The inverse of operator matrix A where A≥I and A> 0

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الخلاصة

ليكن كل من H,K فضاء هلبرت وليكنH⊕Kهو الضرب الديكارتي لهما وليكن

B(K,H),B(H,K) B(K),B(H),B(H⊕K) فضاءات باناخ لكل المؤثرات المقيده(المستمره)على H,K, H⊕K ، ومنK الى H ومنH الىK على الترتيب.في هذا البحث سنجد معكوس مصفوفة المؤثر $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ ϵ B(H⊕K) حيث أن BϵB(H),CϵB(K,H),DϵB(H,K),EϵB(K) وأن $A\geq I_{H\oplus K}$, $A>0$ حيث $I_{H\oplus K}$ هي المؤثر المحايد علىH⊕K

ABSTRACT

Let H and K be Hilbert spaces and let H⊕K be the cartesian product of them.Let B(H),B(K),B(H⊕K),B(K,H),B(H,K) be the Banach spaces of bounded(continuous) operators on H,K,H⊕K,and from K into H and from H into K respectively.In this paper we find the inverse of operator matrix $A=\begin{bmatrix} B & C \\ D & E \end{bmatrix}$ ЄB(H⊕K) where BϵB(H) ,CϵB(K,H), DϵB(H,K), EϵB(K) and $A\geq I_{H\oplus K}$, $A>0$where $I_{H\oplus K}$ is the identity operator on H⊕K

Introduction

Let $<,>$ denotes an inner product on a Hilbert space, and we will denote Hilbert spaces by H, K, H_i, K_i and $H\oplus K$ denotes the Cartesian product of the Hilbert spaces H, K ,and B(H) ,B(H⊕K),B(K,H), be the Banach spaces of bounded(continuous) operators on H, H⊕K,and from K into H respectively[see2]. The inner product on H⊕K is define by:

$< (x,y),(w,z) \geq < x,w > + < y,z > \quad x,w\in H, y,z\in K$

we say that A is positive operator on H and denote that by $A\geq 0$ if $< Ax,x >\geq 0$ for all x in H,and in this case it has a unique positive square root ,we denote this square root by $\sqrt{A}$ [see2],it is easy to check that A is invertible if and only if $\sqrt{A}$ is invertible.A^*denotes the adjoint of A and I_H denotes the identity operator on the Hilbert space H.We define the operator matrix $A=\begin{bmatrix} B & C \\ E & D \end{bmatrix}$ϵB($H\oplus K$,L⊕M) where BϵB(H,L), C ϵB(K,L), E ϵB(H,M), D ϵB(K,M) as following $A\begin{pmatrix} x \\ y \end{pmatrix}=\begin{bmatrix} B & C \\ E & D \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{bmatrix} Bx+Cy \\ Ex+Dy \end{bmatrix}$,where $\begin{pmatrix} x \\ y \end{pmatrix}\in H\oplus K$,and similar for the case $m\times n$ operator matrix [see 1&3&6].

If $A=\begin{bmatrix} B & C \\ E & D \end{bmatrix}$then $A^*=\begin{bmatrix} B^* & E^* \\ C^* & D^* \end{bmatrix}$.

If $A = \begin{bmatrix} B & C \\ E & D \end{bmatrix} \geq 0$ then A is a self- adjoint and so has the form $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix}$ and similar for the case $n \times n$ operator matrix [see 1&3]. For related topics[see 7&8]. For elementary facts about matrices [see5 &9] and for elementary facts about Hilbert spaces and operator theory [see 2&6].

Remark: we will sometimes denote $I_{H\oplus K}$(the identity on $H\oplus K$)or I_H(the identity on H)or I_K(the identity on K) or any identity operator by I, and also we will sometimes denote any zero operator by 0

1)Preliminaries:

Proposition1.1.: Let T϶B(H,K) then

1)if $T^*T \geq I$ and $TT^* \geq I$ then T is invertible,

2)if T is self-adjoint, $T^2 \geq I$ then T is invertible,

3)if $T \geq 0$ then T is invertible if and only if $\sqrt{T}$ is invertible, and in this case we have $((\sqrt{T})^2)^{-1} = ((\sqrt{T})^{-1})^2$,

4)if T is self-adjoint then T is invertible from right if and only if it is invertible from left,

5)if $T \geq I$ then T is invertible,

6) if $T \geq 0$ and it is invertible then $T^{-1} \geq 0$,and in this case we have $\sqrt{T^{-1}} = (\sqrt{T})^{-1}$

7) $T \geq I$ if and only if $0 \leq T^{-1} \leq I$.

Proof:1)see[2]p.156

2)from 1)

3) if T is invertible then there exists an operator S such that ST =TS=I ,so $(S\sqrt{T})\sqrt{T} = \sqrt{T}(\sqrt{T}S) = I$ i.e. $\sqrt{T}$ is invertible . Conversely if $\sqrt{T}$ is invertible then there exists an operator R such that $R\sqrt{T} = \sqrt{T}R = I$,so $I = I.I = (\sqrt{T}R)(\sqrt{T}R) = \sqrt{T}(R\sqrt{T})R = \sqrt{T}(\sqrt{T}R)R = TR^2 = R^2T$,hence T is invertible, and in this case we have $((\sqrt{T})^2)^{-1} = T^{-1} = R^2 = ((\sqrt{T})^{-1})^2$.□

4) if T is self-adjoint then $T = T^*$,but T is invertible from right if and only if T^* is invertible from left.□

5) if $T \geq I$ then $T \geq 0$,so $\sqrt{T}$ exists and it is self-adjoint and $(\sqrt{T})^2 \geq I$,so $\sqrt{T}$ is invertible and hence T is invertible.□

6)if $T \geq 0$ and it is invertible then $\langle Tx, x\rangle \geq 0$,so $\langle TT^{-1}x, T^{-1}x\rangle \geq 0$. i.e. $\langle x, T^{-1}x\rangle \geq 0, \forall x$.Hence $T^{-1} \geq 0$.Now $\sqrt{I}$=I,because $\sqrt{I}.\sqrt{I} = I$,and $I.I = I$,but the positive square root is unique(see[2]p.149) so $\sqrt{I} = I$.and since $T \geq 0$, $T^{-1} \geq 0$, $T^{-1}T = I \geq 0$,we have$\sqrt{T^{-1}}\sqrt{T} = \sqrt{T^{-1}T}$(see[2]p.149),so$\sqrt{T^{-1}}\sqrt{T} = \sqrt{I}$=I ,hence $\sqrt{T^{-1}} = \left(\sqrt{T}\right)^{-1}$.□

7)if $T \geq I$ then $T \geq 0$ and it is invertible .so[from 6)]we have $T^{-1} \geq 0$.Now $T^{-1} \geq 0$ &$T - I \geq 0$ & $T^{-1}(T - I) = (T - I)T^{-1}$ [because $T^{-1}(T - I) = T^{-1}T - T^{-1} = I - T^{-1}$ and $(T - I)T^{-1} = TT^{-1} - T^{-1} = I - T^{-1}$].So, $T^{-1}(T - I) \geq 0$ (see[2]p.149) ,hence $T^{-1} \leq I$.□ Conversely if $0 \leq T^{-1} \leq I$ then [from 6)]we have $T \geq 0$ but $I - T^{-1} \geq 0$ and $T(I - T^{-1}) = (I - T^{-1})T$, so $T(I - T^{-1}) \geq 0$,hence $T \geq I$.□

Proposition1.2:1)if $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \geq 0$ then C=D^* and B≥0 & E≥0

2)if $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \geq I$ then C=D^* and B≥ I & E≥I

3) if $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \leq I$ then C=D^* and B≤ I & E≤ I

Proof:1)see[1]p.18.□

2)if $A \geq I$ then $A - I \geq 0$ but $I = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$,so $\begin{bmatrix} B & C \\ D & E \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} B - I & C \\ D & E - I \end{bmatrix} \geq 0$ .Then from 1) we have that C=D^* ,$B - I \geq 0, E - I \geq 0$ i.e. B≥ I & E≥ I.□

3)Similar to 2)

Proposition1.3.: if $A = \begin{bmatrix} B & C \\ C^* & E \end{bmatrix}$ is invertible, $A \geq I$ then B,E are invertible

Proof: from Proposition1.2. 2) we have B≥ I & E≥ I,so B,E are invertible.□

To show that the converse is not true we need the following theorem from[1]p.19:-

Theorem1.4.:Let B∊B(H),E∊B(K),C∊B(K,H) such that B≥0 & E≥0 then:

$\begin{bmatrix} B & C \\ C^* & E \end{bmatrix} \geq 0$ if and only if there exists a contraction X∊B(K,H) such that $C = \sqrt{B}\, X \sqrt{E}$.

Now the following example show that the converse of proposition1.3. is not true

Example1.5: Let $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$,so B=2≥1,E=2≥1 and they are invertible but A is not invertible[since det A=0].Note that $A \geq$0[since C=2=$\sqrt{2}\,\sqrt{2}$=$\sqrt{B}\, X \sqrt{E}$ where X =1,hence$|X| \leq 1$],but $A \ngeq I$[since $A - I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$,and if ∃$X$ such that 2=$\sqrt{1}\, X \sqrt{1}$,so X =2,hence$|X| \nleq 1$ i.e. $A - I \ngeq 0$,hence $A \ngeq I$].

Remark1.6.: it is easy to check that:

1)if A is invertible $m \times n$ operator matrix (i.e. ∃ an $n \times m$ operator matrix B s.t. $AB = I_m$ & $BA = I_n$,

Where I_m & I_n are the $m \times m$ & the $n \times n$ identity operator matrices respectively) and if matrix C results from A by interchanging two rows(columns) of A then C is also invertible.

2)if two rows(columns) of an $m \times n$ operator matrix A are equal then A is not invertible.

3)if a row(column) of an $m \times n$ operator matrix A consists entirely of zero operators then A is not invertible.

4) $A = \begin{bmatrix} B & 0 \\ 0 & E \end{bmatrix}$ is invertible if and only if B,E are invertible, and in this case $A^{-1} = \begin{bmatrix} B^{-1} & 0 \\ 0 & E^{-1} \end{bmatrix}$.

Remark1.7.:from remark1.6. 1) we can conclude :if $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ ∊B(H⊕K, L⊕M) then

$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} C & B \\ E & D \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} D & E \\ B & C \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} E & D \\ C & B \end{bmatrix}$ is invertible.

2) The inverse of a 2×2 operator matrix A where A≥I

Theorem2.1.:1)if $A = \begin{bmatrix} B & C \\ C^* & E \end{bmatrix} \geq I$ then $B, E, B - CE^{-1}C^*, E - C^*B^{-1}C$ are invertible and $A^{-1} = \begin{bmatrix} (B - CE^{-1}C^*)^{-1} & -(B - CE^{-1}C^*)^{-1}CE^{-1} \\ -E^{-1}C^*(B - CE^{-1}C^*)^{-1} & (E - C^*B^{-1}C)^{-1} \end{bmatrix}$

In fact :2)if $A=\begin{bmatrix} B & C \\ C^* & E \end{bmatrix} \geq I$ then $B \geq I, E \geq I, B - CE^{-1}C^* \geq I, E - C^*B^{-1}C \geq I$

Proof 1) if $A=\begin{bmatrix} B & C \\ C^* & E \end{bmatrix} \geq I$ then A is invertible[proposition1.1.5)] and

$B \geq I, E \geq I$[proposition1.2.2)] ,so B,E are invertible [proposition1.1.5)]

Now, let $A^{-1}=\begin{bmatrix} J & G \\ G^* & F \end{bmatrix}$ i.e. $AA^{-1}=I=\begin{bmatrix} I_H & 0 \\ 0 & I_K \end{bmatrix}$

,then $J \geq 0, F \geq 0$ since $A^{-1} \geq 0$ .And

i)$BJ + CG^* = I_H$, ii)$BG + CF = 0$,iii)$C^*J + EG^* = 0$, iv)$C^*G + EF = I_K$.

So from iii) we have $JC + GE = 0$.So,

$G = -JCE^{-1} = -B^{-1}CF$.

Then we have from iv) that

$(E - C^*B^{-1}C)F = I_K$ i.e. $E - C^*B^{-1}C$ is invertible , $F = (E - C^*B^{-1}C)^{-1}$,

and from i) we have $J(B - CE^{-1}C^*) = I_H$,so $B - CE^{-1}C^*$ is invertible, and

$J = (B - CE^{-1}C^*)^{-1}$, $G = -(B - CE^{-1}C^*)^{-1}CE^{-1} = -B^{-1}C(E - C^*B^{-1}C)^{-1}$.

Then it is clear that,$A^{-1}=\begin{bmatrix} (B - CE^{-1}C^*)^{-1} & -(B - CE^{-1}C^*)^{-1}CE^{-1} \\ -E^{-1}C^*(B - CE^{-1}C^*)^{-1} & (E - C^*B^{-1}C)^{-1} \end{bmatrix}$

2) if$A=\begin{bmatrix} B & C \\ C^* & E \end{bmatrix} \geq I$ then

$$0 \leq A^{-1}=\begin{bmatrix} (B - CE^{-1}C^*)^{-1} & -(B - CE^{-1}C^*)^{-1}CE^{-1} \\ -E^{-1}C^*(B - CE^{-1}C^*)^{-1} & (E - C^*B^{-1}C)^{-1} \end{bmatrix} \leq I$$

,so from proposition1.2.1&3)We have$0 \leq (B - CE^{-1}C^*)^{-1} \leq I, 0 \leq (E - C^*B^{-1}C)^{-1} \leq I$,

then from proposition1.1.7)

$B - CE^{-1}C^* \geq I, E - C^*B^{-1}C \geq I$,

also from proposition 1.2.2) We have that B≥ I & E≥I.▫

Remark2.2.:it is easy to check that if $B, E, B - CE^{-1}C^*, E - C^*B^{-1}C$ are invertible then $A = \begin{bmatrix} B & C \\ C^* & E \end{bmatrix}$ is invertible and

$$A^{-1} = \begin{bmatrix} (B - CE^{-1}C^*)^{-1} & -(B - CE^{-1}C^*)^{-1}CE^{-1} \\ E^{-1}C^*(B \quad CE^{-1}C^*)^{-1} & (E \quad C^*B^{-1}C)^{-1} \end{bmatrix}.$$

Remark2.3.:since $(B - CE^{-1}C^*)^{-1}CE^{-1} = B^{-1}C(E - C^*B^{-1}C)^{-1}$,and since $A = \begin{bmatrix} B & C \\ C^* & E \end{bmatrix} \geq I$,hence $A - I \geq 0$ and $A \geq 0$,therefore there exists a contraction X and a contraction Y such that

$$C = \sqrt{B}\, X \sqrt{E} = \sqrt{B - I}\, Y \sqrt{E - I}$$

then we have alternative forms of A^{-1} such:

$$1)\ A^{-1} = \begin{bmatrix} (B - CE^{-1}C^*)^{-1} & -B^{-1}C(E - C^*B^{-1}C)^{-1} \\ -E^{-1}C^*(B - CE^{-1}C^*)^{-1} & (E - C^*B^{-1}C)^{-1} \end{bmatrix} \text{ or}$$

$$2) A^{-1} = \begin{bmatrix} (\sqrt{B})^{-1}(I - XX^*)^{-1}(\sqrt{B})^{-1} & -(\sqrt{B})^{-1}(I - XX^*)^{-1}X(\sqrt{E})^{-1} \\ -(\sqrt{E})^{-1}X^*(I - XX^*)^{-1}(\sqrt{B})^{-1} & (\sqrt{E})^{-1}(I - X^*X)^{-1}(\sqrt{E})^{-1} \end{bmatrix} \ldots \text{etc.}$$

Remark2.4.:the second form of A^{-1} above show that $I - XX^*$, $I - X^*X$ are invertible and this is easy to check.

Remark2.5.:we know that if a ,c , e are complex numbers(the complex number is a special case of an operator) and

$A = \begin{bmatrix} b & c \\ c^* & e \end{bmatrix}$ where c^* is the conjugate of c then $A^{-1} = \begin{bmatrix} \frac{e}{be - |c|^2} & \frac{-c}{be - |c|^2} \\ \frac{-c^*}{be - |c|^2} & \frac{b}{be - |c|^2} \end{bmatrix}$ but from above:

$$A^{-1} = \begin{bmatrix} (b - ce^{-1}c^*)^{-1} & -(b - ce^{-1}c^*)^{-1}ce^{-1} \\ -e^{-1}c^*(b - ce^{-1}c^*)^{-1} & (e - c^*b^{-1}c)^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{b - \frac{|c|^2}{e}} & -\frac{1}{b - \frac{|c|^2}{e}} c \frac{1}{e} \\ -\frac{1}{e} c^* \frac{1}{b - \frac{|c|^2}{e}} & \frac{1}{e - \frac{|c|^2}{b}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e}{be - |c|^2} & \frac{-c}{be - |c|^2} \\ \frac{-c^*}{be - |c|^2} & \frac{b}{be - |c|^2} \end{bmatrix}. \square$$

Remark2.6.:of course we can generalize the 2×2 case to the $n \times n$ case by iteration. For example: if $A = \begin{bmatrix} B & C & D \\ C^* & E & G \\ D^* & G^* & F \end{bmatrix} \geq I$,then

$A = \begin{bmatrix} B & C & D \\ C^* & E & G \\ D^* & G^* & F \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} B & C \\ C^* & E \end{bmatrix} & \begin{bmatrix} D \\ G \end{bmatrix} \\ [D^* & G^*] & F \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} B & C \\ C^* & E \end{bmatrix} & \begin{bmatrix} D \\ G \end{bmatrix} \\ \begin{bmatrix} D \\ G \end{bmatrix}^* & F \end{bmatrix}$,and we can first find the inverse of $\begin{bmatrix} B & C \\ C^* & E \end{bmatrix} \geq I$,then find the inverse of A.

Remark2.7.:there is no general relation between the invertiblity of $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ and the invertiblity of B, C, D, E ,and all the 32 cases can be hold,for example

1)$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible but B, C, D, E are invertible

2) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is invertible and also B, C, D, E are invertible

3)$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}$is not invertible

[sincedet A=0]and$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

is not invertible, but $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, E = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ are invertible.

And so on.

of course, $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} C & B \\ E & D \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} D & E \\ B & C \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} E & D \\ C & B \end{bmatrix}$ is invertible, is useful here

3) The inverse of a 2×2 operator matrix A where $A > 0$

In this section we generalize the results of **A≥I to A**> 0 .

Theorem 3.1.:if $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} > 0$ is an invertible then so are B&D .

Proof: $C = \sqrt{B}\, X \sqrt{D}$, $C^* = \sqrt{D}\, X^* \sqrt{E}$ and $\exists M = \begin{bmatrix} E & G \\ G^* & F \end{bmatrix}$ s.t. $AM = \mathbf{I} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ then

$BE + \sqrt{B}\, X \sqrt{D} G^* = I$, $\sqrt{D}\, X^* \sqrt{B} G + DF = I$.Hence,

$\sqrt{B}\left(\sqrt{B}\, E + X \sqrt{D} G^*\right) = I$, $\sqrt{D}\,(X^* \sqrt{B} G + \sqrt{D}\, F) = I$ .So, $\sqrt{B}$, $\sqrt{D}$ are invertible ,then B, D are invertible

Remark 3.2.:the converse of theorem 3.1.is not true as we can see by the following example.

Example 3.3.:let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} > 0$ (since $C = \mathbf{1} = \sqrt{1}\, X \sqrt{1}$ where $X = 1$ and $\|X\| = |1| = 1$ so $A > 0$), then $B = 1$, $D = 1$ are invertible but A is not an invertible ($\det A = 0$).

Remark 3.4.:if A is not positive then it is may be that $A = \begin{bmatrix} B & C \\ E & D \end{bmatrix}$ is an invertible but B, D are not , as we can see by the following example.

Example 3.5.:let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then A is not positive($C = 1 \neq \sqrt{B}\, X \sqrt{D} = 0$)and A is an invertible ($\det A \neq 0$)but $B = 0$, $D = 0$ are not invertible.

The main result in this section is the following:

Theorem 3.6.: $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} > 0$ is an invertible if and only if
B ,D, $B - CD^{-1}C^*$,$D - C^*B^{-1}C$ are invertible, and in this case we have:

$$A^{-1} = \begin{bmatrix} (B - CD^{-1}C^*)^{-1} & -(B - CD^{-1}C^*)^{-1}CD^{-1} \\ -D^{-1}C^*(B - CD^{-1}C^*)^{-1} & (D - C^*B^{-1}C)^{-1} \end{bmatrix}.$$

Proof: $\Rightarrow$)If $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} > 0$ is an invertible $A^{-1} > 0$

so let $A^{-1} = \begin{bmatrix} E & G \\ G^* & F \end{bmatrix}$ then

i)$BE + CG^* = I$,ii) $BG + CF = 0$,iii)$C^*E + DG^* = 0$,

iv)$C^*G + DF = I$,then we have

$G = -ECD^{-1} = -B^{-1}CF$ .Hence

$E(B - CD^{-1}C^*) = I$,i.e. $B - CD^{-1}C^*$ is an invertible and $E = (B - CD^{-1}C^*)^{-1}$.

$(D - C^*B^{-1}C)F = I$, i.e. $D - C^*B^{-1}C$ is an invertible and

$F = (D - C^*B^{-1}C)^{-1}$.Then it is clear that

$$A^{-1} = \begin{bmatrix} (B - CD^{-1}C^*)^{-1} & -(B - CD^{-1}C^*)^{-1}CD^{-1} \\ -D^{-1}C^*(B - CD^{-1}C^*)^{-1} & (D - C^*B^{-1}C)^{-1} \end{bmatrix}.$$

$\Leftarrow$)if we let $M = \begin{bmatrix} (B - CD^{-1}C^*)^{-1} & -(B - CD^{-1}C^*)^{-1}CD^{-1} \\ -D^{-1}C^*(B - CD^{-1}C^*)^{-1} & (D - C^*B^{-1}C)^{-1} \end{bmatrix}$ then it is easy to check that $AM = I$ i.e. $M = A^{-1}$ □

From the proof of theorem 3.6 we can prove that

Theorem 3.7.:if B&D are invertible then

$A = \begin{bmatrix} B & C \\ E & D \end{bmatrix} \in B(H\oplus K, L\oplus M)$ is an invertible if and only if $B - CD^{-1}E, D - EB^{-1}C$ are invertible and in this case we have

$$A^{-1} = \begin{bmatrix} (B - CD^{-1}E)^{-1} & -(B - CD^{-1}E)^{-1}CD^{-1} \\ -(D - EB^{-1}C)^{-1}EB^{-1} & (D - EB^{-1}C)^{-1} \end{bmatrix} \in B(L\oplus M, H\oplus K)$$

Proof: Similar to proof of theorem 3.6..

Remark 3.8.: Also we can get the following alternative forms of A^{-1}

1) $A^{-1} = \begin{bmatrix} (B - CD^{-1}E)^{-1} & -B^{-1}C(D - EB^{-1}C)^{-1} \\ -(D - EB^{-1}C)^{-1}EB^{-1} & (D - EB^{-1}C)^{-1} \end{bmatrix}.$

2) $A^{-1} = \begin{bmatrix} (B - CD^{-1}E)^{-1} & -(B - CD^{-1}E)^{-1}CD^{-1} \\ -D^{-1}E(B - CD^{-1}E)^{-1} & (D - EB^{-1}C)^{-1} \end{bmatrix}.$

3) $A^{-1} = \begin{bmatrix} (B - CD^{-1}E)^{-1} & -B^{-1}C(D - EB^{-1}C)^{-1} \\ -D^{-1}E(B - CD^{-1}E)^{-1} & (D - EB^{-1}C)^{-1} \end{bmatrix}.$

Remark 3.9.: $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} \geq I$ is special case of $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} > 0$(because $A \geq I > 0$).And if $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} \geq I$ then it is necessary that A is invertible then B ,D,$B - CD^{-1}C^*$,$D - C^*B^{-1}C$ are invertible , in fact $B \geq I$,$D \geq I$, $B - CD^{-1}C^* \geq I$,$D - C^*B^{-1}C \geq I$,(and hence they are invertible).And if they are invertible then $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix}$ is an invertible. So we may ask the following question :

Question 3.10.:is it true that if $B \geq I$,$D \geq I$,$B - CD^{-1}C^* \geq I$,$D - C^*B^{-1}C \geq I$ then $A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix} \geq I$?

But the following example show that this is not true:-

Example 3.11.: $A = \begin{bmatrix} 5 & 4.1 \\ 4.1 & 5 \end{bmatrix}$ then $B \geq 1$, $D \geq 1$,

$B - CD^{-1}C^* = B - \frac{|C|^2}{D} = 5 - \frac{16.81}{5} = 1.638 \geq 1$,

$D - C^*B^{-1}C = D - \frac{|C|^2}{B} = 5 - \frac{16.81}{5} = 1.638 \geq 1$ but,

$A = \begin{bmatrix} 5 & 4.1 \\ 4.1 & 5 \end{bmatrix} \geq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if and only if $\begin{bmatrix} 4 & 4.1 \\ 4.1 & 4 \end{bmatrix} \geq 0$ but this is not true(because it is true if and only if there exists X , $|X| \leq 1$ such that $4.1 = \sqrt{4}X\sqrt{4}$,but then

$|X| = \frac{4.1}{4} > 1$,a contradiction).

Remark3.12: If T∈B(H,K) then it is easy to check
T is an invertible if and only if $T^*T \in B(H,H)$& $TT^* \in B(K,K)$ are invertible and in this case we have $T^{-1} = (T^*T)^{-1}T^* = T^*(TT^*)^{-1}$

Also from [2] we have:

i)$T^*T \geq 0$ and $TT^* \geq 0$

ii)$T \neq 0$ if and only if $T^*T \neq 0$ if and only if $T^*T \neq 0$
so we have that
T is an invertible if and only if $T^*T > 0$ & $TT^* > 0$ are invertible and in this case we have $T^{-1} = (T^*T)^{-1}T^* = T^*(TT^*)^{-1}$. Hence we can use this fact to find the inverse of $A = \begin{bmatrix} B & C \\ E & D \end{bmatrix}$ (if it exists) by first find the inverses of $AA^* > 0$ & $A^*A > 0$ and use them to find the inverse of A,so

Theorem3.13: $A = \begin{bmatrix} B & C \\ E & D \end{bmatrix}$∈B(H⊕K,L⊕M) is an invertible if and only if
1)$a = BB^* + CC^*$ 2) $b = EE^* + DD^*$ 3)$c = a - (BE^* + CD^*)b^{-1}(BE^* + CD^*)^*$
4)$d = b - (EB^* + DC^*)a^{-1}(EB^* + DC^*)^*$ 5) $e = B^*B + E^*E$ 6) $f = C^*C + D^*D$
7) $g = e - (B^*C + E^*D)f^{-1}(B^*C + E^*D)^*$ 8) $h = f - (C^*B + D^*E)e^{-1}(C^*B + D^*E)^*$
are invertible and in this case we have

$$A^{-1} = \begin{bmatrix} g^{-1}(B^* - (B^*C + E^*D)f^{-1}C^*) & g^{-1}(E^* - (B^*C + E^*D)f^{-1}D^*) \\ h^{-1}C^* - f^{-1}(C^*B + D^*E)g^{-1}B^* & h^{-1}D^* - f^{-1}(C^*B + D^*E)g^{-1}E^* \end{bmatrix}$$

$$= \begin{bmatrix} (B^* - E^*b^{-1}(EB^* + DC^*))c^{-1} & E^*d^{-1} - B^*c^{-1}(BE^* + CD^*)b^{-1} \\ (C^* - D^*b^{-1}(EB^* + DC^*))c^{-1} & D^*d^{-1} - C^*c^{-1}(BE^* + CD^*)b^{-1} \end{bmatrix}$$

Proof: A is an invertible if and only if $AA^* > 0$ & $A^*A > 0$ are invertible if and only if a, b, c, d, e, f, g, h are invertible and we have

$$A^{-1} = (A^*A)^{-1}A^* = \begin{bmatrix} g^{-1}(B^* - (B^*C + E^*D)f^{-1}C^*) & g^{-1}(E^* - (B^*C + E^*D)f^{-1}D^*) \\ h^{-1}C^* - f^{-1}(C^*B + D^*E)g^{-1}B^* & h^{-1}D^* - f^{-1}(C^*B + D^*E)g^{-1}E^* \end{bmatrix}$$

$$= A^*(AA^*)^{-1}$$

$$= \begin{bmatrix} (B^* - E^*b^{-1}(EB^* + DC^*))c^{-1} & E^*d^{-1} - B^*c^{-1}(BE^* + CD^*)b^{-1} \\ (C^* - D^*b^{-1}(EB^* + DC^*))c^{-1} & D^*d^{-1} - C^*c^{-1}(BE^* + CD^*)b^{-1} \end{bmatrix}$$

Remark3.14: we can generalize theorem3.13 and find the inverse of the $m \times n$ operator matrix A by first we find the inverse of $AA^* > 0$ & $A^*A > 0$ by iteration as we did in remark2.6.then we find A^{-1} by the relation $A^{-1} = (A^*A)^{-1}A^* = A^*(AA^*)^{-1}$

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