimization to solve thick plate bending problems using trigonometric functions.

Nwoji et al. (2018b), Aginam (2011a, 2011b) and Aginam et al. (2012) used the DVM for the analysis of Kirchhoff plates under various transverse loadings and boundary conditions. In similar studies, Osadebe and Aginam (2011), and Mbakogu and Pavlovic (2000) presented variational symbolic solutions to clamped rectangular thin plate bending problems for isotropic and orthotropic material conditions respectively. Emma and Sule (2013) have also presented variational solutions to thin plate bending under uniform loads and simple supports.

The application of Galerkin methods to thin plate flexural analysis were studied by Osadebe et al. (2016), Nwoji et al. (2017a), Okoye et al. (2019), Aginam et al. (2018) and Ike (2023a), and satisfactorily accurate solutions were obtained by the researchers. Kantorovich methods and their variants were applied to plate bending analysis by Nwoji et al. (2017b), Ike (2017b, 2023b), Ike et al. (2017b), Onah et al. (2017), Ike and Mama (2018) and exact analytical solutions that satisfied both boundary conditions and domain equations were obtained by the researchers. Integral transformation methods have also been used for plate problems by An, Gu, and Su (2011).

Finite sine transform method (FSTM) was applied to simply supported thin plate bending analysis by Mama et al. (2017, 2020) and exact analytical solutions were obtained. Ike et al. (2021) used generalized integral transform method (GITM) to develop analytical solutions for bending problems of rectangular thin plate with two opposite simply supported edges and the other edges clamped. Ike (2023c) used the double finite sine transform method to develop closed form solutions for Kirchhoff plate under parabolic load distribution for Dirichlet boundary conditions. The sinusoidal kernel satisfied the boundary conditions and the integral transformation converted the domain equation from a differential equation to an algebraic equation which is more readily solved. Finite difference methods (FDMs) which are based on discretizing the domain equations have been used for plate bending problems by Ezeh et al. (2013), and Ergün and Kumbasar (2011). They obtained satisfactorily accurate solutions and demonstrated the simplicity of the FDMs.

Ibearugbulem et al. (2013) used work principle technique to solve isotropic thin rectangular plate bending problems. Symplectic elasticity methods have been applied to derive exact bending solutions to thin rectangular plates for various loading and boundary conditions by Cui (2007), Lim et al. (2007), Zhong and Li (2009), and Ma (2008). Kant (1981) presented approximate analysis of plate problems with two opposite simply supported edges using segmentation method. Other important studies on thin plate bending are found in Delyavskyy and Rosinski (2020), Alcybeev et al. (2022), Singh and Prashanth (2022), and Boussem and Belouriar (2020).

Literature review reveals that single finite sine transform method (SFSTM) has not been used for the flexural analysis of simply supported thin rectangular plate under parabolic load distribution. This paper applies the SFSTM to develop exact analytical bending solutions to simply supported thin plate under parabolic load distribution. The innovative aspect of the paper is the first principles, systematic derivation of the bending solutions.

2. Governing Partial Differential Equation (GPDE)

The GPDE for the Kirchhoff plate blending problem is the fourth order equation (Timoshenko and Woinowksy-Krieger, 1959):

$$D\left(\frac{\partial^4 v}{\partial x^4}+2\frac{\partial^4 v}{\partial x^2\partial y^2}+\frac{\partial^4 v}{\partial y^4}\right)=D\nabla^4 v(x,y)=q_z(x,y) \quad (1)$$

∇^4 is the biharmonic operator.

$$\nabla^4=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4} \quad (1a)$$

where $v(x,y)$ is the transverse deflection x and y are the in-plane coordinates,

$q_z(x,y)$ is the transversely distributed load intensity,

D is the modulus of flexural rigidity. D is given in terms of the elastic parameters of the plate material by:

$$D=\frac{Eh^3}{12(1-\mu^2)} \quad (2)$$

Wherein,
E is the Young's modulus of elasticity
μ is the Poisson's ratio
h is the plate thickness
The bending moments M_{xx}, M_{yy} are (Szilard, 2004):

$$M_{xx}=-D\left(\frac{\partial^2 v}{\partial x^2}+\mu\frac{\partial^2 v}{\partial y^2}\right) \quad (3)$$

$$M_{yy}=-D\left(\frac{\partial^2 v}{\partial y^2}+\mu\frac{\partial^2 v}{\partial x^2}\right) \quad (4)$$

The twisting moment M_{xy} is:

$$M_{xy}=-D(1-\mu)\frac{\partial^2 v}{\partial x\partial y} \quad (5)$$

The shear force distributions Q_x, Q_y are (Szilard, 2004):

$$Q_x=-D\frac{\partial}{\partial x}\nabla^2 v(x,y)=-D\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad (6)$$

$$Q_y=-D\frac{\partial}{\partial y}\nabla^2 v(x,y)=-D\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad (7)$$

∇^2 is the Laplace operator.

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad (7a)$$

3. Method

3.1. Application of the single finite sine transform method to the GPDE

The considered problem is shown in Figure 1.

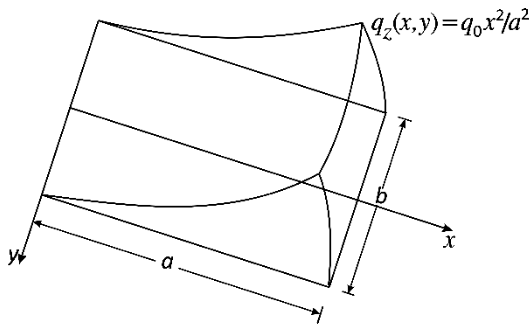

Fig. 1 Kirchhoff plate under parabolic load distribution

The GPDE is:

$$\frac{\partial^4 v}{\partial x^4} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4} - \frac{q_z(x,y)}{D} = 0 \quad (8)$$

where $0 \le x \le a; \quad \frac{-b}{2} \le y \le \frac{b}{2}$

The finite sine transformation of the GPDE is:

$$\int_0^a \left(\frac{\partial^4 v}{\partial x^4} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial x^4} - \frac{q_z(x,y)}{D} \right) \sin\frac{n\pi x}{a}\, dx = 0 \quad (9)$$

From the linearity properties of the single finite sine transform (SFST),

$$\int_0^a \frac{\partial^4 v}{\partial x^4} \sin\frac{n\pi x}{a} dx + 2\int_0^a \frac{\partial^4 v}{\partial x^2 \partial y^2} \sin\frac{n\pi x}{a} dx + \int_0^a \frac{\partial^4 v}{\partial y^4} \sin\frac{n\pi x}{a} dx - \int_0^a \frac{q_z}{D} \sin\frac{n\pi x}{a} dx = 0 \quad (10)$$

Using the properties of the SFST,

$$\int_0^a \frac{\partial^4 v}{\partial x^4} \sin\frac{n\pi x}{a} dx = \left(\frac{n\pi}{a}\right)^4 \int_0^a v(x,y) \sin\frac{n\pi x}{a} dx + \left(\frac{n\pi}{a}\right)^3 \left(v|_{x=a,y} (-1)^n - v|_{x=0,y} \right) - \left(\frac{n\pi}{a}\right) \left((-1)^n \left.\frac{\partial^2 v}{\partial x^2}\right|_{x=a,y} - \left.\frac{\partial^2 v}{\partial x^2}\right|_{x=0,y} \right) \quad \ldots (11)$$

$$\int_0^a \frac{\partial^4 v}{\partial x^2 \partial y^2} \sin\frac{n\pi x}{a} dx = \frac{\partial^2}{\partial y^2} \int_0^a \frac{\partial^2 v}{\partial x^2} \sin\frac{n\pi x}{a} dx = \frac{d^2}{dy^2} \left(-\left(\frac{n\pi}{a}\right)^2 \int_0^a v(x,y) \sin\frac{n\pi x}{a} dx - \left(\frac{n\pi}{a}\right) \left((-1)^n v|_{x=a,y} - v|_{x=0,y} \right) \right) \quad (12)$$

$$\int_0^a \frac{\partial^4 v}{\partial y^4} \sin\frac{n\pi x}{a} dx = \frac{d^4}{dy^4} \int_0^a v(x,y) \sin\frac{n\pi x}{a} dx = \frac{d^4}{dy^4} V(x,y) \quad (13)$$

where $V(n,y) = \int_0^a v(x,y) \sin\frac{n\pi x}{a} dx \quad (14)$

$V(n,y)$ is the single finite sine transform (SFST) of $v(x,y)$

$$\int_0^a \frac{q_z}{D} \sin\frac{n\pi x}{a} dx = \frac{1}{D} \int_0^a q_z \sin\frac{n\pi x}{a} dx = \frac{Q_z(n,y)}{D} \quad (15)$$

$$Q_z(n,y)=\int_0^a q_z(x,y)\sin\frac{n\pi x}{a}dx \tag{16}$$

$Q_z(n,y)$ is the single finite sine transform (SFST) of $q_z(x,y)$
Using the boundary conditions, the FST for this problem simplifies,

$$\int_0^a \frac{\partial^4 v}{\partial x^4}\sin\frac{n\pi x}{a}dx=\left(\frac{n\pi}{a}\right)^4\int_0^a v(x,y)\sin\frac{n\pi x}{a}dx=\left(\frac{n\pi}{a}\right)^4 V(n,y) \tag{17}$$

$$\int_0^a \frac{\partial^4 v}{\partial x^2\partial y^2}\sin\frac{n\pi x}{a}dx=-\left(\frac{n\pi}{a}\right)^2\frac{d^2}{dy^2}V(n,y) \tag{18}$$

$$\int_0^a \frac{\partial^4 v}{\partial y^2}\sin\frac{n\pi x}{a}dx=\frac{d^4}{dy^4}V(n,y) \tag{19}$$

The SFST equation is:

$$\left(\frac{n\pi}{a}\right)^4 V(n,y)-2\left(\frac{n\pi}{a}\right)^2\frac{d^2}{dy^2}V(n,y)+\frac{d^4}{dy^4}V(n,y)-\int_0^a\frac{q_z}{D}\sin\frac{n\pi x}{a}dx=0 \tag{20}$$

Hence,

$$\left(\frac{n\pi}{a}\right)^4 V(n,y)-2\left(\frac{n\pi}{a}\right)^2\frac{d^2}{dy^2}V(n,y)+\frac{d^4}{dy^4}V(n,y)=\frac{1}{D}\int_0^a q_z\sin\frac{n\pi x}{a}=\frac{Q_z}{D}(n,y) \tag{21}$$

$$Q_z(n,y)=\int_0^a\frac{q_0x^2}{a^2}\sin\frac{n\pi x}{a}dx=\frac{q_0}{a^2}\int_0^a x^2\sin\frac{n\pi x}{a}dx \tag{22}$$

q_0 is the magnitude of the parabolic load intensity at $x=a,\ y$.
Using integrator software,

$$\int_0^a x^2\sin\frac{n\pi x}{a}dx=I_1 \tag{23}$$

$$I_1=\frac{a^3}{(\pi n)^3}\left(2n\pi\sin(n\pi)+(2-(n\pi)^2)\cos(n\pi)-2\right)=\frac{a^3}{(n\pi)^3}((2-(n\pi)^2)(-1)^n-2) \tag{24}$$

$$\frac{Q_z}{D}=\frac{q_0}{Da^2}I_1 \tag{25}$$

$$\frac{Q_z}{D}=\frac{q_0a}{D(n\pi)^3}\left((2-(n\pi)^2)(-1)^n-2\right) \tag{26}$$

n is even, $\frac{Q_z}{D}=\frac{q_0a}{D(n\pi)^3}\left(2-(n\pi)^2-2\right)=\frac{-q_0a}{D(n\pi)}$ (27)

n is odd, $\frac{Q_z}{D}=\frac{q_0a}{D(n\pi)^3}\left(-2+(n\pi)^2-2\right)=\frac{q_0a((n\pi)^2-4)}{D(n\pi)^3}$ (28)

4. Results and Discussion

4.1. Homogeneous solution for $V(n,y), V_h$

This found by solving the homogeneous part of Equation (21), namely

$$\left(\frac{n\pi}{a}\right)^4 V(n,y)-2\left(\frac{n\pi}{a}\right)^2\frac{d^2}{dy^2}V(n,y)+\frac{d^4}{dy^4}V(n,y)=0 \tag{29}$$

Let us assume an exponential form of homogeneous solution, V_h

$$V_h = A_n \exp \lambda y \tag{30}$$

where A_n are constants, and λ are parameters to be found.
Then,

$$\left(\frac{n\pi}{a}\right)^4 A_n e^{\lambda y} - 2\left(\frac{n\pi}{a}\right)^2 \lambda^2 A_n e^{\lambda y} + \lambda^4 A_n e^{\lambda y} = 0 \tag{31}$$

Simplifying,

$$\left(\left(\frac{n\pi}{a}\right)^4 - 2\left(\frac{n\pi}{a}\right)^2 \lambda^2 + \lambda^4\right) A_n e^{\lambda y} = 0 \tag{32}$$

For nontrivial solutions,

$$A_n e^{\lambda y} \neq 0$$

Hence,

$$\left(\frac{n\pi}{a}\right)^4 - 2\left(\frac{n\pi}{a}\right)^2 \lambda^2 + \lambda^4 = 0 \tag{33}$$

Factorising,

$$\left(\lambda^2 - \left(\frac{n\pi}{a}\right)^2\right)^2 = 0 \tag{34}$$

Solving,

$$\lambda = \pm\frac{n\pi}{a} \text{ twice} \tag{35}$$

Hence the four eigenvalues (roots) are given by Equations (35a) and (35b)

$$\lambda = +\frac{n\pi}{a} \text{ twice} \tag{35a}$$

$$\lambda = -\frac{n\pi}{a} \text{ twice} \tag{35b}$$

The basis of linearly independent homogeneous solutions become:

$$V_{n_1}(n, \lambda) = A_{n_1} \exp\frac{n\pi y}{a} \tag{36a}$$

$$V_{n_2}(n, \lambda) = \frac{n\pi y}{a} A_{n_2} \exp\frac{n\pi y}{a} \tag{36b}$$

$$V_{n_3}(n, y) = A_{n_3} \exp\left(\frac{-n\pi y}{a}\right) \tag{36c}$$

$$V_{n_4}(n, y) = \frac{-n\pi y}{a} A_{n_4} \exp\left(\frac{-n\pi y}{a}\right) \tag{36d}$$

The general solution is expressed using hyperbolic functions as

$$V_n(n, y) = C_{1n} \cosh\frac{n\pi y}{a} + C_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a} + C_{3n}\frac{n\pi y}{a}\cosh\frac{n\pi y}{a} + C_{4n}\sinh\frac{n\pi y}{a} \tag{37}$$

The problem considered is symmetrical about the x axis since

$$v(x, y) = v(x, -y) \tag{38}$$

Therefore, expectedly,

$$V_n(n, y) = V_n(n, -y) \tag{39}$$

$$C_{1n} \cosh\frac{n\pi y}{a} + C_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a} + C_{3n}\frac{n\pi y}{a}\cosh\frac{n\pi y}{a} + C_{4n}\sinh\frac{n\pi y}{a} = C_{1n}\cosh\left(\frac{-n\pi y}{a}\right) +$$
$$C_{2n}\left(\frac{-n\pi y}{a}\right)\sinh\left(\frac{-n\pi y}{a}\right) + C_{3n}\left(\frac{-n\pi y}{a}\right)\cosh\left(\frac{-n\pi y}{a}\right) + C_{4n}\sinh\left(\frac{-n\pi y}{a}\right) =$$
$$C_{1n} \cosh\frac{n\pi y}{a} + C_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a} - C_{3n}\frac{n\pi y}{a}\cosh\frac{n\pi y}{a} - C_{4n}\sinh\frac{n\pi y}{a} \tag{40}$$

Hence,

$$C_{3n}\frac{n\pi y}{a}\cosh\frac{n\pi y}{a}+C_{4n}\sinh\frac{n\pi y}{a}=-C_{3n}\frac{n\pi y}{a}\cosh\frac{n\pi y}{a}-C_{4n}\sinh\frac{n\pi y}{a} \quad (41)$$

Which implies, $C_{3n}=0$ (42a)

$C_{4n}=0$ (42b)

Hence, the general solution for $V_h(x, y)$ which satisfies symmetry of the problem is:

$$V_h(n,y)=\sum_{n=1}^{\infty}\left(C_{1n}\cosh\frac{n\pi y}{a}+C_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}\right) \quad (43)$$

By inversion,

$$v_h(x,y)=\frac{2}{a}\sum_{n=1}^{\infty}V_h(n,y)\sin\frac{n\pi x}{a} \quad (44)$$

4.2. Particular solution, $V_p(n,y)$

The applied load distribution does not depend upon the y coordinate variable. Hence,

$$\frac{d^4}{dy^4}V_p(n,y)=0 \quad (45)$$

$$\frac{d^2}{dy^2}V_p(n,y)=0 \quad (46)$$

where $V_p(n,y)$ is the particular integral.

Then, substitution of Equations (49) and (50) into the non-homogeneous Equation (21) gives:

$$\left(\frac{n\pi}{a}\right)^4 V_p(n,y)=\frac{Q_z(n,y)}{D} \quad (47)$$

$$V_p(n,y)=\left(\frac{a}{n\pi}\right)^4\frac{Q_z(n,y)}{D} \quad (48)$$

4.3. General solution $V_g(n,y)$

The general solution $V_g(n, y)$ is the sum of the homogeneous solution and the particular solution.

$$V_g(n,y)=V_p(n,y)+V_h(n,y)=\left(\frac{a}{n\pi}\right)^4\frac{Q_z(n,y)}{D}+\sum_{n=1}^{\infty}\left(C_{1n}\cosh\frac{n\pi y}{a}+C_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}\right) \quad (49)$$

4.4. General solution in the problem coordinate variables v(x, y)

The general solution in the physical problem coordinate variables $v(x, y)$ is obtained by inversion of $V_g(n, y)$ as:

$$v(x,y)=\frac{2}{a}\sum_{n=1}^{\infty}V_g(n,y)\sin\frac{n\pi x}{a} \quad (50)$$

$$v(x,y)=\sum_{n=1}^{\infty}\frac{2}{a}\left(C_{1n}\cosh\frac{n\pi y}{a}+C_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}+\frac{a^4}{D(n\pi)^4}Q_z(n,y)\right)\sin\frac{n\pi x}{a} \quad (51)$$

Let $\bar{C}_{1n}=\frac{2}{a}C_{1n}$ (52a)

$$\bar{C}_{2n}=\frac{2}{a}C_{2n} \quad (52b)$$

$$v(x,y)=\sum_{n=1}^{\infty}\left(\bar{C}_{1n}\cosh\frac{n\pi y}{a}+\bar{C}_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}+\bar{V}_p\right)\sin\frac{n\pi x}{a} \quad (53)$$

where $\bar{V}_{pn}=\frac{2q_0a^4}{D(n\pi)^7}[(2-(n\pi)^2)(-1)^n-2]$ (54)

When n is even,

$$\bar{V}_{pn}=\frac{2q_0a^4}{D(n\pi)^7}((2-(n\pi)^2)-2) \quad (55)$$

$$\bar{V}_{pn}=\frac{2q_0a^4}{D(n\pi)^7}\cdot-(n\pi)^2=\frac{-2q_0a^4}{D(n\pi)^5} \quad (56)$$

$n=2,4,6,8\ldots$

When n is odd, $(-1)^n=-1$

$$\bar{V}_{pn}=\frac{2q_0a^4}{D(n\pi)^7}(-(2-(n\pi)^2)-2)=$$

$$\bar{V}_{pn}=\frac{2q_0a^4}{D(n\pi)^7}(-2+n\pi^2-2)=$$

$$\bar{V}_{pn}=\frac{2q_0a^4}{D(n\pi)^7}((n\pi)^2-4) \qquad n=1,3,5,7\ldots \quad (57)$$

Hence,

$$v(x,y)=\sum_{n=1}^{\infty}\left(\bar{C}_{1n}\cosh\frac{n\pi y}{a}+C_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a}+\sum_{n=1,3,5\ldots}^{\infty}\frac{2q_0a^4}{D(n\pi)^7}((n\pi)^2-4)\sin\frac{n\pi x}{a}+\sum_{n=2,4,6\ldots}^{\infty}\frac{-2q_0a^4}{D(n\pi)^5}\sin\frac{n\pi x}{a} \quad (58)$$

$$v(x,y)=\sum_{n=1}^{\infty}\left(\bar{C}_{1n}\cosh\frac{n\pi y}{a}+\bar{C}_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a}+\frac{2q_0a^4}{D\pi^7}\sum_{n=1,3,5}^{\infty}\frac{1}{n^7}((n\pi)^2-4)\sin\frac{n\pi x}{a}-\frac{2q_0a^4}{D\pi^5}\sum_{n=2,4,6}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{a} \quad (59)$$

Thus,

$$v(x,y)=\sum_{n=1}^{\infty}\left(\bar{C}_{1n}\cosh\frac{n\pi y}{a}+\bar{C}_{2n}\frac{n\pi y}{a}\sinh\frac{n\pi y}{a}\right)\sin\frac{n\pi x}{a}+V_{pn} \quad (60)$$

where $V_{pn}=\frac{2q_0a^4}{D\pi^7}\sum_{n=1,3,7}^{\infty}\frac{((n\pi)^2-4)}{n^7}\sin\frac{n\pi x}{a}$ (61)

for $n=1,3,5,7,9,\ldots$

and $V_{pn}=\frac{2q_0a^4}{D\pi^5}\sum_{n=2,4\ldots}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{a}$ (62)

for $n=2,4,6,8,\ldots$

4.5. Enforcement of Boundary Conditions

The integration constants $\bar{C}_{1n}$ and $\bar{C}_{2n}$ are found by the enforcement of boundary conditions at the edges $y=\pm b/2$. The boundary conditions at $y=\pm b/2$ are

$v(x,y=\pm b/2)=0$ (63a)

$v''(x, y = \pm b/2) = 0$ (63b)

For each value of n,

$$V_{pn} + \bar{C}_{1n} \cosh\frac{n\pi b}{2a} + \bar{C}_{2n}\frac{n\pi b}{2a}\sinh\frac{n\pi b}{2a} = 0 \quad (64)$$

$$\bar{C}_{1n}\left(\frac{n\pi}{a}\right)^2 \cosh\left(\frac{n\pi b}{2a}\right) + \bar{C}_{2n}\left(\frac{n\pi}{a}\right)^2\left(\frac{n\pi b}{2a}\sinh\frac{n\pi b}{2a} + 2\cosh\frac{n\pi b}{2a}\right) = 0 \quad (65)$$

Solving,

$$\bar{C}_{1n} = \frac{-\bar{C}_{2n}\left(\frac{n\pi b}{2a}\sinh\frac{n\pi b}{2a} + 2\cosh\frac{n\pi b}{2a}\right)}{\cosh\frac{n\pi b}{2a}} \quad (66)$$

$$\bar{C}_{1n} = -\bar{C}_{2n}\left(2 + \frac{n\pi b}{2a}\tanh\frac{n\pi b}{2a}\right) \quad (67)$$

Then, substitution of Equation (73) into the second boundary condition Equation (70) yields:

$$\bar{C}_{1n} = -V_{pn}\left(\frac{2 + \left(\frac{n\pi b}{2a}\right)\tanh\left(\frac{n\pi b}{2a}\right)}{2\cosh\left(\frac{n\pi b}{2a}\right)}\right) = -V_{pn}\left(\frac{1 + \left(\frac{n\pi b}{4a}\right)\tanh\left(\frac{n\pi b}{2a}\right)}{\cosh\left(\frac{n\pi b}{2a}\right)}\right) \quad (68)$$

$$\bar{C}_{2n} = \frac{V_{pn}}{2\cosh\left(\frac{n\pi b}{2a}\right)} \quad (69)$$

Then the deflection equation becomes:

$$v(x,y) = \sum_{n=1}^{\infty}\left(-V_{p_n}\left(\frac{2 + \left(\frac{n\pi b}{2a}\right)\tanh\left(\frac{n\pi b}{2a}\right)}{2\cosh\left(\frac{n\pi b}{2a}\right)}\right)\cosh\left(\frac{n\pi y}{a}\right) + \frac{V_{pn}}{2\cosh\left(\frac{n\pi b}{2a}\right)}\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)\right)\sin\left(\frac{n\pi x}{a}\right) +$$
$$\frac{2q_0 a^4}{D\pi^7}\sum_{n=1,3,5}^{\infty}\frac{((n\pi)^2 - 4)}{n^7}\sin\left(\frac{n\pi x}{a}\right) + \frac{2q_0 a^4}{D\pi^5}\sum_{n=2,4,6,8}^{\infty}\frac{1}{n^5}\sin\left(\frac{n\pi x}{a}\right) \quad (70)$$

Simplifying,

$$v(x,y) = \frac{2q_0 a^4}{D\pi^7}\sum_{n=1,3,5\ldots}^{\infty}\frac{((n\pi)^2 - 4)}{n^7}K(n,y)\sin\frac{n\pi x}{a} - \frac{2q_0 a^4}{D\pi^5}\sum_{n=2,4,\ldots}^{\infty}\frac{K(n,y)}{n^5}\sin\frac{n\pi x}{a} \quad (71)$$

where $K(n,y) = 1 - \left(\frac{2\cosh\lambda_n\cosh\gamma_n + \lambda_n\sinh\lambda_n\cosh\gamma_n - \gamma_n\sinh\gamma_n\cosh\lambda_n}{1 + \cosh 2\lambda_n}\right)$ (72)

$$\lambda_n = \frac{n\pi b}{2a} \quad (73a)$$

$$\gamma_n = \frac{n\pi y}{a} = \alpha_n y \quad (73b)$$

$$\alpha_n = \left(\frac{n\pi}{a}\right) \quad (74)$$

It is observed that

$v(x = 0, y) = 0$ (75a)

$v(x = a, y) = 0$ (75b)

$K(n, y = \pm b/2) = 0$ (75c)

When $y = \pm\frac{b}{2}$

$$\gamma_n = \frac{\pm n\pi b/2}{a} = \pm\frac{n\pi b}{2a} \tag{75d}$$

$$K(n, y=\pm b/2) = 1 - \frac{(2\cosh\lambda_n \cosh\lambda_n + \lambda_n \sinh\lambda_n \cosh\lambda_n - \lambda_n \sinh\lambda_n \cosh\lambda_n)}{1+\cosh 2\lambda_n} \tag{76}$$

$$K(n, y=\pm b/2) = 1 - \frac{2\cosh\lambda_n \cosh\lambda_n}{1+\cosh 2\lambda_n} = \frac{1+\cosh 2\lambda_n - 2\cosh\lambda_n \cosh\lambda_n}{1+\cosh 2\lambda_n} \tag{76a}$$

$$\cosh 2x \equiv 2\cosh^2 x - 1 \tag{77}$$

$$1+\cosh 2\lambda_n - 2\cosh^2\lambda_n = 0 \tag{77a}$$

$$\therefore \quad K(n, y=\pm b/2) = 0 \tag{78}$$

Hence, $v(x, y=\pm b/2) = 0$

Solutions at the Center of Square Plate

For square plates, $b = a$, and at the center, $x = a/2$, $y = 0$

$$v(x=a/2, y=0) = \frac{2q_0a^4}{D\pi^7}\sum_{n=1,3,5\dots}^{\infty}\left(\frac{(n\pi)^2-4}{n^7}\right)K(n, y=0)\sin\frac{n\pi}{2} - \frac{2q_0a^4}{D\pi^5}\sum_{n=2,4,\dots}^{\infty}\frac{K(n,y=0)}{n^5}\sin\frac{n\pi}{2} \tag{79}$$

$$K(n, y=0) = 1 - \left(\frac{2\cosh\lambda_n \cosh 0 + \lambda_n \sinh\lambda_n \cosh 0 - 0}{1+\cosh 2\lambda_n}\right) \tag{80}$$

For square plates,

$$\lambda_n = \frac{n\pi}{2} \tag{81}$$

$$K(n, y=0) = 1 - \left(\frac{2\cosh\frac{n\pi}{2} + \frac{n\pi}{2}\sinh\frac{n\pi}{2}}{1+\cosh 2\left(\frac{n\pi}{2}\right)}\right) \tag{82}$$

$$K(n, y=0) = 1 - \left(\frac{2\cosh\frac{n\pi}{2} + \frac{n\pi}{2}\sinh\frac{n\pi}{2}}{1+\cosh n\pi}\right) \tag{83}$$

$$K(n=1, y=0) = 1 - \left(\frac{2\cosh\left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\sinh\left(\frac{\pi}{2}\right)\right)}{1+\cosh\pi}\right) \tag{84a}$$

$$K(n=1, y=0) = 1 - \left(\frac{8.63322882}{12.59195328}\right) \tag{84b}$$

$$K(n=1, y=0) = 0.314385256 \tag{84c}$$

$$v(x,y) = \frac{2q_0a^4}{D\pi^7}\left(\left(\frac{\pi^2-4}{1}\right)(0.314385)\right) = \frac{q_0a^4}{D}(1.221945\times 10^{-3}) \tag{85}$$

$$v(x=a/2, y=0) = 1.221945\times 10^{-3}\,\frac{q_0a^4}{D} \tag{86}$$

This is 2% greater from the solution found in Szilard (2004) who used a Levy method.

$$K(n=2, y=0) = 1 - \left(\frac{2\cosh\pi + \pi\sinh\pi}{1+\cosh 2\pi}\right) \tag{87}$$

$$K(n=2, y=0) = 1 - 0.221269052 \tag{88}$$

$$K(n=2, y=0) = 0.778731 \tag{89}$$

Two term solution increment Δv_c is:

$$\Delta v_c = \frac{-2q_0a^4}{D\pi^5}\left(\frac{0.778731}{2^5}\right)\sin\pi = 0 \tag{90}$$

For two term solution

$$v(x = a/2, y = 0) = 1.221945\times10^{-3}\frac{q_0a^4}{D} \tag{91}$$

Three term solution

$$K(n = 3, y = 0) = 1-\left(\frac{2\cosh1.5\pi + 1.5\pi\sinh1.5\pi}{1+\cosh3\pi}\right) \tag{92}$$

$$K(n = 3, y = 0) = (1-0.060287646) = 0.929712353 \tag{93}$$

The increment in center deflection Δv_c due to the third term is:

$$\Delta v_c = \frac{2q_0a^4}{D\pi^7}\left(\frac{(3\pi)^2-4}{3^7}\right)0.939712353\times\sin1.5\pi \tag{94}$$

Evaluating,

$$\Delta v_c = -2.413561\times10^{-5}\frac{q_0a^4}{D} \tag{95}$$

Three term solution for center deflection v_c is

$$v_c = 1.221945\times10^{-3}\frac{q_0a^4}{D} - 2.413561\times10^{-5}\frac{q_0a^4}{D} \tag{96}$$

$$v_c = 1.19780939\times10^{-3}\left(\frac{q_0a^4}{D}\right) \tag{97}$$

The analytical solution for center deflection is identical with the solution obtained by Szilard (2004) using Levy method.

4.6. Bending Moment Expressions

By partial differentiation of $v(x.y)$,

$$\frac{\partial^2 v}{\partial x^2} = \frac{2q_0a^4}{D\pi^7}\sum_n^\infty\left(\frac{(n\pi)^2-4}{n^7}\right)K(n,y)\cdot-\left(\frac{n\pi}{a}\right)^2\sin\frac{n\pi x}{a} - \frac{2q_0a^4}{D\pi^5}\sum_n^\infty\frac{K(n,y)}{n^5}\cdot-\left(\frac{n\pi}{a}\right)^2\sin\frac{n\pi x}{a} \tag{98}$$

Similarly,

$$\frac{\partial^2 v}{\partial y^2} = \frac{2q_0a^4}{D\pi^7}\sum_n^\infty\left(\frac{(n\pi)^2-4}{n^7}\right)\frac{d^2K(n,y)}{dy^2}\sin\frac{n\pi x}{a} - \frac{2q_0a^4}{D\pi^5}\sum_n^\infty\frac{d^2K(n,y)}{dy^2}\frac{1}{n^5}\sin\frac{n\pi x}{a} \tag{99}$$

Simplifying Equation (104) gives:

$$\frac{\partial^2 v}{\partial x^2} = \frac{-2q_0a^2}{D\pi^5}\sum_{n=1}^\infty\left(\frac{(n\pi)^2-4}{n^5}\right)K(n,y)\sin\frac{n\pi x}{a} + \frac{2q_0a^2}{D\pi^3}\sum_n^\infty\frac{K(n,y)}{n^3}\sin\frac{n\pi x}{a} \tag{100}$$

Simplifying Equation (105) gives:

$$\frac{\partial^2 v}{\partial y^2} = \frac{-2q_0a^2}{D\pi^5}\sum_{n=1}^\infty\frac{(n\pi)^2-4}{n^5}K_1\sin\left(\frac{n\pi x}{a}\right) - \frac{2q_0a^2}{D\pi^3}\sum_{n=1}^\infty\frac{K_1}{n^3}\sin\frac{n\pi x}{a} \tag{101}$$

where $\frac{d^2K}{dy^2} = \alpha_n^2K_1(n,y)$ (102)

$$\alpha_n = \left(\frac{n\pi}{a}\right)$$

Substitution of Equations (106) and (107) into the bending moment-deflection equations-Equations (3) and (4) – and simplification gives:

$$M_{xx} = \frac{2q_0a^2}{\pi^5}\sum_{n=1,3,5}^\infty\left(\frac{(n\pi)^2-4}{n^5}\right)(K(n,y)-\mu K_1(n,y))\sin\frac{n\pi x}{a} - \frac{2q_0a^2}{\pi^3}\sum_{n=2,4}^\infty\frac{1}{n^3}(K(n,y)+\mu K_1(n,y))\sin\frac{n\pi x}{a}$$

…(103)

where $K_1(n,y) = \frac{\lambda_n \sinh\lambda_n \cosh\gamma_n - \gamma_n \sinh\gamma_n \cosh\lambda_n}{1+\cosh 2\lambda_n} = \frac{1}{\alpha_n^2} K''(n,y)$ (104)

Similarly,

$$M_{yy} = \frac{2q_0a^2}{\pi^5} \sum_{n=1,3,5}^{\infty} \left(\frac{(n\pi)^2 - 4}{n^5}\right)(K_1(n,y) - \mu K(n,y)) \sin\frac{n\pi x}{a} - \frac{2q_0a^2}{\pi^3} \sum_{n=2,4}^{\infty} \frac{1}{n^3}(K_1(n,y) + \mu K(n,y)) \sin\frac{n\pi x}{a}$$

...(105)

At the plate center, $y = 0$, $x = a/2$, $\gamma_n = 0$

$$K_1(n, y=0) = \frac{\lambda_n \sinh\lambda_n}{1+\cosh 2\lambda_n} \quad (106)$$

where,

$\lambda_n = \frac{n\pi}{2}$ for square plate

$$K_1(n, y=0) = \frac{\lambda_n \sinh\left(\frac{n\pi}{2}\right)}{1+\cosh n\pi} \quad (107)$$

$$K_1(n=1, y=0) = \frac{\pi/2}{1+\cosh\pi} = 0.287078 \quad (108)$$

The solution for center bending moments using the expressions are

$M_{xx} = 1.319 \times 10^{-2} q_0a^2$

$M_{yy} = 1.389 \times 10^{-2} q_0a^2$

The single finite sine series solutions are in agreement with the Levy solutions present by Szilard (2004).

5. Conclusions

In this paper, the single finite sine transform method has been used for closed form flexural solutions of simply supported Kirchhoff plate under parabolic load. The sinusoidal kernel of the finite sine transform satisfies the simply supported boundary conditions along the edges $x = 0$, and $x = a$; and is a suitable transformation method for the considered problem.

In conclusion,

- the single finite sine transform method simplifies the governing equation of equilibrium to a fourth order ODE in the transformed space variables.
- the general solution $V_g(n, y)$ in the transformed space variable is obtained using methods for solving ODEs as the sum of the homogeneous solution and the particular solution.
- the general solution for the deflection in the physical problem coordinate variables $v(x, y)$ is obtained by inversion of $V_g(n, y)$ as a rapidly convergent single series of infinite terms.
- the enforcement of boundary conditions along the simply supported edges $y = 0$, $y = b$ is used to determine the unknown constants of integration in the expression for $v(x, y)$.
- a three-term truncation of the infinite series for $v(x, y)$ gives solutions for the center deflection that is equal to the exact solution showing convergence at the third term.
- bending moments are obtained using the bending moment deflection relations as convergent single series with infinite terms.

Nomenclature

x, y	in-plane Cartesian coordinates
z	transverse Cartesian coordinate
h	thickness
a	least in-plane dimension

b	in-plane dimension (length)
$v(x, y)$	transverse deflection
$q_z(x, y)$	transversely distributed load intensity
D	modulus of flexural rigidity
E	Young's modulus of elasticity
μ	Poisson's ratio
M_{xx}, M_{yy}	bending moments
M_{xy}	twisting moment
Q_x, Q_y	shear force distributions
n	single finite sine integral transform parameter
$V(n, y)$	single finite sine transform of the transverse deflection $v(x, y)$
$Q_z(n, y)$	single finite sine transform of the distributed transverse loading intensity.
I_1	integral defined in terms of sine transform of x^2
V_h	homogeneous solution for $V(n, y)$
A_n	constants used in the trial function solution for V_h
λ	parameters used in the trial function solution for V_h
C_{1n}, C_{2n}, C_{3n}, C_{4n},	integration constants
$V_p(n, y)$	particular solution for $V(n, y)$
$V_g(n, y)$	general solution
$\bar{V}_p$	expression for V_p when n is even or when n is odd.
$\bar{C}_{1n}, \bar{C}_{2n}$	integration constants re-expressed in terms of C_{1n} and $2/a$ and C_{2n} and $2/a$ respectively.
3D	three-dimensions
2D	two-dimensions
FSDT	first order shear deformation theory
FSDTs	first order shear deformation theories
KLPT	Kirchhoff-Love plate theory
TSDT	trigonometric shear deformation theory
DVM(s)	direct variational method(s)
FSTM	finite sine transform method
GITM	generalized integral transform method
FDM(s)	finite difference method(s)
SFSTM	single finite sine transform method
GPDE	governing partial differential equation
ODE(s)	ordinary differential equations
$\frac{\partial^2}{\partial y^2}$	operator for second partial derivative with respect to x
∇^2	Laplace operator
∇^4	biharmonic operator
$\sum$	summation
$\int$	integral
$\frac{\partial^2}{\partial x^2}$	operator for second partial derivative with respect to x
γ_n	parameter defined in terms of n, π, a and y
α_n	parameter defined in terms of n, π, and a
λ_n	parameter defined in terms of n, π, b and a
$K(n, y)$	parameter defined in terms of λ_n, γ_n
%	percent (percentage)
Δ	change in
q_0	value of parabolic load intensity at $x = a, y$
v_0	transverse deflection of the plate at the plate center

$\frac{d^2K(n,y)}{dy^2}$	second derivative of $K(n,y)$ with respect to y
$K_1(n,y)$	parameter defined in terms of the second derivative of $K(n,y)$ with respect to y

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